$\mathbb{A}^1$-homotopy theory and the Weil Conjectures:

new work in these lectures is joint with T. Bachmann, M. Bilu, W. Ho, P. Srinivasan, I. Vogt

$\mathbb{F}_q$ = finite field with q elements, q prime power

$\mathbb{F}_q \to R$ Fröbenius $f: R \to R$ ring homomorphism
$r \mapsto r^q$

$X \to \operatorname{spec} \mathbb{F}_q$ Fröbenius $f: X \to X$ by gluing

$$\begin{tikzcd}
X \arrow[dr, dashed, "F"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right] & & \\
& X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
& \operatorname{spec} \mathbb{F}_q \arrow[r, "f"] & \operatorname{spec} \mathbb{F}_q
\end{tikzcd}$$

F relative Fröbenius

Weil (1949) studied

$$X(\mathbb{F}_q) = \{ (x_1, \dots, x_n) \in \mathbb{F}_q^m \mid a x_1^{c_1} + \dots + a_n x_n^{c_n} = b \}$$

X sm, proper $/ \mathbb{F}_q$

Connection $|X(\mathbb{F}_{q^m})|$ and topology $X(\mathbb{C})$

Thm: (Dwork, Grothendieck, Deligne)

$$\sum_{m=1}^{\infty} |X(\mathbb{F}_{q^m})| t^{m-1} = \frac{d}{dt} \log Z_X(t)$$

with $Z_X(t) = \dfrac{P_1(t) P_3(t) \dots P_{2d-1}(t)}{P_0(t) P_2(t) \dots P_{2d}(t)}$ $d = \dim X$

zeta function

$$P_r(t) = \det\left(1 - tF \mid_{H^r_{\acute{e}t}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)}\right)$$

$$= \prod_{i=1}^{B_r} (1 - \alpha_{r,i} t)$$

$\alpha_{r,i}$ algebraic integers

$|\alpha_{r,i}| = q^{r/2}$ Riemann hypothesis

Moreover,

Functional equation:

$$Z\left(\frac{1}{q^d t}\right) = \pm q^{\frac{d\chi}{2}} t^{\chi} Z(t)$$

Application: (Ellenberg–Venkatesh–Westerland)

Cohen-Lenstra heuristics for class groups of function fields

$b_i(\mathbb{C})$ $\updownarrow$ $|X(\mathbb{F}_{q^m})|$ for X Hurwitz spaces of branched covers of $\mathbb{P}^1$

Stability results on $b_i(\mathbb{C}) \rightsquigarrow$

<u>Thm</u> (EVW) ℓ odd prime. A finite abelian ℓ-group. As $q \to \infty$ ($q \not\equiv 1 (\ell)$) the upper and lower densities of quadratic extensions of $\mathbb{F}_q(t)$ ramified at ∞ with ℓ-class group iso to A converge to

$$\frac{\prod_{i \geq 1} (1-\ell^{-i})}{\mathrm{Aut}(A)}$$

Proof of rationality: $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ symmetric monoidal category

A dual of X is $\mathbb{D}X$ s.t. $\exists$

$$X \otimes \mathbb{D}X \xrightarrow{\varepsilon} \mathbb{1}$$

$$\mathbb{1} \xrightarrow{\eta} \mathbb{D}X \otimes X$$

$$X \xrightarrow{\varepsilon} X \otimes \mathbb{D}X \otimes X$$

s.t. $X \xrightarrow{\eta} X \otimes \mathbb{D}X \otimes X \xrightarrow{\varepsilon} X$

and $\mathbb{D}X \xrightarrow{\eta} \mathbb{D}X \otimes X \otimes \mathbb{D}X \xrightarrow{\varepsilon} \mathbb{D}X$

are identity on X and $\mathbb{D}X$

Then: $\mathrm{map}(X \otimes Y, Z) = \mathrm{map}(Y, \mathbb{D}X \otimes Z)$

you expect objects to be dualizable under some finiteness condition

$X \in \mathcal{C}$ dualizable with dual $\mathbb{D}X$

$X \xrightarrow{f} X$

$\mathrm{Tr}(f) \in \mathrm{End}_{\mathcal{C}}(1)$

$\mathrm{Tr}(f): 1 \longrightarrow \mathbb{D}X \otimes X \xrightarrow{1 \otimes f} \mathbb{D}X \otimes X \overset{\tau}{\cong} X \otimes \mathbb{D}X \longrightarrow 1$

<u>Ex 1</u>: • X finite dimensional vector space over k field

$$k \longrightarrow X \otimes \mathbb{D}X \xrightarrow{f \otimes 1} X \otimes \mathbb{D}X \overset{\tau}{\cong} \mathbb{D}X \otimes X \longrightarrow 1$$

$$1 \mapsto \sum e_i \otimes e_i^* \xrightarrow{f \otimes 1} \sum f(e_i) \otimes e_i^* \cong \sum e_i^* \otimes f(e_i) \longmapsto \sum e_i^*(f(e_i))$$

$$\overset{\shortparallel}{\mathrm{Tr}(f)}$$

Ex2: $C_* \in D^{Perf}(R)$ with $f \circlearrowright C_*$

$$\mathbb{D} C_* = \underline{Hom}(C_*, R)$$

$$Tr(f) = \sum_i (-1)^i Tr\, f|_{C_i}$$

Ex3: $Spaces_*$ = homotopy theory of pointed spaces

$$X \wedge Y = \frac{X \times Y}{X \times * \cup * \times Y}$$

$V \to X$ vector bundle

trivial bundle rk 1 real v.b.

