Billiards and the arithmetic of non-arithmetic groups

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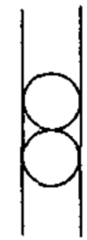
Weil, Manin, Birch, Leutbecher, Veech, Masur, Forni, Möller, Viehweg, Hubert, Lanneau, Filip, Davis, Lelievre, Smillie, Ulcigrai, F. Calegari, …

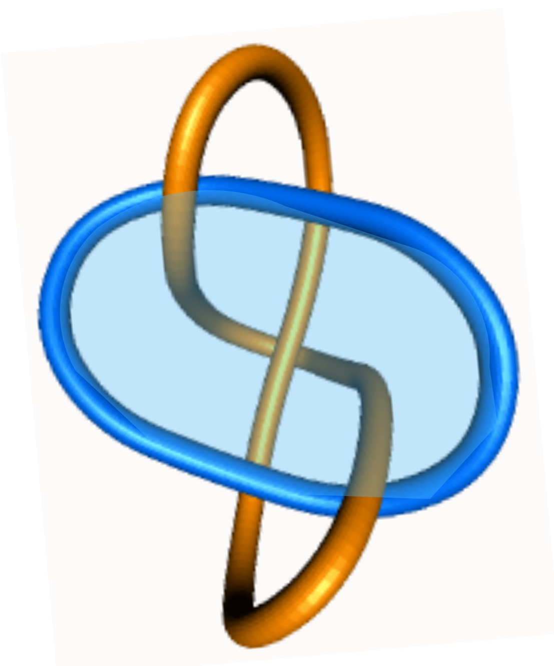
Bass notes

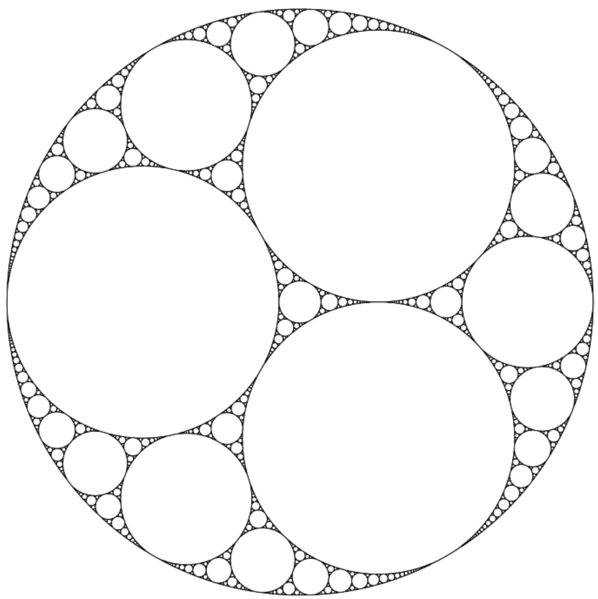
$\Omega_2: C_1, C_2, C_3, C_5$

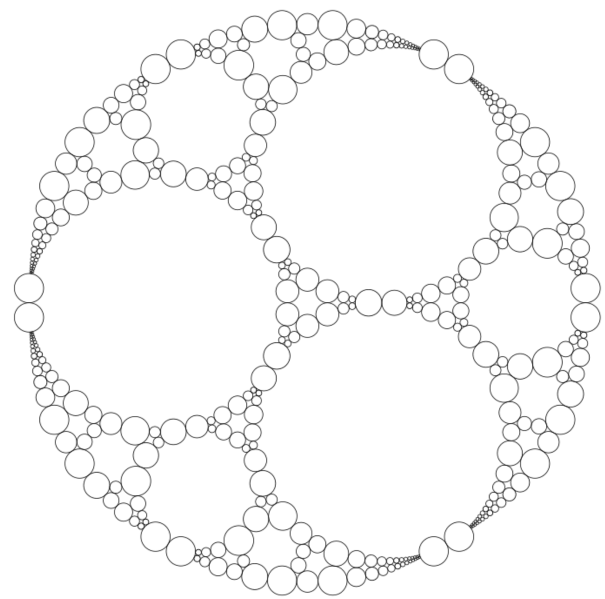
s	$N(s, j)$	$\log[N(s, j)]$	$\Delta(s)$
5	151	5.0173	Two circles
6	529	6.2710	1.254
7	1 915	7.5575	1.287
8	6 832	8.8294	1.272
9	25 375	10.1415	1.312
10	94 135	11.4525	1.311
11	347 380	12.7582	1.306
12	1 278 563	14.0613	1.303



Phillips and Sarnak, ca. 1983





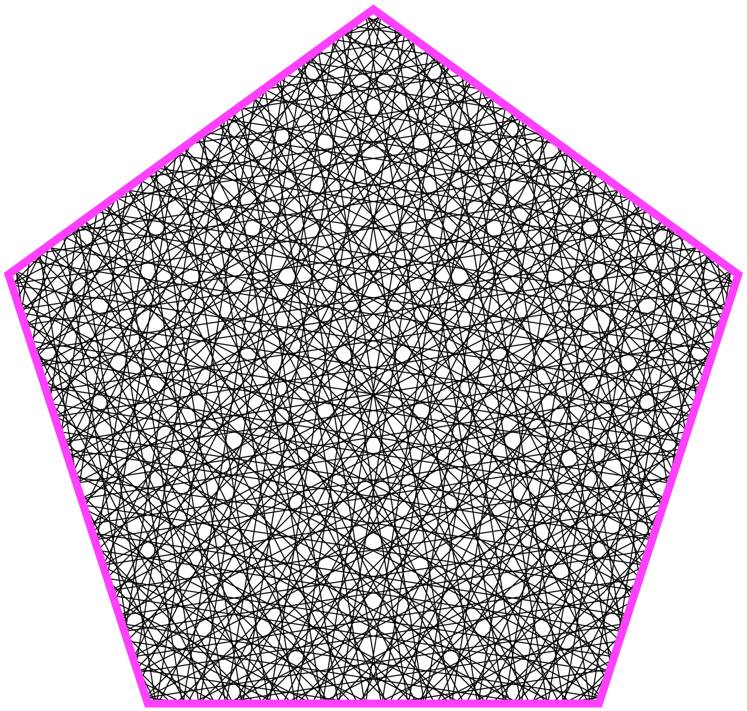


dim = 1.305688

Billiards I

Periodic trajectories and Hilbert modular surfaces

Billiards in a regular pentagon

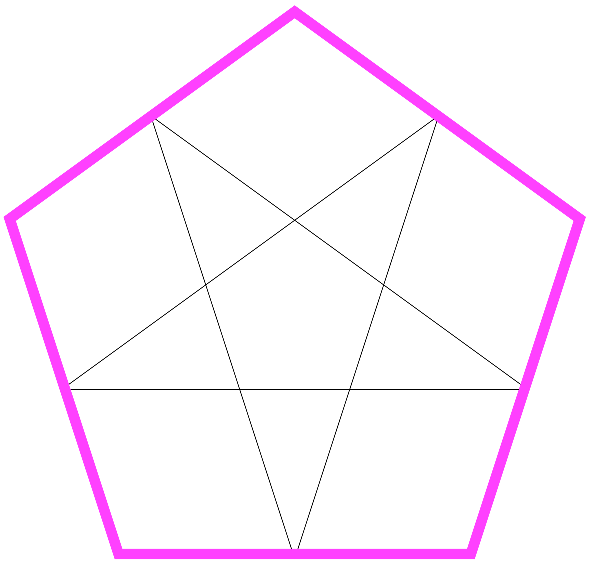


A dense set of slopes are periodic.

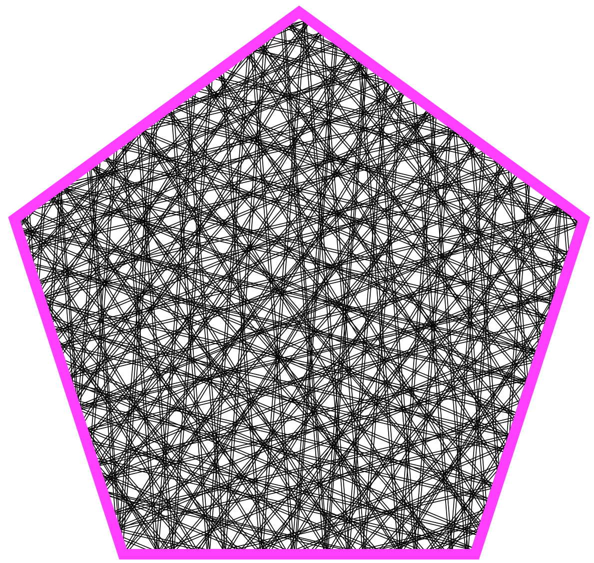
Which ones?

How do the periodic trajectories behave?

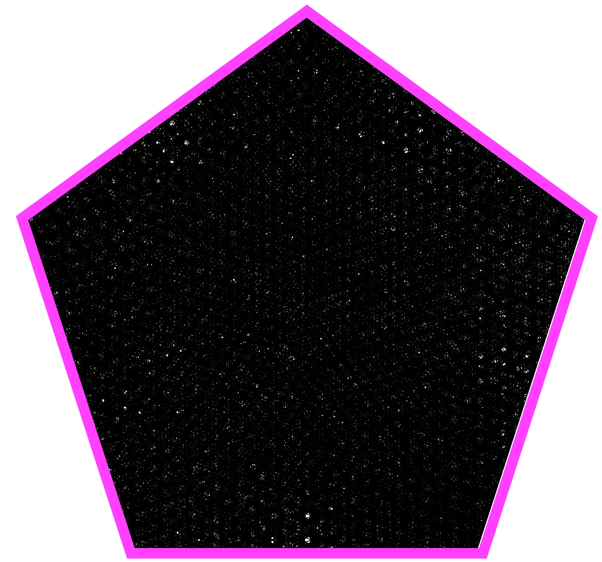
Lengths: Experiments



L(s) = 5



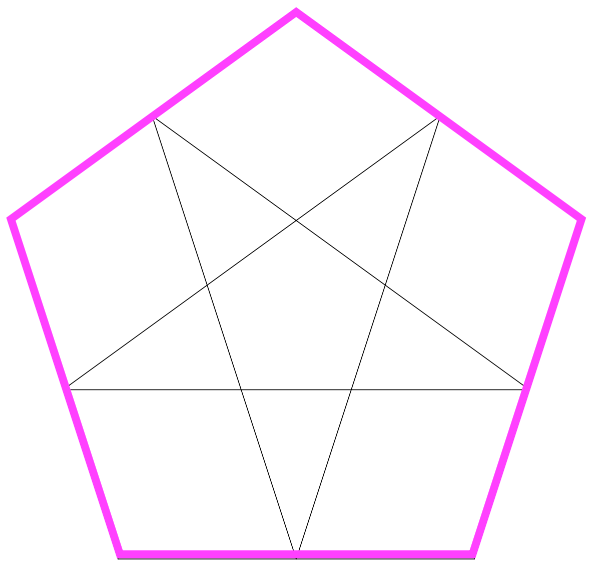
L(4s) = 469



L(20s) = 2338

L(6765s) = 1.734×10^{25}

Lengths and heights



Theorem

The periodic slopes coincide with $\mathbb{Q}(\sqrt{5})s$, and $\log L(xs) = O(h(x)^2)$.

