3. Lecture 3: Plane tropical curves

Exercise 3.1 (Inspired by Exercise 3.7(1) in Maclagan-Sturmfels)**.** Find the associated tropical polynomials and draw the tropical vanishing loci of the following polynomials in $k\{\{t\}\}[x, y]$

(1) $f = (t^{-1} + 1)z_1 + (t^2 - 3t^3)z_2 + 5t^4$
(2) $f = t^3z_1^2 + 2z_1z_2 + tz_2^2 + (t + t^3)z_1 - 5z_2 + 1$

Exercise 3.2. Use the dual subdivision to show that plane tropical curves satisfy the *balancing condition*: At every vertex $v \in \Gamma$ and every edge e with vertex v, let $u_e \in \mathbb{Z}^2$ be the vector pointing away from v in direction e with entries coprime (the u_e are drawn as arrows in Figure 1). Then for a fixed vertex v

$$\sum_{v \in e} w(e)u_e = 0.$$

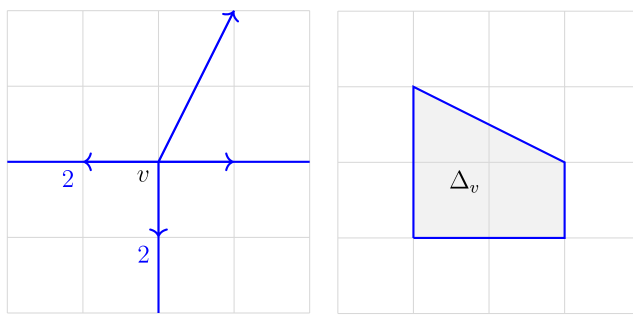

Figure 1. Balancing condition

Exercise 3.3 (Exercise 2.4(4) in Brugallé-Shaw)**.** Show that a tropical curve with Newton polygon Δ_d has at most d^2 vertices.

Theorem 3.4 (Pick's theorem)**.** Suppose that a polygon has integer coordinates for all of its vertices. Let i be the number of integer points interior to the polygon, and let b be the number of integer points on its boundary (including both vertices and points along the sides), and let A be its area. Then

$$A = i + \frac{b}{2} - 1.$$

Exercise 3.5. A plane tropical curve is *smooth* if its dual subdivision is a unimodular triangulation, meaning that every polygon in the subdivision is a triangle with no lattice points (not on the boundary or in the interior) besides its vertices. The *genus* of a smooth tropical plane curve is its first Betti number.

(1) Use Pick's theorem to show that every triangle in the dual subdivision of a plane smooth tropical curve has area $\frac{1}{2}$.
(2) Recall that

$$\Delta_d = \text{Conv}\{(0, 0), (d, 0), (0, d)\} \subset \mathbb{R}^2$$

and that the combinatorial type of a tropical curves is determined by its dual subdivision. Find all possible combinatorial types of smooth plane tropical curves with Newton polygon Δ_2. If you enjoyed this, try the same for smooth plane tropical curves with Newton polygon Δ_3.

(3) Compute the genus if a smooth plane tropical curve with Newton polygon Δ_2 and Δ_3.

(4) Show that the genus of a smooth plane tropical curve with Newton polygon Δ_d equals the number of interior lattice points of Δ_d. What is this number in terms of d?