$$Th_X(V) = \frac{\mathbb{P}(V \oplus \mathcal{O})}{\mathbb{P}(V)} \simeq \frac{Disk(V)}{Sphere(V)} \simeq V / V - X$$

$$Th_X(V \oplus \mathcal{O}) \simeq S^1 \wedge Th_X(V)$$

spectra

$$Sp = Spaces_*[(\wedge S^1)^{-1}]$$

allows us to represent cohomology and make finite CW complexes dualizable

$\mathbb{1}$ is the sphere spectrum

$$V \oplus W \simeq \mathcal{O}^n$$

$$Th_X(V' - V) := (S^1)^{-n} \wedge Th_X(V' \oplus W)$$

$\rightsquigarrow$

$$\mathrm{Th} : K_0(X) \longrightarrow \mathrm{Sp}$$

$K_0(X)$ = group completion iso classes vector bundles

Purity $Z \hookrightarrow X$ embedding closed mflds

$$\mathrm{Th}_Z(N_Z X) \simeq X/X-Z$$

Abbreviation: $X^V = \mathrm{Th}_X V$

<u>Atiyah duality</u>: X sm, compact manifold

$$\mathbb{D}\, X^V \simeq X^{-TX-V}$$

so $\mathbb{D}\, X_+ \simeq X^{-TX}$

$X_+ \simeq \mathrm{Th}_X(0)$

<u>pf sketch</u>: construct ε:

$$X^V \otimes X^{(-TX-V)} \simeq X\times X^{\overbrace{(\pi_1^* V \oplus \pi_2^*(-TX-V))}^{W}}$$

$$\downarrow$$

$$(X\times X)^W \big/ (X\times X - \Delta_X)^W$$

$N_\Delta \simeq TX$

diagonal

$$\Big\| \text{ Purity}$$

$$\mathrm{Th}_X(W + TX)$$

$$\cong$$
$$Th_X(V - TX - V + TX)$$
$$\cong$$
$$X_+ \longrightarrow S^0 = 1$$

Construct η: $V=0$

$$1 \xrightarrow[\text{Thom collapse}]{} X^{-TX} \xrightarrow[\text{Thom diagonal}]{} X^{-TX} \otimes X$$

Thom diag $\quad X^V \xrightarrow{(id, proj)} X^V \wedge X_+$

Thom collapse

$$S^N \to \frac{S^N}{S^N - X} \simeq S^N \wedge X^{-TX}$$

Show perfect pairing $\square$

Lefschetz fixed point thm

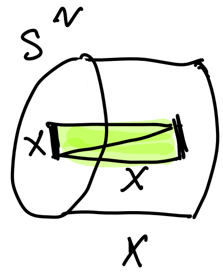
(ref. Cor 4.6 Dold-Puppe)
Hoyois §4

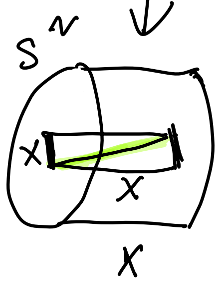
Given $X \circlearrowleft f$ compact smooth manifold such that f has isolated fixed points

Then $\displaystyle \mathrm{Tr}(f) = \sum_{\substack{x \in X \\ f(x)=x}} \mathrm{ind}_x f \in \mathbb{Z}$

<u>pf sketch</u>: $\mathrm{Tr}(f)$ is the composition

$$\begin{tikzcd}
S^N \arrow[r] & \frac{S^N}{S^N-X} \arrow[r, "\text{Thom diag}"] \arrow[dd] & \frac{S^N\times X}{(S^N-X)\times X} \arrow[r, "(f{,}1)"] \arrow[d, phantom, "\cong"] & \frac{S^N\times X}{(S^N-X)\times X} \arrow[d] \\
 & & \frac{S^N}{S^N-X}\wedge X_+ & \frac{S^N\times X}{(S^N\times X)-\Delta_X} \arrow[d, "\text{Purity}"] \\
 & \frac{S^N}{S^N-X^f} \arrow[drr] & & \frac{S^N}{S^N-X} \simeq S^N\wedge X_+ \arrow[d] \\
 & & & S^N
\end{tikzcd}$$





$X^f = \{x \in X \mid f(x) = x\}$

So $\mathrm{Tr}(f)$ is a sum over fixed points $\square$

<u>Ex 4</u>: $\mathbb{A}^1$-homotopy theory (Morel-Voevodsky)

