PCMI: $\mathbb{A}^1$-HOMOTOPY THEORY AND THE WEIL CONJECTURES PROBLEM SESSIONS

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1. Problem Session for Lecture 1

Problem 1. *Let X be a dualizable object of a symmetric monoidal category $(\mathcal{C}, \otimes, 1, \tau)$ with dual $\mathbb{D}X$. Show there is a natural bijection*

$$\mathrm{Map}(X \otimes Y, Z) = \mathrm{Map}(Y, \mathbb{D}X \otimes Z)$$

Problem 2. *Compute the categorical trace of a rational function on $\mathbb{P}^1$, e.g. $z \mapsto z^3$, $z \mapsto z^2+z+1$, $z \mapsto \frac{z^5+1}{z^2+2z+1}$*

Problem 3. *Compute the Zeta function of $\mathbb{P}^n$ over a finite field.*

Problem 4. *Show that $\mathrm{Th}(\mathbf{P}^1, \mathcal{O}(2)) \simeq \mathrm{Th}(\mathbf{P}^1, \mathcal{O}(-2)) \simeq (\mathbf{P}^1)^{\wedge 2} \vee \mathbf{P}^1$.*

Problem 5. [Sil09, V §2] *Let E be an elliptic curve over a finite field and let F denote the relative Fröbenius. Show the Zeta function of E is of the form*

$$\frac{(1 - aT + qT^2)}{(1-T)(1-qT)}$$

as follows. Let $T_\ell E$ denote the Tate module. Let $\mathrm{End}(E) \to \mathrm{End}(T_\ell E)$ be denoted by $\psi \mapsto \psi_\ell$. One can use the Tate pairing to show that $\det \psi_\ell = \deg \psi$. $E(\mathbb{F}_{q^m}) = \deg(1 - F^m)$. Compute $\det(T - F_\ell^m)$ in terms of roots of $\det(T - F_\ell)$.

2. Problem Session for Lecture 2

Problem 6. *Let u be a non-square in $\mathbf{F}_{q^m}$. Compute $\mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_q} \langle u \rangle$ for $m = 1, 2, 3, \ldots$*

Problem 7. *Compute the logarithmic $\mathbb{A}^1$-zeta function of $\mathbb{P}^n$ over a finite field.*

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Problem 8. *Check that a symmetric monoidal functor (also called a $\otimes$-functor) $H : \mathcal{C} \to \mathcal{D}$ takes dualizable objects to dualizable objects and* $\mathrm{Tr}\, H = H\, \mathrm{Tr}$.

Problem 9. *(See e.g.* [Mil13, Lemma 27.5]*) Let P be the characteristic polynomial of an endomorphism F of a finite dimensional vector space. Suppose $P(t) = \prod_r (1 - \alpha_r t)$. Show that* $\mathrm{Tr}(F^m) = \sum \alpha_i^m$ *and*

$$d \log P(t) = -\sum_m \mathrm{Tr}(F^m) t^{m-1},$$

where $d \log P(t) = P'(t)/P(t)$.

Problem 10. [Mor12, Lemma 3.5] *We've asserted that* $\mathrm{GW}(k) \cong K_0^{MW}(k)$. *This exercise is to help get comfortable with that. We define $K_*^{MW}(k)$ as the graded associative ring generated by symbols $[a]$ for each a in k^* of degree 1 and a symbol η of degree -1 subject to the relations*

(1) $[a][1-a] = 0$
(2) $[ab] = [a] + [b] + \eta[a][b]$
(3) $\eta[a] = [a]\eta$
(4) $h\eta = 0$ *where* $h = 1 + 1 + \eta[-1]$.

Define $\langle a \rangle$ in $K_0^{MW}(k)$ by $\langle a \rangle = 1 + \eta[a]$. Show that $[ab] = [a] + \langle a \rangle [b] = [a]\langle b \rangle + [b]$, and that $\langle ba \rangle = \langle a \rangle \langle b \rangle$.

