PCMI 2024

$\mathbb{A}^1$- algebraic topology

(Following F. Morel)

Lecture 1: Introduction.

The goal of this mini course is to give an introduction to unstable motivic homotopy theory (aka. $\mathbb{A}^1$-homotopy theory)

the main reference is Morel's book "$\mathbb{A}^1$-algebraic topology over a field".

We start with an overview of the

kind of results we will discuss.

§1.1. Basic setup.

We fix once and for all a ground field k. (Many results will require k to be perfect, so we may want to assume this from the beginning.) We denote by Sm_k the category of smooth k-varieties, which we endow with the Nisnevich topology.

In motivic homotopy theory, we study motivic spaces. We start by introducing these:

Definition:

1) A <u>space</u> (or <u>k-space</u>) is a Nisnevich sheaf of Kan complexes on Sm_k. Explicitly, it is a functor

$$F : (Sm_k)^{op} \longrightarrow \mathcal{S}$$

to the ∞-category of spaces (aka., presheaf) such that, for every Nisnevich square in Sm_k,

$$\begin{tikzcd} V \arrow[r, "j'"] \arrow[d, "e'"'] & Y \arrow[d, "e"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

the associated square

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ F(Y) \arrow[r] & F(V) \end{tikzcd}$$

is (homotopy) cartesian.

2) A k-space $\mathcal{X}$ is said to <u>motivic</u> if it is $\mathbb{A}^1$-invariant, i.e., if $\mathcal{X}(U) \to \mathcal{X}(\mathbb{A}^1 \times U)$, induced by the obvious projection, is an equivalence for any $U \in Sm_k$.

<u>Notations</u>: We denote by $\mathcal{P}(Sm_k)$ the ∞-category of presheaves of Kan complexes on Sm_k. We let

$$Spc_k := \mathcal{P}_{Nis}(Sm_k)$$

be the full subcategory of spaces, and $\mathcal{H}(k) \subset Spc_k$ the full subcat. of motivic spaces, aka., the Morel–Voevodsky category.

__Example:__ An easy example of a motivic space is the sheaf $\mathcal{O}^\times$, also denoted by $\mathbb{G}_m$. A non example is $\mathbb{G}_a = \mathcal{O}$.

__Remark:__ It is hard to write down explicitly motivic spaces. Instead, we have a (highly inexplicit) way of turning any presheaf into a motivic space. these are the localisation functors:

$$\begin{tikzcd} \mathcal{P}(Sm_k) \arrow[r, "L_{Nis}"] \arrow[rr, bend right] & \mathcal{P}_{Nis}(Sm_k) \arrow[r, "L_{mot}"] & \mathcal{H}(k) \end{tikzcd}$$

An explicit model for L_{Nis} is given by the Godement resolution. We will see later how to construct L_{mot}

Remark: $\mathcal{H}(k)$ admits an initial object $\emptyset$ given by the empty sheaf. It admits also a final object $*$ given sectionwise by the one-point set. A <u>pointed motivic space</u> $(\mathcal{X}, x)$ is a motivic space $\mathcal{X}$ with a morphism $x : * \longrightarrow \mathcal{X}$. We denote by $\mathcal{H}_*(k)$ the category of ptd motivic spaces. We define similarly the notion of pointed presheaf / space.

Notation: Let $\mathcal{X}$ be a k-space. We denote by $\pi_0(\mathcal{X})$ the Nisnevich sheaf of connected components.

If $(\mathcal{X}, x)$ is a pointed k-space, and $n \geq 0$, we denote by $\pi_n(\mathcal{X}, x)$ the sheafification of

$$U \in Sm_k \mapsto \pi_n(\mathcal{X}(U), x)$$

Notation: If $\mathcal{X}$ is a k-space, we let $\pi_0^{\mathbb{A}^1}(\mathcal{X}) = \pi_0(L_{\mathbb{A}^1}(\mathcal{X}))$. If $(\mathcal{X}, x)$ is a pointed k-space, we let $\pi_n^{\mathbb{A}^1}(\mathcal{X}, x) = \pi_n(L_{\mathbb{A}^1}(\mathcal{X}), x)$.

§ 1.2. the results we will discuss.

A basic problem in $\mathbb{A}^1$-algebraic topology is:

Problem: Given a pointed motivic space $\mathcal{X}$:

1) Understand the sheaves $\pi_n^{\mathbb{A}^1}(\mathcal{X})$.
2) Understand how $\mathcal{X}$ is "built" from the $\pi_n^{\mathbb{A}^1}(\mathcal{X})$'s.

Experience from classical topology shows that the $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ can be very difficult to compute. thus, we will mainly focus on their general properties: what kind of sheaves are they? Question 2 admits a very satisfactory answer using the machinery of Postnikov towers and obstruction theory. Answering 1) and 2) is the main goal of this mini-course.

Definition:

1) A sheaf of sets $\mathcal{F}$ is said to be **$\mathbb{A}^1$-invariant** if $\mathcal{F}(U) \simeq \mathcal{F}(\mathbb{A}^1_U)$ for every $U \in Sm_k$.

2) A sheaf of groups $\mathcal{F}$ is said to be **strongly $\mathbb{A}^1$-invariant** if

$$H^*(U; \mathcal{F}) \xrightarrow{\sim} H^*(\mathbb{A}^1_U; \mathcal{F})$$

for $* \in \{0, 1\}$ and every $U \in Sm_k$.

3) A sheaf of abelian groups $\mathcal{F}$ is said to be **n-strongly $\mathbb{A}^1$-invariant** if

$$H^*(U; \mathcal{F}) \xrightarrow{\sim} H^*(\mathbb{A}^1_U; \mathcal{F})$$

for $0 \leq * \leq n$ and every $U \in Sm_k$.
If this happens for all $n \geq 1$, we say that $\mathcal{F}$ is **strictly $\mathbb{A}^1$-invariant**.

We can now state the following theorems of Morel.

Theorem (Morel)

Let $\mathcal{X}$ be a pointed k-space.

1) $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant.

2) For $n \geq 2$, $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is strictly $\mathbb{A}^1$-invariant.

It would have been natural to expect that $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is "only" n-strongly $\mathbb{A}^1$-invariant. But it turned out that this is the same, due to the following difficult result of Morel.

theorem (Morel)

Let F be a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. Then F is strictly $\mathbb{A}^1$-invariant.

This is one of the major theorems in Morel's book. We will try to sketch a proof. Clearly, it is enough to show that

$$n\text{-strongly} \Rightarrow (n+1)\text{-strongly}.$$

Remark: It turned out that the case $n=0$ doesn't hold, i.e, there are motivic spaces $\mathcal{X}$ such that $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is not $\mathbb{A}^1$-inv.

For this reason, one often assume that $\mathcal{X}$ is 0-connected (i.e., 1-connective) as in the next result.

<u>theorem</u> (Morel)

Let $\mathcal{X}$ be a pointed connected k-space. TFAE:

(1) $\mathcal{X}$ is motivic

(2) $\pi_n(\mathcal{X})$ is strongly $\mathbb{A}^1$-inv. for all $n \geq 1$.

Moreover, in this case, all the stages of the Postnikov tower

$$\mathcal{X} \to \cdots \tau_{\leq n}\mathcal{X} \to \tau_{\leq n-1}\mathcal{X} \to \cdots \to \tau_{\leq 0}\mathcal{X} = *$$

are motivic spaces.