$h(n) = \log(n)$

can have $L(10^n) \sim 10^{n^2}$

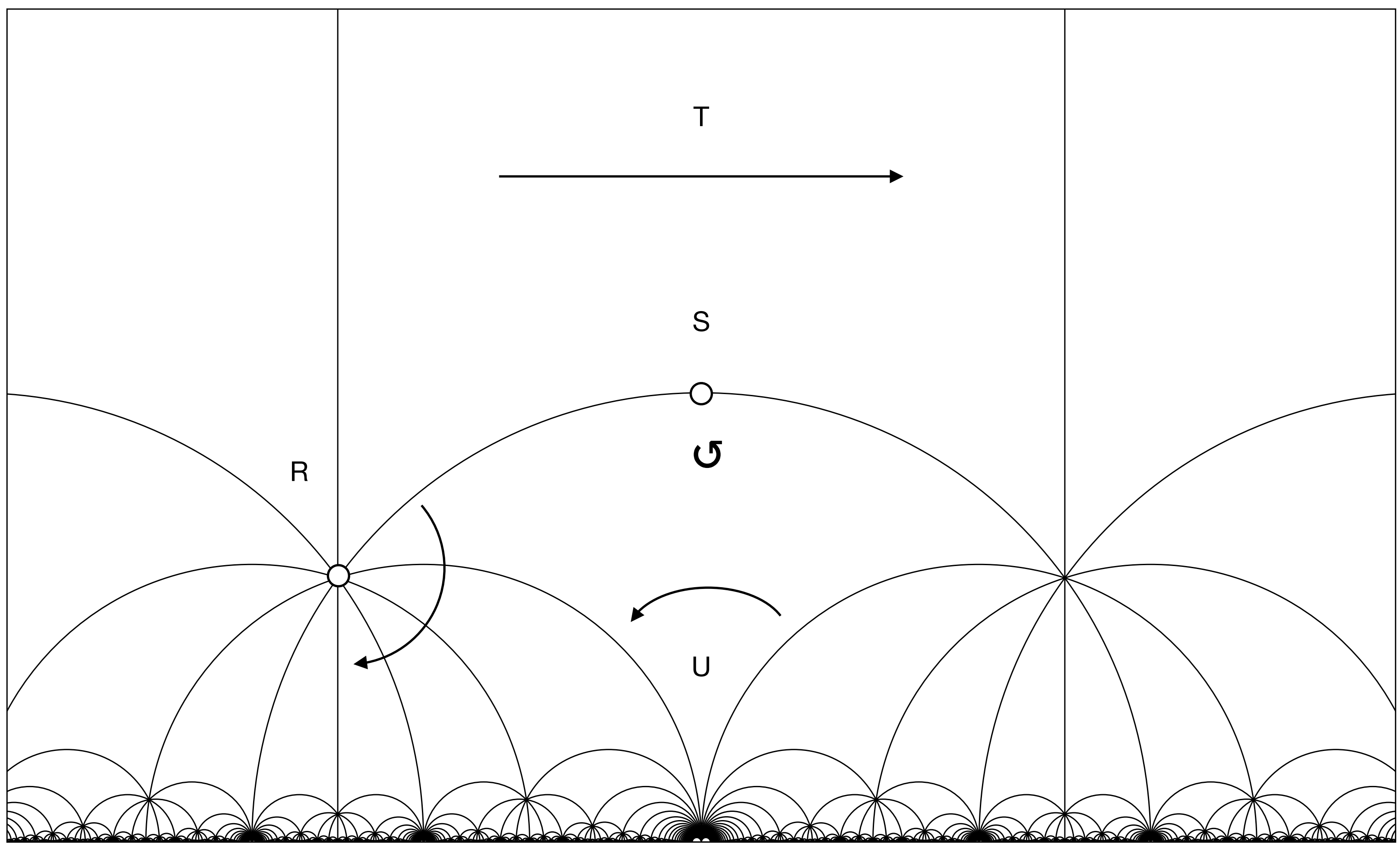
exponent 2 is sharp

Renormalization

Theorem (Veech)

The periodic slopes for billiards in a regular pentagon correspond to the cusps of the triangle group $\Delta_5 \subset SL_2(\mathbb{R})$.

Renormalization group Δ_5 for the pentagon

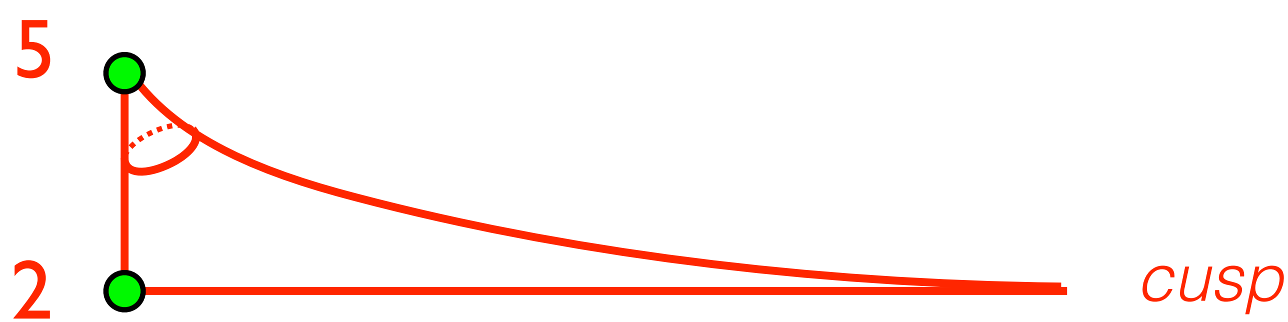


Power of renormalization

$\left(\begin{array}{c} \textit{Up to renormalization:} \\ \text{There is only 1 type of} \\ \text{periodic billiard in a pentagon} \end{array} \right)$

↕

(cusp of $V = \mathbb{H}/\Delta_5$)



Thin group perspective

$$K = \mathbb{Q}(\sqrt{5}), \quad \mathscr{O}_K = \mathbb{Z}[\gamma], \quad \gamma = (1+\sqrt{5})/2$$

$$\mathbf{SL}_2(\mathscr{O}_K) \subset \mathbf{SL}_2(\mathbb{R})^2$$

is an arithmetic lattice.

$$\Delta_5 \subset \mathbf{SL}_2(\mathscr{O}_K)$$

is a thin, nonarithmetic subgroup.

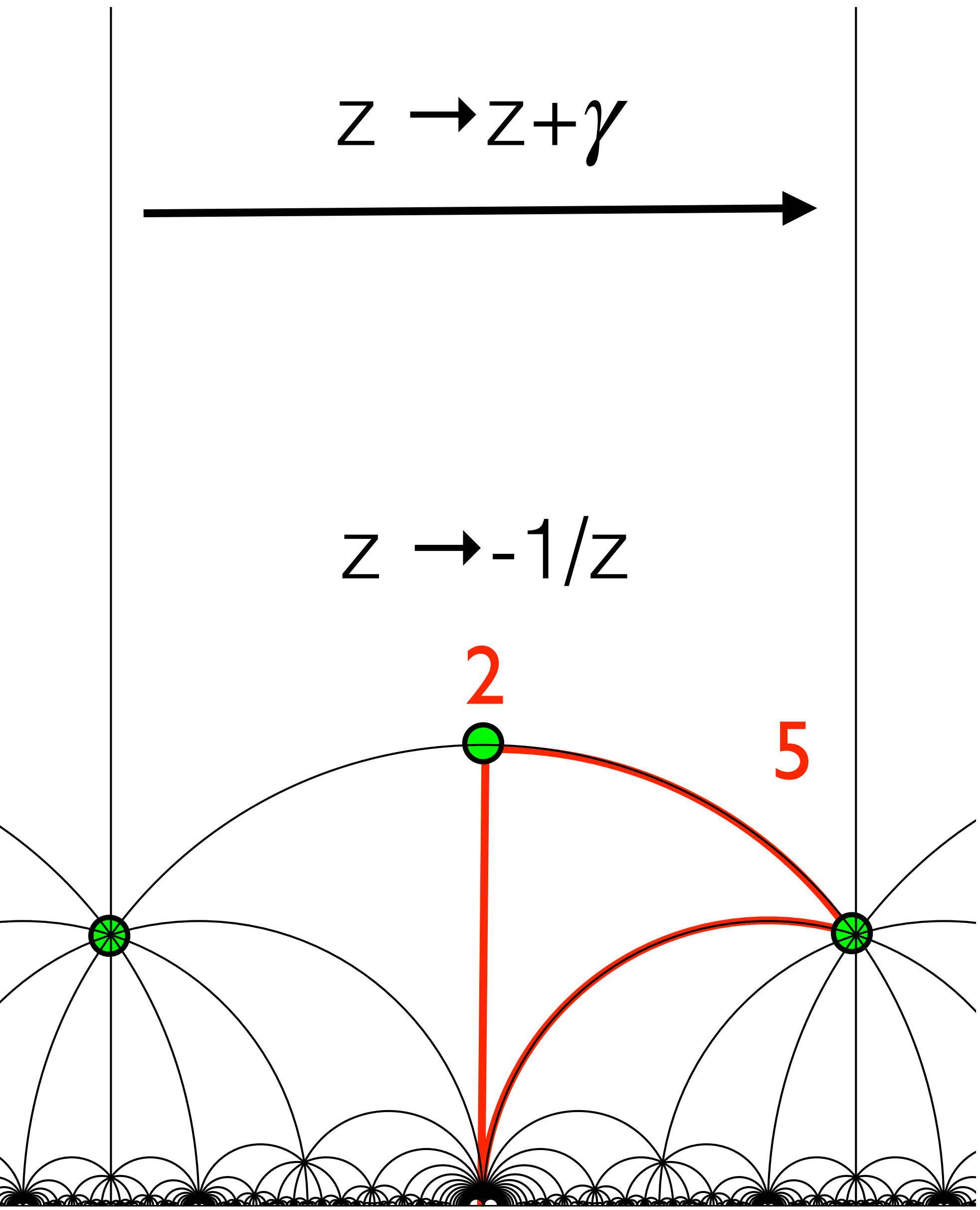
Non-arithmetic groups

are mysterious!

