Thm (Rationality of $Z_X(t)$: Dwork, Grothendieck, Deligne)

X sm, proper $/\mathbb{F}_q$

$$\sum_{m=1}^{\infty} |X(\mathbb{F}_{q^m})| t^{m-1} = \frac{d}{dt}\log Z_X(t)$$

with $Z_X(t) = \dfrac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)P_2(t)\cdots P_{2d}(t)}$ $\quad d=\dim X$

zeta function

$$P_r(t) = \det\left((1-tF)|_{H^r_{et}(X_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell)}\right)$$

- F relative Fröbenius
- étale cohomology

$H^*_{et} = H^*_{et}((-)_{\bar{k}};\mathbb{Q}_\ell) : SH(k) \longrightarrow D(\mathrm{Spec}\,k;\mathbb{Q}_\ell)$

tensor functor

Derived category of ℓ-adic sheaves on Spec k

- Rmk: $H:\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ tensor functor $\Rightarrow$ H sends dualizable objects to dualizable objects and $H\,\mathrm{Tr} = \mathrm{Tr}\,H$

Pf: $\overline{\mathbb{F}}_q^{\,f^m} = \mathbb{F}_{q^m}$ Galois theory

$\Rightarrow X(\mathbb{F}_{q^m}) = X^{F^m}$

Lefschetz Fixed pt theorem: $\mathrm{Tr}(F^m) = \sum_{x \in X^{F_m}} \mathrm{ind}_x F^m$

$dF^m_x = 0 \overset{\text{Fact}}{\Longrightarrow} \mathrm{ind}_x F^m = 1 \quad \forall\ x \in X(\mathbb{F}_{q^m})$

$$\sum_{m=1}^{\infty} |X(\mathbb{F}_{q^m})| t^{m-1} = \sum_{m=1}^{\infty} \mathrm{Tr}(F^m)\, t^{m-1}$$ Lefschetz

$$= \sum_{m=1}^{\infty} H^*_{ét}(\mathrm{Tr}(F^m)\, t^{m-1})$$ $H^*_{ét}: \mathrm{End}(1_{SH}) \xrightarrow{\cong} \mathrm{End}(1_D)$

$$= \sum_{m=1}^{\infty} \mathrm{Tr}(H^*_{ét}(F^m))\, t^{m-1}$$ $H^*_{ét}$ $\otimes$-functor

$$= \sum_{m=1}^{\infty} \sum_i (-1)^i \mathrm{Tr}\, F^m|_{H^i_{ét}}\, t^{m-1}$$ Ex 2

$$= \sum_i (-1)^i \sum_{m=1}^{\infty} \mathrm{Tr}\, F^m|_{H^i_{ét}}\, t^{m-1}$$

$$= \sum_i (-1)^{i+1} \frac{d}{dt} \log P_i(t)$$ algebraic lemma

$$P_i(t) = \det(1 - tF|_{H^i_{ét}})$$

In these lectures we will consider the following enrichment of the logarithmic derivative Z_X

joint with M. Bilu, W. Ho, P. Srinivasan, I. Vogt

Def: For $X \in Sm_B$ dualizable and

$F: X \to X$ endomorphism, let

$$d\log Z_X^{\mathbb{A}^1}(t) = \sum_{m=1}^{\infty} Tr_{SH(B)}(F^m)\, t^{m-1}$$

$$\cap$$

$\cdot\ Tr_{SH(B)}(F^m) \in End_{SH(B)}(\mathbb{1})$ $\qquad End_{SH(B)}(\mathbb{1})[[t]]$

Thm (Morel) k field, $End_{SH(k)}(\mathbb{1}) \cong [\mathbb{P}^n, \mathbb{P}^n]_{n\geq 2} \cong$

$\rightsquigarrow GW(k)$

Grothendieck-Witt group

ring A $\quad GW(A) :=$ Group completion iso classes symmetric non-degenerate bilinear forms