B quasi-compact, quasi-separated scheme

Sm_B = Smooth schemes over B

Nisnevich topology

$$\mathrm{Spc}(B) \underset{\xleftarrow{L_{mot}}}{\subset} \mathrm{Fun}(\mathrm{Sm}_k^{op}, \mathrm{Spaces})$$

"Spaces in $\mathbb{A}^1$ homotopy theory"

$\mathbb{A}^1$-invariant Nisnevich sheaves

presheaves of spaces

$$\mathrm{Th}_X(\mathcal{O}_X) \simeq \frac{\mathbb{A}^1 \times X}{\mathbb{G}_m \times X} \simeq \frac{\mathbb{P}^1 \times X}{\mathbb{A}^1 \times X} \simeq \frac{\mathbb{P}^1 \times X}{\infty \times X} \simeq \mathbb{P}^1 \wedge X_+$$

$\rightsquigarrow$ so we invert $\wedge \mathbb{P}^1$

$$SH(B) := \mathrm{Spc}_*(B)[(\wedge \mathbb{P}^1)^{-1}]$$

symmetric monoidal

ref: Robalo thesis

$$\mathrm{Spc}(B) \xrightarrow{\Sigma^\infty_{+,\mathbb{P}^1}} SH(B) \qquad \otimes\text{-functor}$$

<u>Thm</u> (Hu, Riou x, ...) X sm, proper over k (k field)

$V \to X$ vector bundle

$\mathrm{Th}_X(V)$ is dualizable in $SH(k)$

with dual X^{-V-TX}

Pf: $\mathcal{E}$ same as before

$$\eta : \mathbb{1} \dashrightarrow X^{-TX} \xrightarrow[\substack{\text{Thom} \\ \text{diagonal as before}}]{} X^{-TX} \otimes X$$

- X^{-TX} contravariant for $Z \hookrightarrow X$

$$X^{-TX} \to X^{-TX} / (X - Z)^{-TX} \underset{\text{Purity}}{\simeq} Z^{N_Z X - TX} \simeq Z^{-TZ}$$

- Suffices to construct η for $\mathbb{P}^n$ ... $\square$

Thm (Hoyois) k field. X sm, proper $/B$

$f : X \to X$ regular fixed points (meaning X^f sm over B and for $X^f \overset{i}{\hookrightarrow} X$ with normal bundle N_i, we have $1 - i^*(df) : N_i \circlearrowleft$ is an iso)

$$Tr(f) = \sum_{x \in X^f} ind_x f$$

- étale cohomology

$$H^*_{et} = H^*_{et}((-)_{\bar{k}}, \mathbb{Q}_\ell) : SH(k) \to D(\text{Spec } k; \mathbb{Q}_\ell)$$

tensor functor

Derived category of ℓ-adic sheaves on Spec k

• Proof of rationality Z_X :

$$\mathbb{F}_{\bar{q}}^{F^m} = \mathbb{F}_{q^m} \quad \text{Galois theory}$$

$$\Rightarrow \quad X(\mathbb{F}_{q^m}) \simeq X^{F^m}$$

$$dF_x = 0 \overset{\text{Fact}}{\Longrightarrow} \operatorname{ind}_x F = 1 \quad \forall \; x \in X(\mathbb{F}_{q^m})$$

$$\sum_{m=1}^{\infty} |X(\mathbb{F}_{q^m})| t^{m-1} = \sum_{m=1}^{\infty} \operatorname{Tr}(F^m)\, t^{m-1} \qquad \text{Lefschetz}$$

$$= \sum_{m=1}^{\infty} H^*_{\bar{e}t}\left(\operatorname{Tr}(F^m)\, t^{m-1}\right) \qquad H^*_{\bar{e}t} : \operatorname{End}(1_{SH}) \xrightarrow{\approx} \operatorname{End}(1_D)$$

$$= \sum_{m=1}^{\infty} \operatorname{Tr}\left(H^*_{\bar{e}t}(F^m)\right) t^{m-1} \qquad H^*_{\bar{e}t} \ \otimes\text{-functor}$$

$$= \sum_{m=1}^{\infty} \sum_i (-1)^i \operatorname{Tr} F^m|_{H^i_{\bar{e}t}}\, t^{m-1} \qquad \text{Ex } 2$$

$$= \sum_i (-1)^i \sum_{m=1}^{\infty} \operatorname{Tr} F^m|_{H^i_{\bar{e}t}}\, t^{m-1}$$

$$= \sum_i (-1)^i \frac{d}{dt} \log P_i(t) \qquad \text{algebraic lemma}$$

$$P_i(t) = \det\left(1 - tF|_{H^i_{\bar{e}t}}\right)$$

$\square$

In these lectures we will consider the following enrichment of the logarithmic derivative Z_X

Def: (BHSVW) For $X \in Sm_B$ dualizable and $F: X \to X$ endomorphism, let

$$d\log Z_X^{\mathbb{A}^1}(t) = \sum_{m=1}^{\infty} \mathrm{Tr}_{SH(B)}(F^m)\, t^{m-1}$$