Δ_5 =

$$\left\langle \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

$\gamma = (1+\sqrt{5})/2$



matrix entries = ?

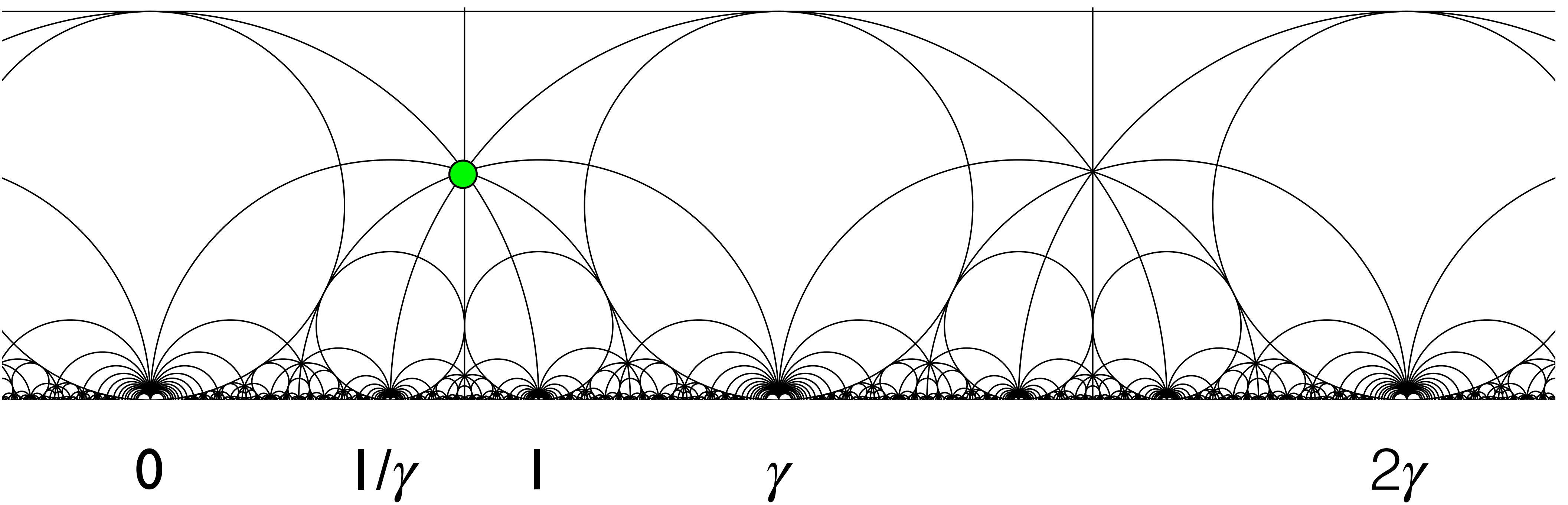
columns = ?

cusps = ?

Theorem

*T*he cusps of Δ_5 coincide with $\mathbb{P}^1(\mathbb{Q}(\sqrt{5}))$, and satisfy quadratic height bounds.

5 packing hits all points in $\mathbb{Q}(\sqrt{5})$



Continued fractions

Every $s \in \mathbb{Q}(\gamma)$ can be expanded as a finite golden continued fraction,

$$s = [a_1, a_2, a_3, \ldots, a_N] = a_1\gamma + \cfrac{1}{a_2\gamma + \cfrac{1}{a_3\gamma + \cdots \cfrac{1}{a_N\gamma}}}$$

with $a_i \in \mathbb{Z}$.

$\gamma = (1 + \sqrt{5})/2$

Height bounds: length N and a_i are $O(1+h(s))$.

Golden Fractions

Corollary

Every x in K= $\mathbb{Q}(\sqrt{5})$ can be written uniquely as a `golden fraction' x = a/c, up to sign.

a,c in $\mathbb{Z}[\gamma]$ relatively prime

(a,c) column of a matrix in Δ_5

Quadratic height bounds: $h(a)+h(c) = O(1+h(x)^2)$.

Complex geodesics

X_V

$V = \mathbb{H}/\Delta_5 \longrightarrow \mathscr{M}_2 \longrightarrow \mathscr{A}_2$

$\mathrm{Jac}(X_V)$

has real multiplication

$$X_K = (\mathbb{H} \times \mathbb{H}) / \mathrm{SL}_2(\mathcal{O}_K)$$

Hilbert modular surface

V = Kobayashi geodesic curve

Curves on a Hilbert modular surface

Assuming K is real quadratic:

Theorem

The cusps of `every' geodesic curve $V = \mathbb{H}/\Delta$ *on* X_K *coincide with* $\mathbb{P}^1(K)$ *, and satisfy quadratic height bounds.*

Corollary

Results on billiards and Δ_n *follow.*

Heights

Heights and descent

Classical: To show the `continued fraction' for x in $\mathbb{P}^1(K)$ terminates, show a suitable height H(x) decreases at each step.

discrete, clever H

Modern : To show a geodesic γ in $V \subset X_K$ heads towards a cusp at x in $\mathbb{P}^1(K)$, show $H_A(x) \to 0$ as $A \in X_K$ moves along γ.

continuous, natural H

Classical height on $\mathbb{P}^n(K)$

$$H(x) = H(x_0 : x_1 : \cdots : x_n) = \prod_v \max_i |x_i|_v. \geq 1$$

comparable to

$$\widetilde{H}(x) = \inf_a \prod_{v|\infty} \max_i |a_i|_v, \qquad [a_0 : \cdots : a_n] = [x].$$

a_i integers

only requires knowledge of integers $\mathcal{O}_K$
and infinite places of K

Real multiplication

A = a polarized abelian variety

K = totally real number field, deg(K) = dim(A)

A has real multiplication by K if we are given a map

$$T : K \longrightarrow \mathrm{End}(A) \otimes \mathbb{Q},$$

and T_k is self-adjoint for all k in K.

The projective line $\mathbb{P}^1_A(K)$

A = abelian variety with real multiplication by K

$$H_1(A, \mathbb{Q}) \cong K^2$$

$\mathbb{P}^1_A(K)$ = space of K-lines in $H_1(X, \mathbb{Q})$

Also get an orthonormal basis of eigenforms

$$\{\omega_v : v \,|\, \infty\} \subset \Omega(A)$$

Hodge height on $\mathbb{P}^1_A(K)$

$$H_A(x) = \inf \left\{ \prod_{v|\infty} \left| \int_C \omega_v \right|^{1/g} : C \in H_1(A, \mathbb{Z}), [C] = x \right\}$$

$$= \inf_{[C]=x} \prod_{v|\infty} |C|_v$$

product of Hodge valuations with C integral

⇒ The classical height and Hodge height are comparable

⇒ The Hodge height is > c(A) > 0.

For Hilbert modular surfaces

K quadratic

$$H_A(x)^2 \leq \left| \int_C \omega \right| \cdot \left| \int_C \omega' \right|$$

Can drive first term to zero like exp(-t)
along a geodesic $\gamma \subset V \subset X_K$.

Second term grows slower than exp(t)

$$\implies \quad H_A(x) \to 0 \text{ along } \gamma$$

Schwarz lemma

$$\implies \gamma \to \infty \text{ in V and } X_K$$

Conclusion: any x in $\mathbb{P}^1(K)$ is a cusp of V
(with quadratic height bounds). QED

beyond quadratic fields… Undecidability?

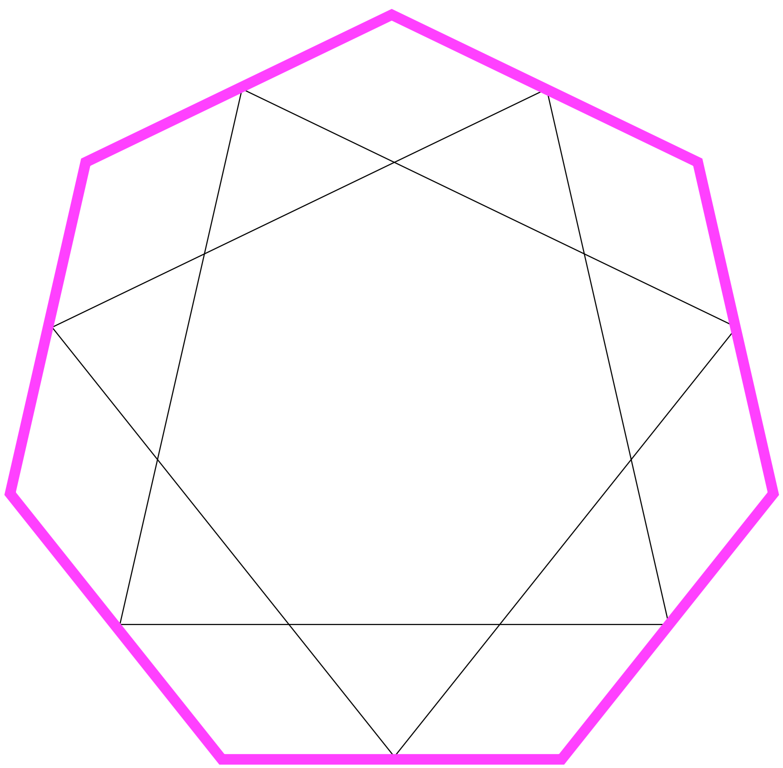
CUSP(n) = Given s = a/b in K, decide if s is a cusp of Δ_n.

