PCMI: $\mathbb{A}^1$-HOMOTOPY THEORY AND THE WEIL CONJECTURES PROBLEM SESSIONS

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CONTENTS

1. Problem Session for Lecture 1

Problem 1. *Let* X *be a dualizable object of a symmetric monoidal category* $(\mathcal{C}, \otimes, 1, \tau)$ *with dual* $\mathbb{D}X$. *Show there is a natural bijection*

$$\mathrm{Map}(X \otimes Y, Z) = \mathrm{Map}(Y, \mathbb{D}X \otimes Z)$$

Problem 2. *Compute the categorical trace of a rational function on* $\mathbb{P}^1$, *e.g.* $z \mapsto z^3$, $z \mapsto z^2+z+1$, $z \mapsto \frac{z^5+1}{z^2+2z+1}$

Problem 3. *Compute the Zeta function of* $\mathbb{P}^n$ *over a finite field.*

Problem 4. *Show that* $\mathrm{Th}(\mathbf{P}^1, \mathcal{O}(2)) \simeq \mathrm{Th}(\mathbf{P}^1, \mathcal{O}(-2)) \simeq (\mathbf{P}^1)^{\wedge 2} \vee \mathbf{P}^1$.

Problem 5. [Sil09, V §2] *Let* E *be an elliptic curve over a finite field and let* F *denote the relative Fröbenius. Show the Zeta function of* E *is of the form*

$$\frac{(1 - aT + qT^2)}{(1-T)(1-qT)}$$

as follows. Let $T_\ell E$ *denote the Tate module. Let* $\mathrm{End}(E) \to \mathrm{End}(T_\ell E)$ *be denoted by* $\psi \mapsto \psi_\ell$. *One can use the Tate pairing to show that* $\det \psi_\ell = \deg \psi$. $E(\mathbb{F}_{q^m}) = \deg(1 - F^m)$. *Compute* $\det(T - F_\ell^m)$ *in terms of roots of* $\det(T - F_\ell)$.

2. Problem Session for Lecture 2

Problem 6. *Let* u *be a non-square in* $\mathbf{F}_{q^m}$. *Compute* $\mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_q}\langle u \rangle$ *for* $m = 1, 2, 3, \ldots$

Problem 7. *Compute the logarithmic* $\mathbb{A}^1$*-zeta function of* $\mathbb{P}^n$ *over a finite field.*

Problem 8. *Check that a symmetric monoidal functor (also called a* $\otimes$*-functor)* $H : \mathcal{C} \to \mathcal{D}$ *takes dualizable objects to dualizable objects and* $\mathrm{Tr}\, H = H\, \mathrm{Tr}$.

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Problem 9. *(See e.g.* [Mil13, Lemma 27.5]*) Let* P *be the characteristic polynomial of an endomorphism* F *of a finite dimensional vector space. Suppose* $P(t) = \prod_r (1 - \alpha_r t)$*. Show that* $\mathrm{Tr}(F^m) = \sum \alpha_i^m$ *and*

$$d \log P(t) = -\sum_m \mathrm{Tr}(F^m) t^{m-1},$$

where $d \log P(t) = P'(t)/P(t)$.

Problem 10. [Mor12, Lemma 3.5] *We've asserted that* $\mathrm{GW}(k) \cong K_0^{MW}(k)$*. This exercise is to help get comfortable with that. We define* $K_*^{MW}(k)$ *as the graded associative ring generated by symbols* $[a]$ *for each* a *in* k^* *of degree* 1 *and a symbol* η *of degree* -1 *subject to the relations*

(1) $[a][1-a] = 0$
(2) $[ab] = [a] + [b] + \eta[a][b]$
(3) $\eta[a] = [a]\eta$
(4) $h\eta = 0$ *where* $h = 1 + 1 + \eta[-1]$.

Define $\langle a \rangle$ *in* $K_0^{MW}(k)$ *by* $\langle a \rangle = 1 + \eta[a]$*. Show that* $[ab] = [a] + \langle a \rangle [b] = [a]\langle b \rangle + [b]$*, and that* $\langle ba \rangle = \langle a \rangle \langle b \rangle$.

3. Problem Session for Lecture 3

Problem 11. *Compute the logarithmic* $\mathbb{A}^1$*-zeta function of* $\mathbb{G}\mathrm{r}(2,4)$ *over a finite field, where* $\mathbb{G}\mathrm{r}(2,4)$ *denotes the Grassmannian of* $\mathbb{P}^1$*'s in* $\mathbb{P}^3$ *or equivalently the Grassmannian of dimension 2 subspaces of a 4 dimensional vector space.*

Problem 12. *The Möbius inversion formula says: Let* $f, g : \mathbb{N} \to \mathbb{C}$ *be functions. Suppose that* $g(n) = \sum_{d|n} f(d)$ *for every* $n \geq 1$*. Then* $f(n) = \sum_{d|n} \mu(d) g(\frac{n}{d})$ *for every integer* $n \geq 1$*. Here* μ *is the Mobius function with takes an integer divisible by a square to* 0*, and takes square free integers* m *to* $(-1)^e$ *where* e *is the number of factors of* m*. Let* X *be a scheme over* $\mathbb{F}_q$*. Let* $\alpha(m)$ *denote the number of closed points of* X *with residue field* $\mathbb{F}_{q^m}$*. Show that the numbers* $|X(\mathbb{F}_{q^m})|$ *for* $m = 1, 2, \ldots$ *determine the numbers* $\alpha(m)$ *for* $m = 1, 2, \ldots$ *and vice versa.*

Problem 13. *Show that the left adjoint to a lax symmetric monoidal functor is oplax. Similarly, the right adjoint to an oplax symmetric monoidal functor is lax.*

Problem 14. *A compact object is an object* x *satisfying* $\mathrm{map}(x, \varinjlim x_i) \simeq \varinjlim \mathrm{map}(x, x_i)$ *for all filtered colimits* $\varinjlim x_i$*. Show that dualizable objects are compact.*

Problem 15. *Show filtered colimits commute with all finite limits.*

References

[Mil13] J.S. Milne, *Lectures on etale cohomology* (2013). Available at `https://www.jmilne.org/math/CourseNotes/LEC.pdf`. ↑9

[Mor12] Fabien Morel, $\mathbb{A}^1$*-algebraic topology over a field*, Lecture Notes in Mathematics, vol. 2052, Springer, Heidelberg, 2012. MR2934577 ↑10

[Sil09] Joseph H. Silverman, *The arithmetic of elliptic curves*, Second, Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009. MR2514094 ↑5

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