CHARACTERICTIC CLASSES IN MOTIVIC HOMOTOPY THEORY (PRELIMINARY VERSION)

F. DÉGLISE

Contents

References

- algebraic geometry: [5] for an introduction to scheme theory (§12-16), [4] for intersection theory (§1, Appendix (A),B will be useful), [more classical [7] (Chap. I, II, III, appendix A)]
- algebraic topology: [8] (Chap. 0-5, +chap. 16)
- the ∞-categorical language: [6] (for example).

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- general introduction to motivic homotopy theory: [1], [2] (stable case, use ∞-categories), [3] (more comprehensive)
- basic algebraic K-theory: [9], chap. IV

Conventions

We fix a base scheme S, which is only assumed to be noetherian finite dimensional.

By convention, smooth S-schemes will mean smooth separated of finite type S-schemes. We let $\mathscr{S}m_S$ be the category of such smooth S-schemes.

1. Oriented spectra and their characteristic classes

1.1. **Stable homotopy theory.** Stable homotopy: P1-spectra

Ring spectra and associated cohomology theory

examples: classical cohomology (=mixed Weil theories) mot. coh. (and Chow groups), K-theory (and KH), cobordism theory (MGL and Levine-Morel)

representability of Picard group

1.2. **Spectra and orientation theory.** definition, first Chern class

examples, beware of non-orientability

1.3. **Projective bundle theorem and Chern classes.** statement of the theorem

definition of Chern classes

properties (pullback, invariance under iso, Whitney sum)

the splitting principle

1.4. **The formal group law of oritend spectra.** quick recall on Formal groups

construction via Segre embedding

examples

1.5. **Thom classes.** The Euler spectra sequence

Thom classes and thom isomorphism (vb case)

formula via Chern classes

2. Fundamental classes and Grothendieck-Riemann-Roch theorem

Intro: fundamental classes and algebraic cycles $\Rightarrow$ cohomology with support. cobordism, cobordism classes. aim for a unification of the two concepts

a preliminary: virtual bundle and associated Thom space

2.1. **Bivariant theory.** a glimpse on the 6 functors formalismm: adjunction and exceptional direct image properties: support, purity, base change

definition of twisted bivariant theory (generalisation of Fulton-MacPherson)

example: cohomology with support, Borel-Moore homology

2.2. **Construction of fundamental classes.** Main theorem

idea of proof: case of closed immersion (deformation to the normal cone) smooth case and glueing

Examples: functorial trace maps, Gysin morphism in cohomology Comparison with other theories (Chow groups, K-theory)

Remark: the 4 theories, singular case.

2.3. **GRR theorems.** Morphisms of ring spectra and change of orientation.

Todd class associated to a change of orientation. Interpretation via stric morphisms of FGL.

Applications: abstract Riemann-Roch formulas: fundamental clasess, Gysin morphisms, duality and purity, residue maps

Example: Chern character, Adams operations

3. Generalized orientation and quadratic classes

3.1. **Generalized cohomology and quadratic forms.**

3.2. **Generalized orientations.** linear algebraic groups and torsors

Panin-Walter's definition

examples: GL, SL, Sp

Associated cobordism theory

stable Thom classes

Euler classes: obstruction theory (Barge-Morel, Asok, Fasel)

3.3. **The symplectic case.** hyperbolic projective plane, symplectic projective bundle formula

Borel classes

example: link with Chern classes in the oriented case

properties of Borel classes, the symplectic splitting principle

3.4. **Formal ternary laws.** definition

examples

the walter ring

height of FTL and computations

3.5. **The quadratic Riemann-Roch theorem.** morphism of spectra and change of orientation, symplectic todd classes

formulation

examples: K3 surfaces

References

[1] B. Antieau and E. Elmanto. A primer for unstable motivic homotopy theory. In *Surveys on recent developments in algebraic geometry*, volume 95 of *Proc. Sympos. Pure Math.*, pages 305–370. Amer. Math. Soc., Providence, RI, 2017.

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[3] B. I. Dundas, M. Levine, P. A. Ø stvær, O. Röndigs, and V. Voevodsky. *Motivic homotopy theory*. Universitext. Springer-Verlag, Berlin, 2007. Lectures from the Summer School held in Nordfjordeid, August 2002.

[4] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

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