Question

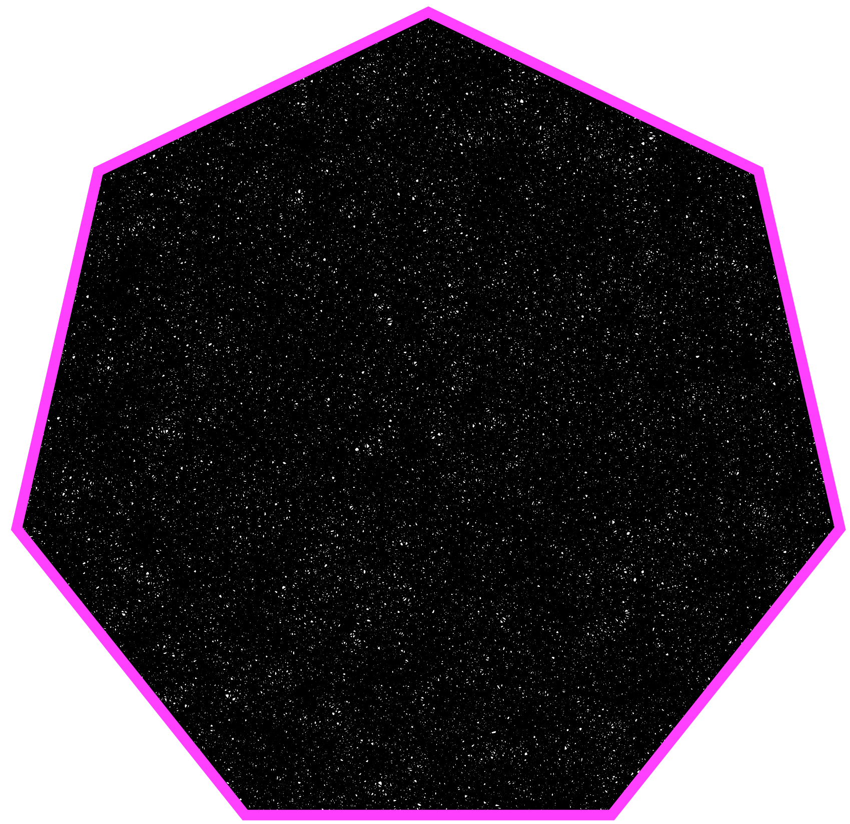
Is there an n = 7, 9, 11, … such that CUSP(n) is undecidable?

Open already for n=7

K = a cubic number field



$L(1)=7$,



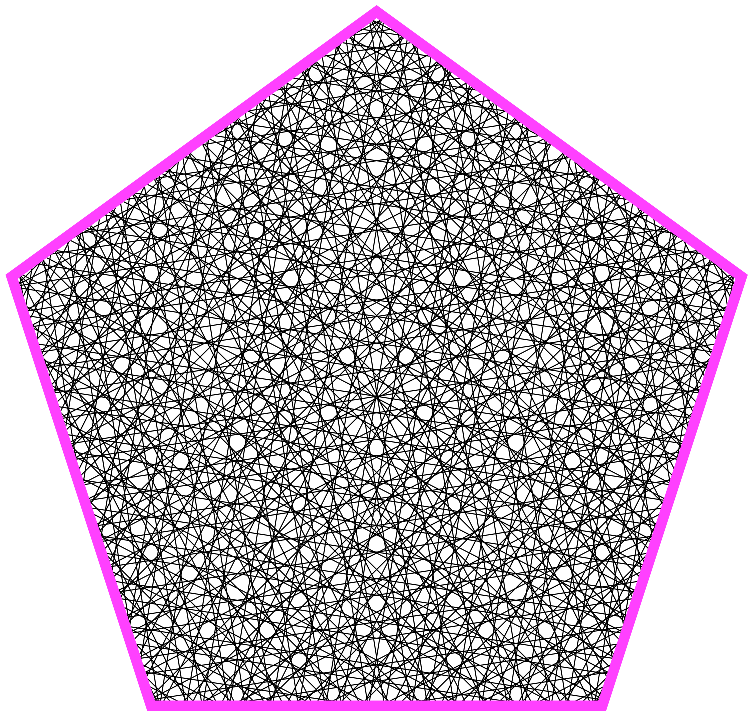
$L(1+14\zeta_7) \approx 10^{40}$.

No known way to test for periodicity of billiards.
How long must we wait for continued fraction to terminate?

Billiards II

modular symbols and equidistribution

Distribution



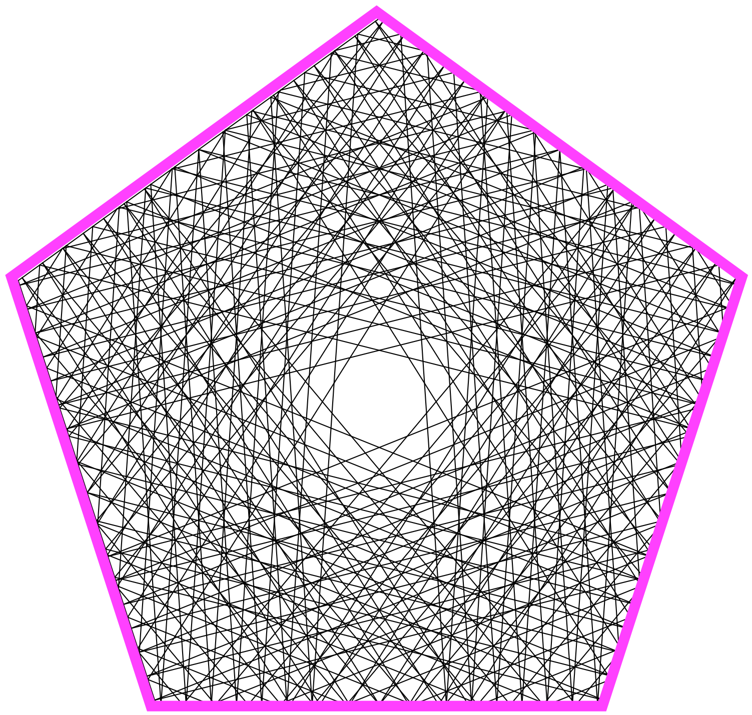
Theorem (Veech)

Every infinite trajectory is uniformly distributed.

Do long periodic trajectories equidistribute?

Davis-Lelievre: Not always!

Distribution



Theorem (Veech)

Every infinite trajectory is uniformly distributed.

Do long periodic trajectories equidistribute?

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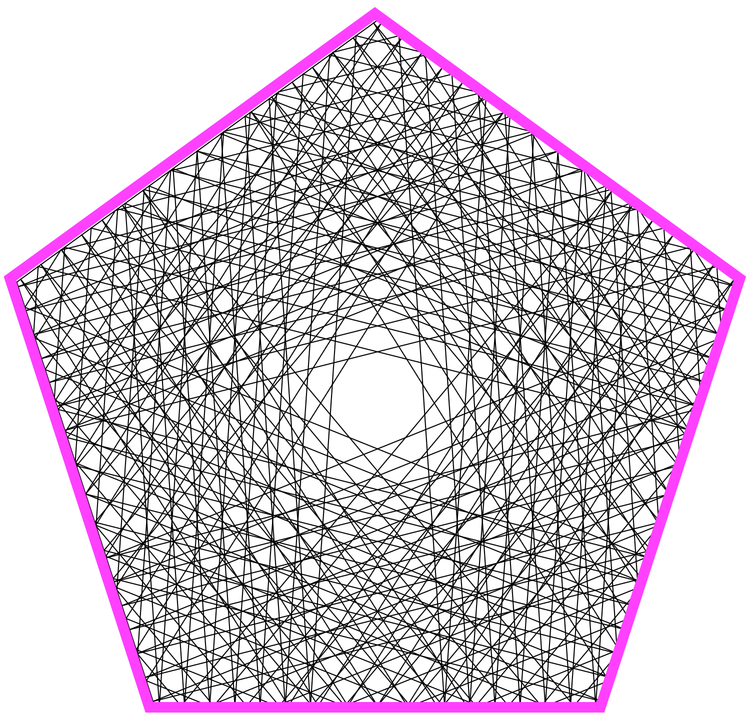
Cantor set?

Countability

Theorem

The limit measures M_s form a countable set, homeomorphic to $\omega^\omega + 1$.

