2. Lecture 2: $\mathbb{A}^1$-degrees

Theorem 2.1. (Sylvester's Law of Inertia): Over a field k of characteristic not equal to two, every symmetric bilinear form $\beta \colon V \times V \to k$ can be *diagonalized*. That is, there exists a basis $e_1, \ldots, e_n$ for V for which the *Gram matrix* $\beta(e_i, e_j)$ is diagonal.

Note 2.2. Observe that this theorem implies that $\mathrm{GW}(k)$ is additively generated by the rank one forms $\langle a \rangle$ for $a \in k^\times$.

Exercise 2.3. Suppose someone hands you the following Gram matrix for a symmetric bilinear form β over $\mathbb{Q}$:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

Diagonalize this form in the sense above. (**warning**: This is not the same as diagonalizing a linear transformation into eigenvalues)

Exercise 2.4. Compute $\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\langle 1 \rangle$.

Exercise 2.5. Show for $a \in k^\times$ that

$$\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle .$$

Exercise 2.6. Compute the (classical) intersection multiplicity of $f(x_1, x_2) = x_1 - x_2^2$ and $g(x_1, x_2) = x_1 + x_2^2$ at $(x_1, x_2) = (0, 0)$.

Exercise 2.7. Compute the enriched intersection multiplicity of $f(x_1, x_2) = x_1 - x_2^2$ and $g(x_1, x_2) = x_1 + x_2^2$ at $(x_1, x_2) = (0, 0)$:

(1) First compute the Bézoutian

$$\text{Béz}(f) = \det \begin{pmatrix} \frac{f(x_1,x_2)-f(y_1,x_2)}{x_1-y_1} & \frac{f(y_1,x_2)-f(y_1,y_2)}{x_2-y_2} \\ \frac{g(x_1,x_2)-g(y_1,x_2)}{x_1-y_1} & \frac{g(y_1,x_2)-g(y_1,y_2)}{x_2-y_2} \end{pmatrix}$$

This lives in the k-algebra

$$\frac{k[x_1, x_2]}{(f(x_1, x_2), g(x_1, x_2))} \otimes \frac{k[y_1, y_2]}{(f(y_1, y_2), g(y_1, y_2))}.$$

(2) Compute a k-vector space basis $\beta_i(x_1, x_2)$ for the local k-algebra

$$Q_0(f, g) = k[x_1, x_2]_{(x_1,x_2)}/(f(x_1, x_2), g(x_1, x_2).$$

(3) Write the Bézoutian as

$$\text{Béz}(f) = \sum_{i,j} a_{ij}\beta_i(x_1, x_2)\beta_j(y_1, y_2),$$

for some $a_{ij} \in k$. Observe that (a_{ij}) is the Gram matrix of a symmetric bilinear form, which is exactly the enriched intersection multiplicity.