3. Problem Session for Lecture 3

Problem 11. *Compute the logarithmic $\mathbb{A}^1$-zeta function of $\mathbb{G}r(2,4)$ over a finite field, where $\mathbb{G}r(2,4)$ denotes the Grassmannian of $\mathbb{P}^1$'s in $\mathbb{P}^3$ or equivalently the Grassmannian of dimension 2 subspaces of a 4 dimensional vector space.*

Problem 12. *The Möbius inversion formula says: Let $f, g : \mathbb{N} \to \mathbb{C}$ be functions. Suppose that $g(n) = \sum_{d|n} f(d)$ for every $n \geq 1$. Then $f(n) = \sum_{d|n} \mu(d) g(\frac{n}{d})$ for every integer $n \geq 1$. Here μ is the Mobius function with takes an integer divisible by a square to 0, and takes square free integers m to $(-1)^e$ where e is the number of factors of m. Let X be a scheme over $\mathbb{F}_q$. Let $\alpha(m)$ denote the number of closed points of X with residue field $\mathbb{F}_{q^m}$. Show that the numbers $|X(\mathbb{F}_{q^m})|$ for $m = 1, 2, \ldots$ determine the numbers $\alpha(m)$ for $m = 1, 2, \ldots$ and vice versa.*

Problem 13. *Show that the left adjoint to a lax symmetric monoidal functor is oplax. Similarly, the right adjoint to an oplax symmetric monoidal functor is lax.*

Problem 14. *A compact object is an object x satisfying $\mathrm{map}(x, \varinjlim x_i) \simeq \varinjlim \mathrm{map}(x, x_i)$ for all filtered colimits $\varinjlim x_i$. Show that dualizable objects are compact.*

Problem 15. *Show filtered colimits commute with all finite limits.*

4. Problem Session for Lecture 4

Problem 16. *Work with 1-categories. Let $y : B \to \mathrm{Fun}(B^{op}, \mathrm{Set})$ denote the Yoneda embedding sending $b \in B$ to $\mathrm{map}(-, B)$. Let $f : B \to A$ be a functor to a category A admitting all colimits. Then the left Kan extension $\mathrm{LKE} : \mathrm{Fun}(B^{op}, \mathrm{Set}) \to A$ sends a presheaf $\mathcal{P}$ to $\varinjlim_{(b, \alpha: y(b) \to \mathcal{P})} f(b)$. Show the left Kan extension preserves colimits.*

Problem 17. *Let* $f : \mathbb{P}^1 \to \mathbb{P}^1$ *be a rational map over a field* k *such and* x *a rational point of* $\mathbb{P}^1$ *such that* $f^{-1}(y)$ *is étale over* k *in the sense that* $f^{-1}(y)$ *consists of finitely many points at which the derivative of* f *does not vanish. We then have* $\deg f = \sum_{x \in f^{-1}(y)} \mathrm{Tr}_{\mathbb{C}/\mathbb{R}} \langle f'(x) \rangle$ *in* $\mathrm{GW}(\mathbb{R})$. *Compute* $\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\langle 1 \rangle$ *and its signature. Then compute the degree of* $f(z) = z(z-1)$ *and* $f(z) = z^3 - 1$ *and verify that the signature of the* $\mathbb{A}^1$*-degree is the topological degree.*

Problem 18. *Compute the real and complex realizations of*

(1) $(\mathbb{P}^1)^{\wedge a} \wedge (S^1)^{\wedge b}$
(2) $\operatorname{Spec} \mathbb{Q}[x,y]/(y^2 - x(x-1)(x-2))$

Problem 19. *Show that* $15 + 12\langle -5 \rangle$ *and* $3 + 12(\langle 2 \rangle + \langle -6 \rangle)$ *are equal in* $\mathrm{GW}(\mathbb{Q})$.

References

[Mil13] J.S. Milne, *Lectures on etale cohomology* (2013). Available at `https://www.jmilne.org/math/CourseNotes/LEC.pdf`. ↑9

[Mor12] Fabien Morel, $\mathbb{A}^1$*-algebraic topology over a field*, Lecture Notes in Mathematics, vol. 2052, Springer, Heidelberg, 2012. MR2934577 ↑10

[Sil09] Joseph H. Silverman, *The arithmetic of elliptic curves*, Second, Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009. MR2514094 ↑5

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