(s periodic slope)

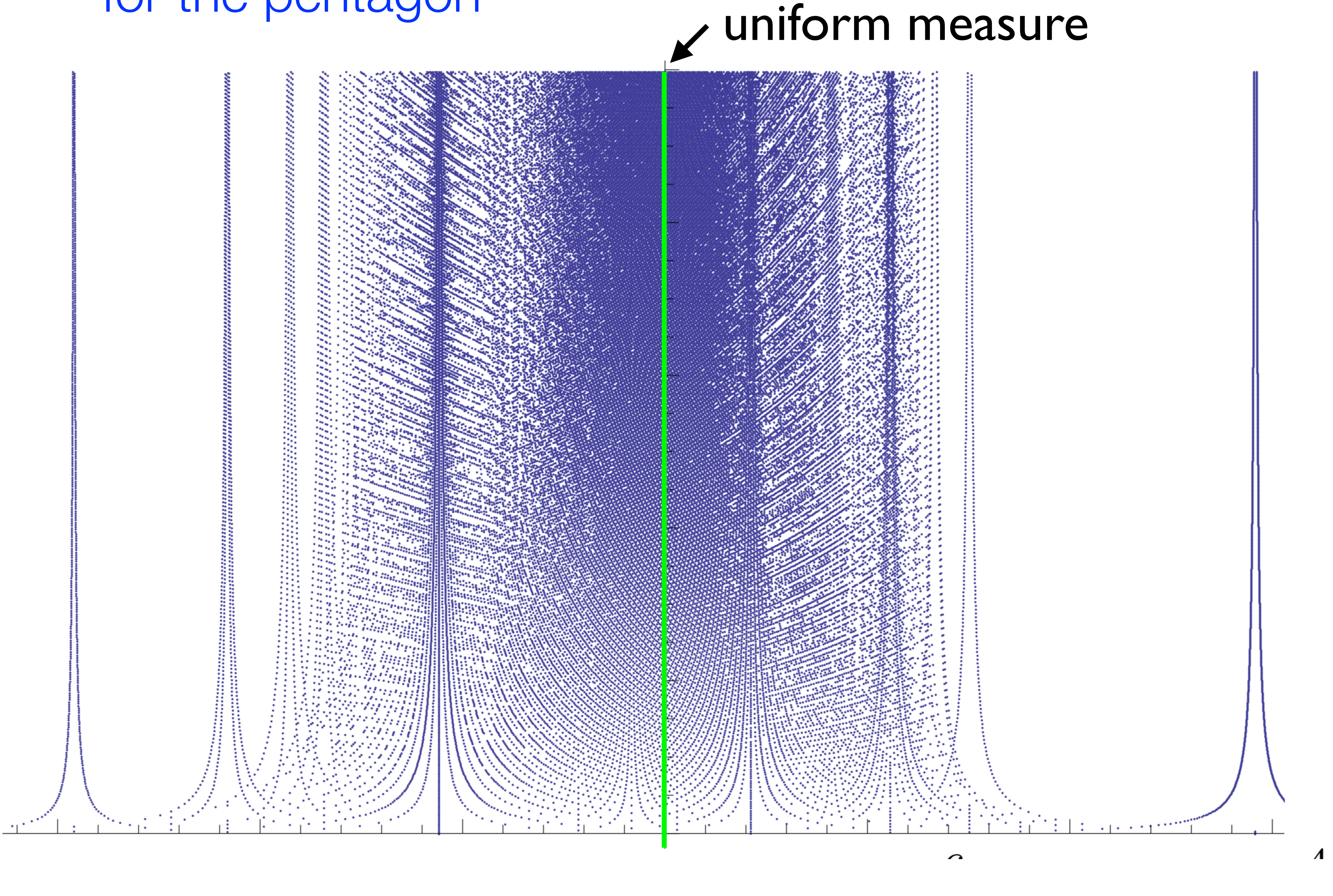


describes scarring

& closure of ergodic measures

Limit Measures M_0 for the pentagon

form a semigroup!



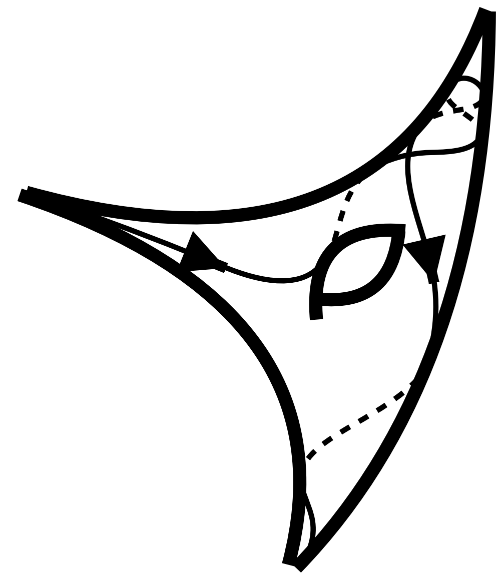
Hidden structure

Let R = $\{x'/x : x$ occurs as a matrix entry in $\Delta_5\}$.

Theorem

The closure of R, rescaled, is a semigroup in [-1,1], homeomorphic to $\omega^{\omega}+1$.

Modular symbols



$V = \mathbb{H}/\Delta_n$ hyperbolic surface

modular symbol =
chain of geodesics between cusps of V

$$a_0 \xrightarrow{\gamma_1} a_1 \xrightarrow{\gamma_2} a_2 \cdots\cdots \xrightarrow{\gamma_i} a_d$$

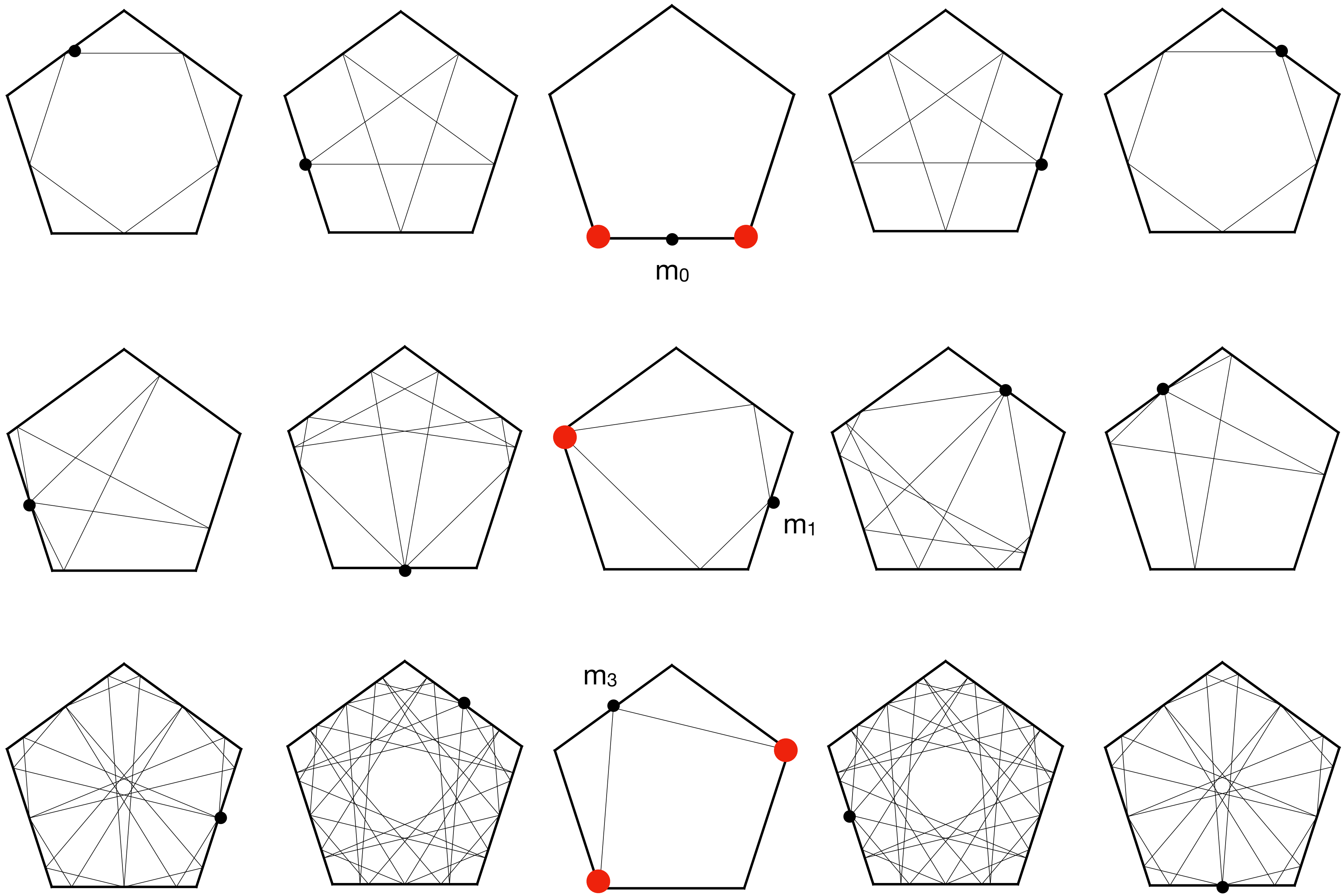
$\mathbb{S}(V) = \{\text{symbols}\} \simeq \omega^{\omega} \rightarrow (\text{limit measures } M_s)$

Source of structure

Billiards III

combinatorics, congruence and chaos

Combinatorics



Given s, which midpoint m_k gives a vertex connection?