$a \in A^*$ $\quad \langle a \rangle : A \times A \to A$

$(x,y) \mapsto axy$

ex: $GW(\mathbb{F}_q) \underset{\text{ring}}{\cong} \mathbb{Z}[\langle u \rangle] / (\langle u \rangle^2 - 1, 2(\langle u \rangle - 1))$

$$\underset{\text{group}}{\overset{\text{rank}\times\text{disc}}{\cong}} \mathbb{Z} \times \mathbb{F}_q^* / (\mathbb{F}_q^*)^2$$

- $\text{rank} : GW(A) \longrightarrow \mathbb{Z}$

  $\text{rank}\,(V \times V \xrightarrow{B} A) = \text{rank}_A V$

It follows from taking étale realizations that

<u>Prop:</u> X sm, proper $/\mathbb{F}_q$

$$\text{rank } d\log Z_X^{\mathbb{A}^1}(t) = \frac{d}{dt} \log Z_X(t) \in \mathbb{Z}[[t]]$$

We can calculate $d\log Z_X^{\mathbb{A}^1}$ from an enrichment of the Lefschetz fixed point theorem due to Hoyois

Last time, we stated this in terms of indices we did not define. Let's give a more specific calculation

$A \subset \tilde{A}$ fin étale extension

$$\text{Tr}_{\tilde{A}/A} : GW(\tilde{A}) \longrightarrow GW(A)$$

$$(V \times V \xrightarrow{B} \tilde{A}) \mapsto V \times V \xrightarrow{B} \tilde{A} \xrightarrow{\text{Tr}_{\tilde{A}/A}} A$$

ex: $k \subset L$ finite separable extension of fields

$$\text{Tr}_{L/k} = \sum_{g \in \text{Emb}_k(L, \bar{k})} g$$

<u>Ex</u>: $\text{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q} \langle 1 \rangle = \begin{cases} d & \text{if } d \text{ odd} \\ d-1 + \langle u \rangle & \text{if } d \text{ is even} \end{cases}$

<u>pf</u>: $\mathbb{F}_q \subseteq \frac{\mathbb{F}_q[x]}{p(x)} \cong \mathbb{F}_{q^d}$ $\quad \{1, x, \dots, x^{d-1}\}$ is $\mathbb{F}_q$-basis $\mathbb{F}_q$ on

$\{1, g_1, \ldots, g_{d-1}\} = \mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$

Let $M = \begin{bmatrix} 1 & g_i 1 \\ x & g_i x \\ \vdots & \vdots \\ x^{d-1} & g_i x^{d-1} \end{bmatrix}$

$\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}\langle 1 \rangle = M M^T$ because $(MM^T)_{ij} = \sum_e g_e(x^i x^j)$

$\Rightarrow \mathrm{disc}\ \mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}\langle 1 \rangle = \det M^2 \underset{\text{Vandermonde}}{=} \prod_{i<j} (g_i x - g_j x)^2 = \mathrm{disc}\ p$

f acts as d-cycle on $\{g_i x : i=1,\ldots,d\}$

so $\mathrm{disc}\ p \in (\mathbb{F}_q^*)^2 \Leftrightarrow$ d-cycle is even $\Leftrightarrow$ d is odd

<u>Thm</u> (Hoyois) Let k be a field, X a smooth, proper k-scheme, $f: X \to X$ k morphism with étale fixed points. Then

$$\mathrm{tr}(f) = \sum_{x \in X^f} \mathrm{Tr}_{k(x)/k} \langle \det(\mathrm{id} - df_x) \rangle$$

$\Rightarrow$ $X / \mathbb{F}_q$ $\circlearrowleft F$ Fröbenius

$\mathrm{ind}_x F = 1 \quad \forall x \in X^F$

$$\Rightarrow \mathrm{Tr}(F^m) = \sum_{x \in X^{F^m}} \mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q} \langle 1 \rangle$$

Thus $d\log Z_X^{\mathbb{A}^1}(t)$ determined by $|X(\mathbb{F}_{q^m})|$

$$\boxed{d\log Z_X^{\mathbb{A}^1}(t) = \sum_{m=1}^{\infty} \Big(\sum_{d|m} \alpha(d) \, \mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q} \langle 1 \rangle \Big) T^{m-1}}$$

