EXERCISES: PCMI 2024

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1. Lecture 1: Enumerative geometry

Exercise 1.1. Show that $\mathcal{O}(d_1) \oplus \ldots \oplus \mathcal{O}(d_n) \to \mathbb{P}^n_k$ is relatively orientable if and only if $n + 1 \equiv \sum_{i=1}^n d_i \mod 2$.

Exercise 1.2.

(1) Argue that there is a bijection
$$\left\{\text{conics in } \mathbb{P}^2\right\} \leftrightarrow \mathbb{P}^5,$$
given by sending a conic to its coefficients.

(2) Argue that a conic is *uniquely* determined by five generic points it passes through.

(3) How many conics are there going through four generic points and tangent to one line (not containing any of the four points) in the plane?

Exercise 1.3. Can you ever draw a non-degenerate conic which crosses itself? That is, if you parametrized it, could it cross itself? If yes, give an example! If no, prove the answer is no.

Exercise 1.4. In this exercise we want to describe the locus $\mathcal{D} \subseteq \mathbb{P}^5$ of "bad conics." This is called the *discriminant locus.*

(1) Let $F(x, y, z) = 0$ be a conic. If we can rewrite
$$F(x, y, z) = \ell_1(x, y, z)^2 - \ell_2(x, y, z)^2,$$
for linear equations ℓ_1 and ℓ_2, then we have that
$$F(x, y, z) = (\ell_1 + \ell_2)(\ell_1 - \ell_2),$$
and therefore $V(F)$ is a pair of lines.

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(2) Take an equation for a general conic:

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 = 0.$$

Restrict our attention to the U_0 patch (meaning set $z = 1$), and suppose that $A \neq 0$. Try to follow out the argument above (hint: group terms on either side of the equals sign and try to complete squares).

(3) Find a sufficient condition on the coefficients $A, \ldots, F$ for this to work. If you found the right condition it turns out that it is necessary and sufficient!

(4) Conclude that the discriminant locus $\mathcal{D} \subseteq \mathbb{P}^5$ is a hypersurface of degree 3.

(5) (Optional) The condition you found above is the determinant of a certain 3×3 matrix. What matrix is it? What does it mean?

(6) Argue a 1-parameter family of conics has three degenerate elements. Bonus: visualize this over $\mathbb{R}$.