Combinatorics

Theorem

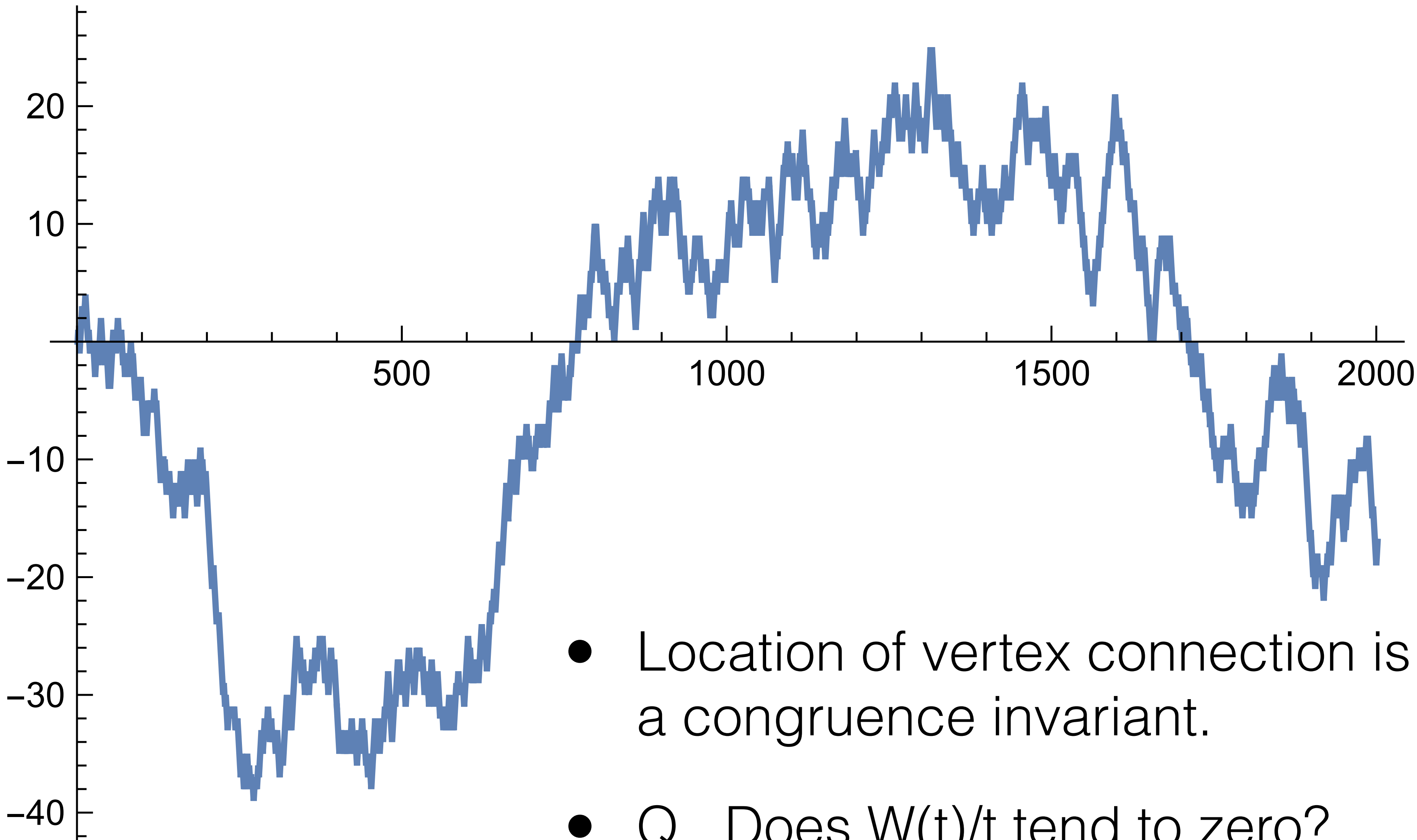
The midpoint m_k gives a vertex connection at slope s

$$\iff [s]_2 = [\zeta_5^k]_2 \ \in \ \mathbb{P}^1(\mathcal{O}_K/2).$$

- Location of vertex connection is a congruence invariant.

Chaos for n=12

$W(t+1) = W(t) + (-1)^k$ = vertex connection at slope t



- Location of vertex connection is *not* a congruence invariant.
- Q. Does W(t)/t tend to zero? ….

Representations

π_1(V) acts on

(edge midpoints of P) ~ (Weierstrass points of X) →

$H^1(X, \mathbb{Z}/2) \cong \left(\mathcal{O}_K/2\right)^2$. Instance of:

$\pi_1(V) = \Delta_n \to \mathbf{SL}_2(\mathcal{O}_K/\ell^i)$ monodromy rep

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{GL}_2(\mathbb{Z}/\ell^i)$ Galois rep associated to E/$\mathbb{Q}$

Adelic perspective

$$[\mathrm{SL}_2(\mathscr{O}_K) : \Delta_n] = \infty$$

$$[\mathrm{SL}_2(\widehat{\mathscr{O}_K}) : \overline{\Delta}_n] \text{ is finite}$$

Q. What *is* the adelic closure of Δ_n?

A. F. Calegari, *The congruence completions of triangle groups*

Corollary

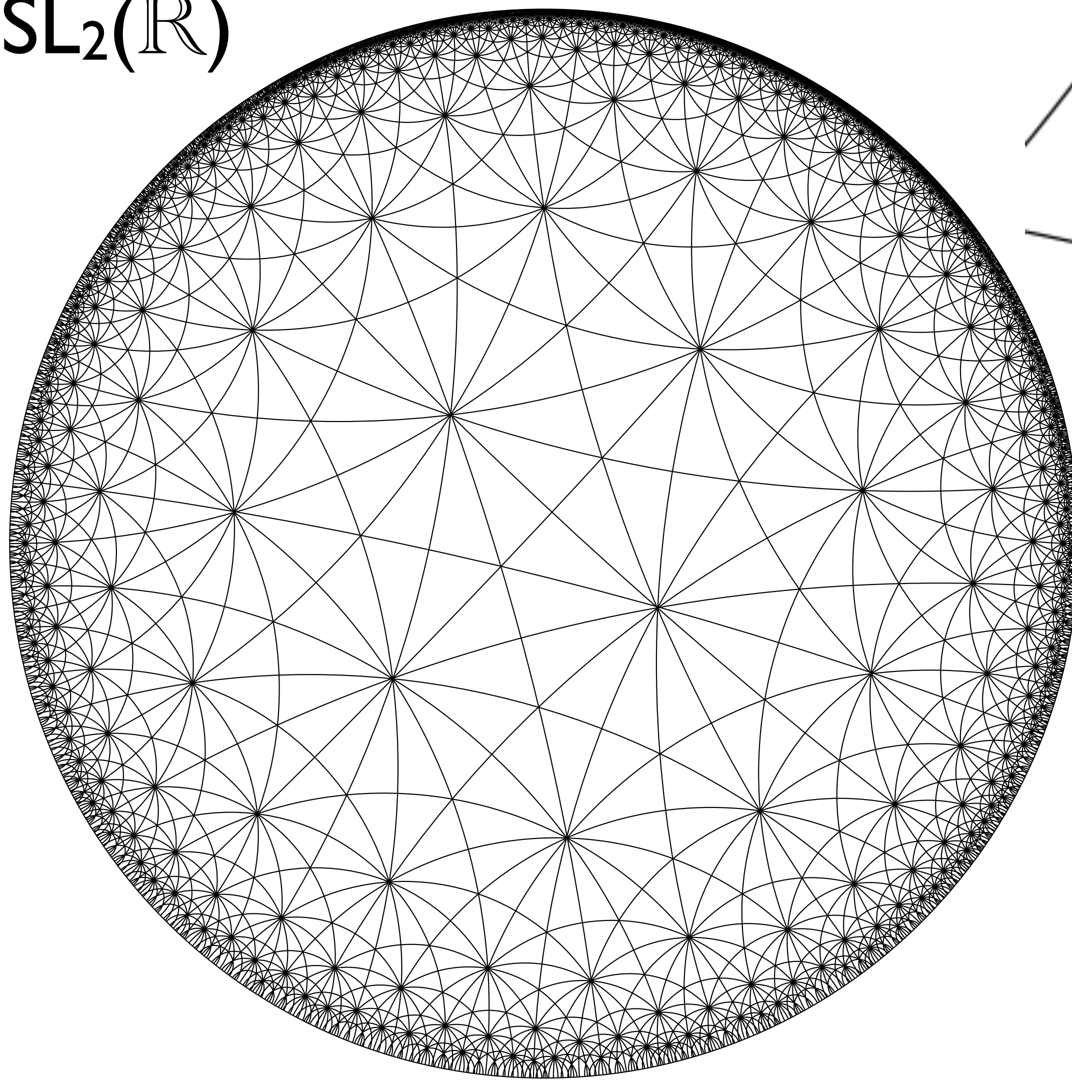
The location of the vertex connections is a congruence invariant unless n=0 mod 4 and $n \neq 2^a$.

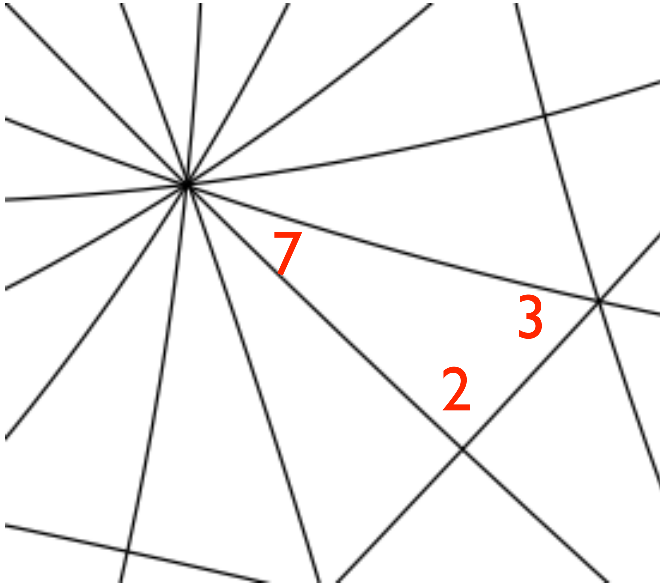
Complément

A spectral gap for triangles

Triangle groups

$\Delta(2,3,7) \subset SL_2(\mathbb{R})$





Moduli space of all triangles

$A = (\mathbb{R}/2\mathbb{Z})^3 \simeq (S^1)^3$

$a = (a_1, a_2, a_3)$ gives angles (πa_i)

Triangle T(a) may be spherical, Euclidean or hyperbolic

Galois orbit: *When a in A is torsion of order n,*

$Gal(a) = (\mathbb{Z}/n)^* \cdot a$.

a_i roots of I

Spectral gap

Ramification density:

$$\rho(a) = \frac{\#(\text{b in Gal(a) : T(b) is spherical})}{\#\text{Gal(a)}}$$

Theorem

There exist constants $0 < \rho_H < \rho_S < 1$, such that

$$\rho(a) \in \{0,1\} \cup [\rho_H , \rho_S].$$

— Probably $[\rho_H , \rho_S] = [1/12, 4/5]$.

— Usually $\rho(a) \approx 1/3$.

