PCMIL: $\mathbb{A}^1$-HOMOTOPY THEORY AND THE WEIL CONJECTURES

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CONTENTS

1. PROBLEM SESSION FOR LECTURE 1

Problem 1. *Let* X *be a dualizable object of a symmetric monoidal category* $(\mathcal{C}, \otimes, 1, \tau)$ *with dual* $\mathbb{D}X$*. Show there is a natural bijection*

$$\mathrm{Map}(X \otimes Y, Z) = \mathrm{Map}(Y, \mathbb{D}X \otimes Z)$$

Problem 2. *Compute the categorical trace of a rational function on* $\mathbb{P}^1$*, e.g.* $z \mapsto z^3$, $z \mapsto z^2+z+1$, $z \mapsto \frac{z^5+1}{z^2+2z+1}$

Problem 3. *Compute the Zeta function of* $\mathbb{P}^n$ *over a finite field.*

Problem 4. *Show that* $\mathrm{Th}(\mathbf{P}^1, \mathcal{O}(2)) \simeq \mathrm{Th}(\mathbf{P}^1, \mathcal{O}(-2)) \simeq S^0 \vee \mathbf{P}^1$.

Problem 5. [Sil09, V §2] *Let* E *be an elliptic curve over a finite field and let* F *denote the relative Fröbenius. Show the Zeta function of* E *is of the form*

$$\frac{(1 - aT + qT^2)}{(1-T)(1-qT)}$$

as follows. Let $T_\ell E$ *denote the Tate module. Let* $\mathrm{End}(E) \to \mathrm{End}(T_\ell E)$ *be denoted by* $\psi \mapsto \psi_\ell$*. One can use the Tate pairing to show that* $\det \psi_\ell = \deg \psi$*.* $E(\mathbb{F}_{q^m}) = \deg(1 - F^m)$*. Compute* $\det(T - F_\ell^m)$ *in terms of roots of* $\det(T - F_\ell)$*.*

REFERENCES

[Sil09] Joseph H. Silverman, *The arithmetic of elliptic curves*, Second, Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009. MR2514094 ↑5

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