$\alpha(d) = \#$ points x of X with $\kappa(x) = \mathbb{F}_{q^d}$ ← Same information as $|X(\mathbb{F}_{q^m})|$ by Möbius inversion

$$|X(\mathbb{F}_{q^m})| = \sum_{d|m} d\,\alpha(d)$$

Thm (Xiaowen Hu) $d\log Z_X^{\mathbb{A}^1}(t)$ is rational

The enrichment depends on congruence conditions on the coefficients of the Weil q-polynomials

You can also get a computer to compute

Ex: $y^2 = x^3 + x + 5$ over $\mathbb{F}_{13}$

$$d\log Z_X^{\mathbb{A}^1}(t) = 9 + (170 + \langle u \rangle)T + 2268 T^2 + (28898 + \langle u \rangle) T^3 + 372069 T^4 + 4{,}826{,}304 \, T^5 + \ldots$$

Since you can compute these, you find out they are not coming from Kapranov's motivic zeta function

Rmk: $\mathrm{Tr}_F^{\mathbb{A}^1}\left(\begin{matrix}\text{Kapranov}\\ \text{motivic}\\ \text{zeta}\end{matrix}\right) \in \mathbb{Z}[[x]]$

$$d\log \chi_C^{\mathbb{A}^1}\left(\begin{matrix}\text{Kapranov}\\ \text{motivic}\\ \text{zeta}\end{matrix}\right) \neq d\log Z_X^{\mathbb{A}^1}$$

Prop: $d\log Z_X^{\mathbb{A}^1} : K_0(\mathrm{Var}_k) \longrightarrow GW(k)[[t]]$ motivic measure

$K_0(\mathrm{Var}_k)$ $=$ free abelian group on varieties over k / $[X] = [Y] + [X \setminus Y]$ $Y \hookrightarrow X$ closed

We were successful computing traces with étale coh
étale coh does not pick up quadratic enrichment, so we look for other cohomology theories

Thm: (BHSVW) Let X be smooth, proj "cellular" scheme over a perfect field and let $f: X \to X$ be an endomorphism

Then
$$d\log Z_X^{\mathbb{A}^1}(t) = \sum_{r=0}^{d} -\langle -1\rangle^r \frac{d}{dt} \log P_r(t)$$

$$P_r(t) = \det\left(1 - tf \,\big|\, C_r^{cell}(X)\right)$$

where $C^{cell}_r(X)$ denotes the cellular $\mathbb{A}^1$-complex of Morel–Sawant and $f|_{C^{cell}_r(X)}$ is a square matrix of elements of $GW(k)$

Ex: $\mathbb{P}^n_{\mathbb{F}_q} \circlearrowleft^F$

- n-cell
- $\vdots$
- 1-cell
- 0-cell

$$F([x_0, \dots, x_n]) = [x_0^q, \dots, x_n^q]$$

$$\deg \wedge^r (\mathbb{P}^1 \xrightarrow{[x_0,x_1] \mapsto [x_0^q, x_1^q]} \mathbb{P}^1) =$$

$$\underbrace{\langle 1 \rangle + \langle -1 \rangle + \dots + \langle 1 \rangle}_{q} =: q_\varepsilon^r$$

$$\Rightarrow$$

$$d\log Z^{\mathbb{A}^1}_{\mathbb{P}^n}(t) = \sum_{r=0}^{n} -\langle -1 \rangle^r \frac{d}{dt} \log(1 - t q_\varepsilon^r)$$

Rmk: Can be verified combinatorially using $|\mathbb{P}^n(\mathbb{F}_{q^m})|$'s, but it is tricky