— Cases $\rho(a) = 0$ or 1 understood, modulo finite set

Proof of spectral gap

— *Equidistribution:*

m(a) = uniform measure on Gal(a)

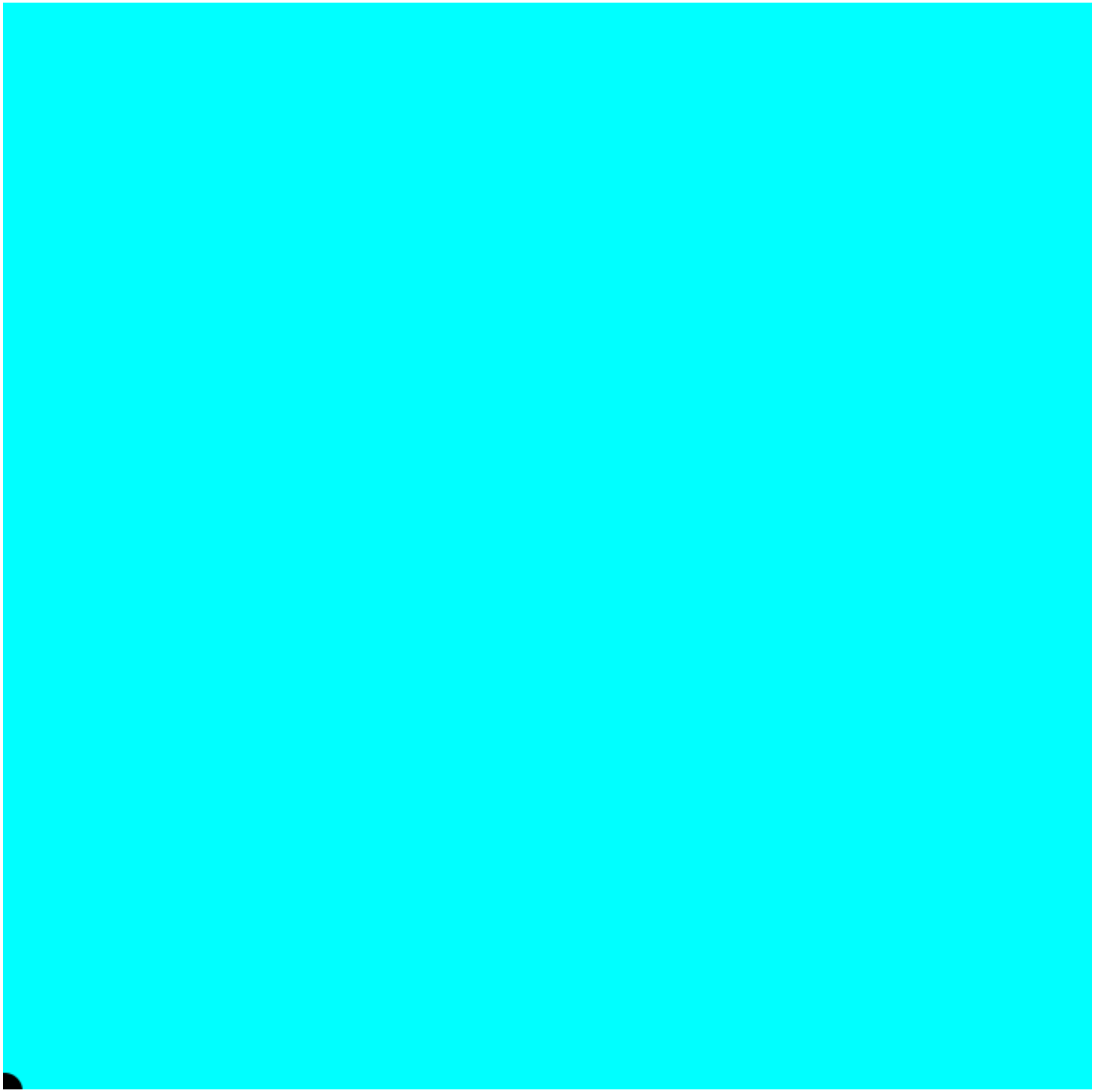
$m(a_n) \to m(B)$ = uniform measure on torus translate

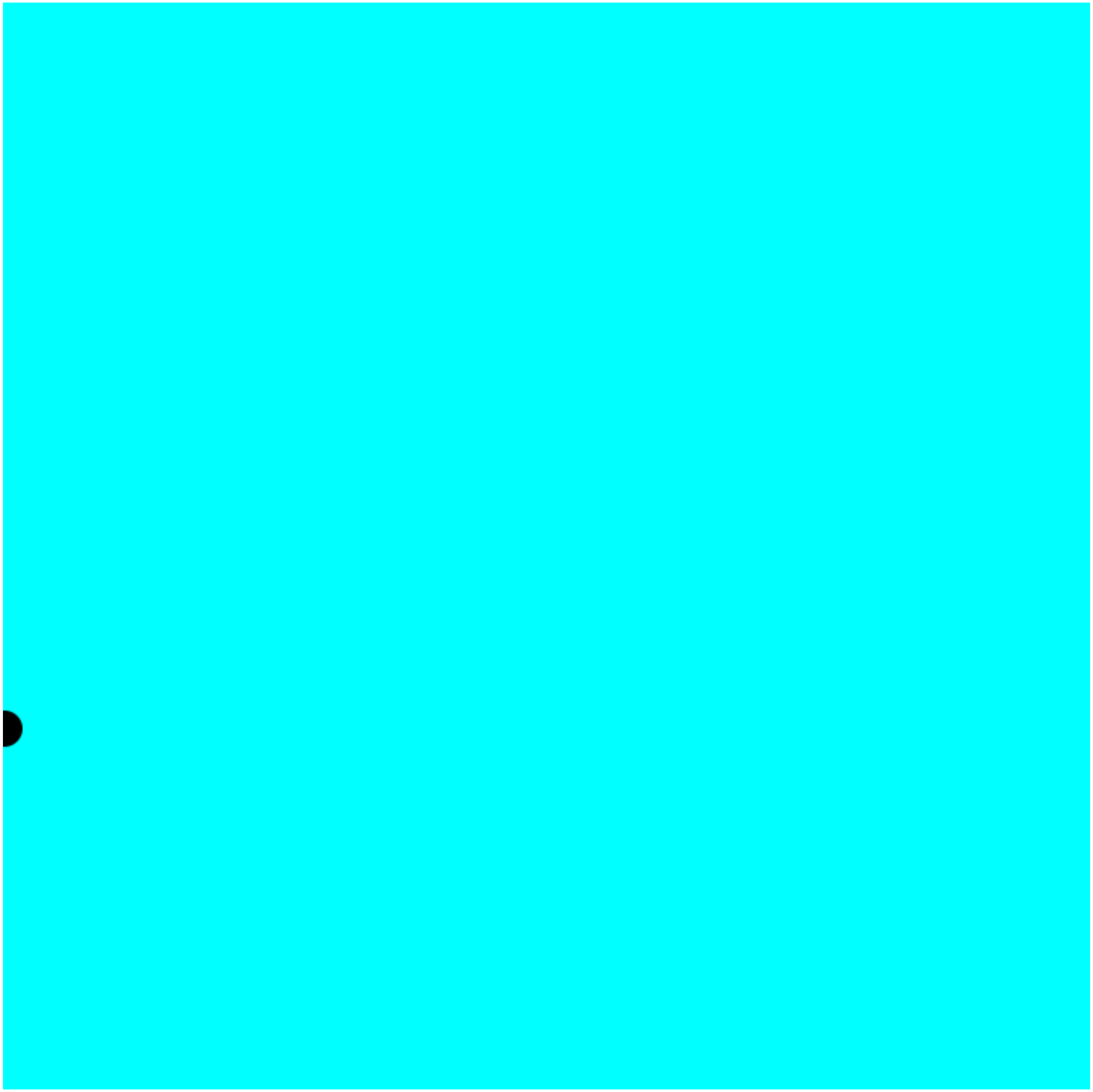
critical: a_n is in B for all $n \gg 0$.

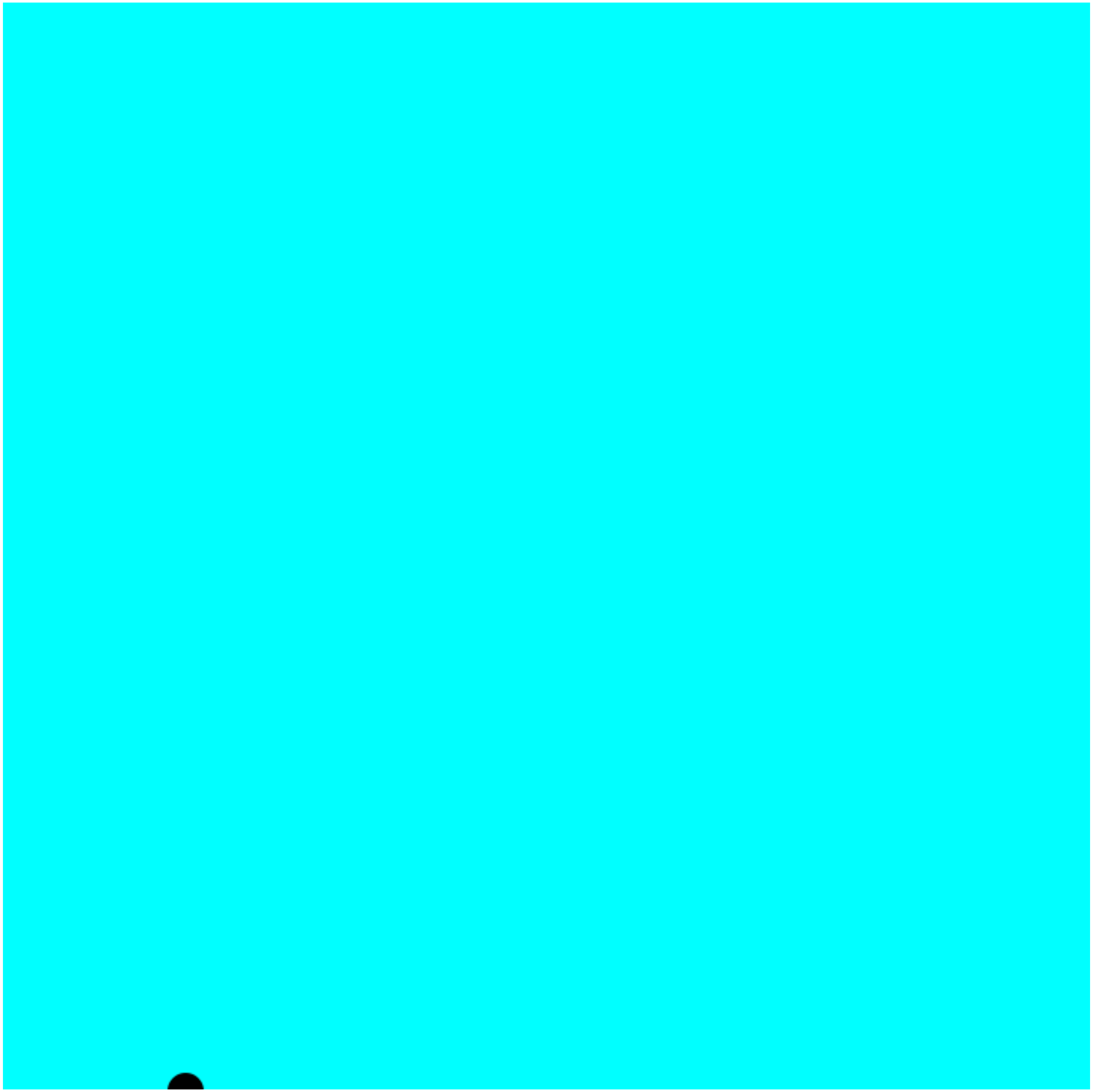
— *Geometry:*

Find possibilities for B

Moving tablecloth game







Spectral gap — *encore*

Theorem

For all but finitely many $\Delta(p,q,r)$,
 # spherical and # hyperbolic places are about the same.