Morel–Sawant $\mathbb{A}^1$-cellular homology, Bondarko has related work

k field

$Ab(k)$ = Nisnevich sheaves abelian groups

$Sm_k^{op} \longrightarrow Ab$

$\cup$

HI = strictly $\mathbb{A}^1$ invariant sheaves, so

- $M(\mathbb{A}^1_U) \cong M(U)$ $\mathbb{A}^1$-invariant
- $H^i_{Nis}(U \times \mathbb{A}^1, M) \cong H^i_{Nis}(U, M)$ strictly $\mathbb{A}^1$-invariant

$\forall\, i, U$

$H_0^{\mathbb{A}^1}(X) :=$ the free strictly $\mathbb{A}^1$-invariant sheaf on X

$n \geq 1$ $\quad H_0^{\mathbb{A}^1}(\mathbb{G}_m^{\wedge n}) = \underline{K}_n^{MW}$ $\quad$ sections over k:

assoc graded alg

generators: $[a] \in K_1^{MW}$ $\quad a \in k^*$

$\eta \in K_{-1}^{MW}$

relations: $[ab] = [a] + [b] + \eta[a][b]$

$[a][1-a] = 0$

$[a]\eta = \eta[a]$

$h\eta = 1 \qquad h = 1 + 1 + \eta[-1]$

$$H_n^{\mathbb{A}^1}(X) := \pi_i^{\mathbb{A}^1}\mathbb{Z}[X]$$

$$\tilde{H}_n^{\mathbb{A}^1}(\underset{\text{pointed}}{X}) := \pi_i^{\mathbb{A}^1}\mathbb{Z}(X)$$

X <u>cellular</u> if there is increasing sequence of opens $\Omega_i \subset X$

$$\emptyset = \Omega_{-1} \subseteq \Omega_0 \subseteq \cdots \subseteq \Omega_d = X$$

such that $\Omega_i(X) - \Omega_{i-1}(X) \cong \coprod \mathbb{A}^{d-i}$

$$\Rightarrow \Omega_i/\Omega_{i-1} \simeq (\mathbb{P}^1)^{\wedge i} \wedge \left(\coprod \mathbb{A}^{d-i}\right)_+$$

$$\cdots \to \tilde{H}_i^{\mathbb{A}^1}(\Omega_i) \to \tilde{H}_i^{\mathbb{A}^1}(\Omega_i/\Omega_{i-1}) \to H_i(\Omega_{i-1}) \to \cdots$$

<u>Def</u>: (Morel–Sawant) $C_*^{cell}(X)$

$$\cdots \to \tilde{H}_i^{\mathbb{A}^1}(\Omega_i/\Omega_{i-1}) \to \tilde{H}_{i-1}^{\mathbb{A}^1}(\Omega_{i-1}/\Omega_{i-2}) \to \cdots$$

To discuss next time: C_*^{cell} functorial up to homotopy

$$L_{mot}\mathbb{Z}[X] \longrightarrow C_*^{cell}(X)$$

$\mathbb{A}^1$-localized $D(Ab(k))$ $\rightsquigarrow$ $D_{\mathbb{A}^1}(k) \longleftarrow D(HI)$ $\leftarrow$ derived category of strictly $\mathbb{A}^1$-inv't sheaves

$$\mathrm{Hom}_{D(HI)}(C_*^{cell}, C_*') \to \mathrm{Hom}_{D_{\mathbb{A}^1}(k)}(L_{mot}\mathbb{Z}[X], C_*')$$

$\Rightarrow C_*^{cell}$ is a functor

<u>pf</u>:

C_*^{cell} is symmetric monoidal $\overset{\text{stabilize}}{\rightsquigarrow} C_*^{sw\text{-}cell}$ symmetric monoidal

$\Rightarrow X \circlearrowleft F$ sm, proj cellular

$$Tr(F) \cong C_*^{sw\text{-}cell} Tr(F) \cong Tr(C_*^{sw\text{-}cell}(F))$$

$$\underset{\substack{\text{explicit} \\ \text{calculation} \\ \text{of dual}}}{\cong} \sum_i \langle -1 \rangle^i \, Tr(C_i^{cell}(F))$$

Then same as proof rationality Z $\square$

Next: joint work with T. Bachmann on a non-cellular version.