Cor (Takeuchi)

There are only finitely many arithmetic triangle groups.

Cor (Waterman—Maclachlan)

There are only finitely many totally hyperbolic triangle groups.

Example

$$a = \left(\frac{1}{2},\frac{1}{3},\frac{1}{7}\right) \sim \left(\frac{1}{2},\frac{1}{3},\frac{2}{7}\right) \sim \left(\frac{1}{2},\frac{1}{3},\frac{3}{7}\right)$$

hyperbolic spherical spherical

Only 1 hyperbolic

$\Delta(2,3,7)$ is arithmetic

$\rho(a) = 2/3$.

Example

$$a = \left(\frac{1}{14},\frac{1}{21},\frac{1}{42}\right) \sim \left(\frac{1}{14},\frac{8}{21},\frac{13}{42}\right) \sim \left(\frac{3}{14},\frac{4}{21},\frac{17}{42}\right) \sim$$

$$\left(\frac{3}{14},\frac{10}{21},\frac{11}{42}\right) \sim \left(\frac{5}{14},\frac{2}{21},\frac{19}{42}\right) \sim \left(\frac{5}{14},\frac{5}{21},\frac{5}{42}\right)$$

all hyperbolic

$\Delta(14,21,42)$ is totally hyperbolic

$\rho(a) = 0$.

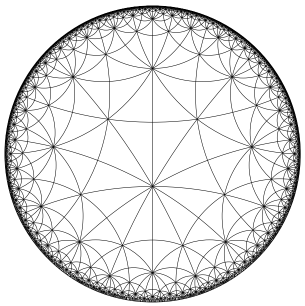
76 cocompact arithmetic triangle groups

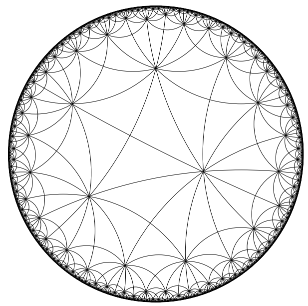
	(e_1, e_2, e_3)	Field	Ram
1	$(2,3,\infty),(2,4,\infty),(2,6,\infty),(2,\infty,\infty),$ $(3,3,\infty),(3,\infty,\infty),(4,4,\infty),$ $(6,6,\infty),(\infty,\infty,\infty)$	$\mathbb{Q}$	$\emptyset$
2	$(2,4,6),(2,6,6),(3,4,4),(3,6,6)$	$\mathbb{Q}$	$2,3$
3	$(2,3,8),(2,4,8),(2,6,8),(2,8,8),(3,3,4),$ $(3,8,8),(4,4,4),\ (4,6,6),(4,8,8)$	$\mathbb{Q}(\sqrt{2})$	$\mathcal{P}_2$
4	$(2,3,12),(2,6,12),(3,3,6),(3,4,12),$ $(3,12,12),(6,6,6)$	$\mathbb{Q}(\sqrt{3})$	$\mathcal{P}_2$
5	$(2,4,12),(2,12,12),(4,4,6),(6,12,12)$	$\mathbb{Q}(\sqrt{3})$	$\mathcal{P}_3$
6	$(2,4,5),(2,4,10),(2,5,5),(2,10,10),$ $(4,4,5),(5,10,10)$	$\mathbb{Q}(\sqrt{5})$	$\mathcal{P}_2$
7	$(2,5,6),(3,5,5)$	$\mathbb{Q}(\sqrt{5})$	$\mathcal{P}_3$
8	$(2,3,10),(2,5,10),(3,3,5),(5,5,5)$	$\mathbb{Q}(\sqrt{5})$	$\mathcal{P}_5$
9	$(3,4,6)$	$\mathbb{Q}(\sqrt{6})$	$\mathcal{P}_2$
10	$(2,3,7),(2,3,14),(2,4,7),(2,7,7),$ $(2,7,14),(3,3,7),(7,7,7)$	$\mathbb{Q}(\cos\pi/7)$	$\emptyset$
11	$(2,3,9),(2,3,18),(2,9,18),(3,3,9),$ $(3,6,18),(9,9,9)$	$\mathbb{Q}(\cos\pi/9)$	$\emptyset$
12	$(2,4,18),(2,18,18),(4,4,9),(9,18,18)$	$\mathbb{Q}(\cos\pi/9)$	$\mathcal{P}_2,\mathcal{P}_3$
13	$(2,3,16),(2,8,16),(3,3,8),$ $(4,16,16),(8,8,8)$	$\mathbb{Q}(\cos\pi/8)$	$\mathcal{P}_2$
14	$(2,5,20),(5,5,10)$	$\mathbb{Q}(\cos\pi/10)$	$\mathcal{P}_2$
15	$(2,3,24),(2,12,24),(3,3,12),(3,8,24),$ $(6,24,24),(12,12,12)$	$\mathbb{Q}(\cos\pi/12)$	$\mathcal{P}_2$
16	$(2,5,30),(5,5,15)$	$\mathbb{Q}(\cos\pi/15)$	$\mathcal{P}_3$
17	$(2,3,30),(2,15,30),(3,3,15),$ $(3,10,30),(15,15,15)$	$\mathbb{Q}(\cos\pi/15)$	$\mathcal{P}_5$
18	$(2,5,8),(4,5,5)$	$\mathbb{Q}(\sqrt{2},\sqrt{5})$	$\mathcal{P}_2$
19	$(2,3,11)$	$\mathbb{Q}(\cos\pi/11)$	$\emptyset$

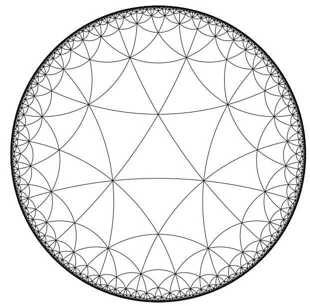
Takeuchi

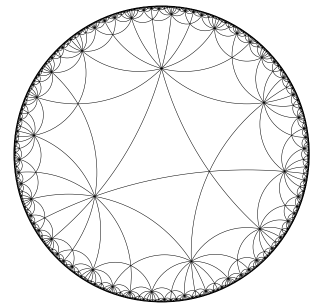
Maclachlan-Reid

11 known totally hyperbolic triangle groups

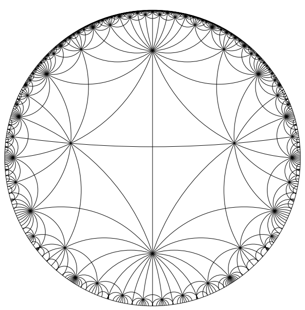

(2,4,6)

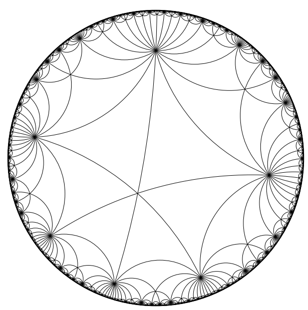

(2,6,6)

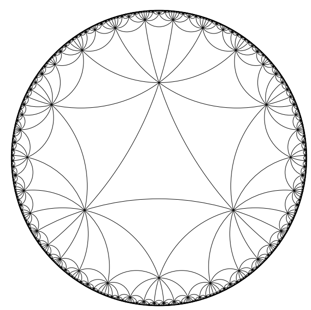

(3,4,4)

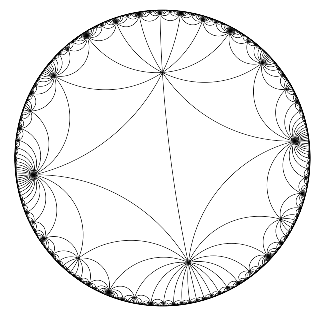

(3,6,6)

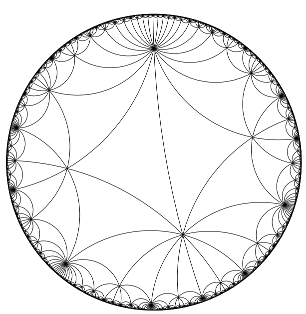
$\rho(a) = 0.$

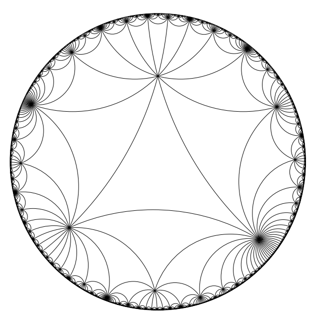

(2,6,10)

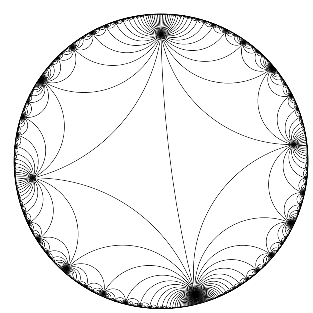

(3,10,10)


(5,6,6)


(6,10,15)


(4,6,12)


(6,9,18)


(14,21,42)

M, Maclachlan-Waterman

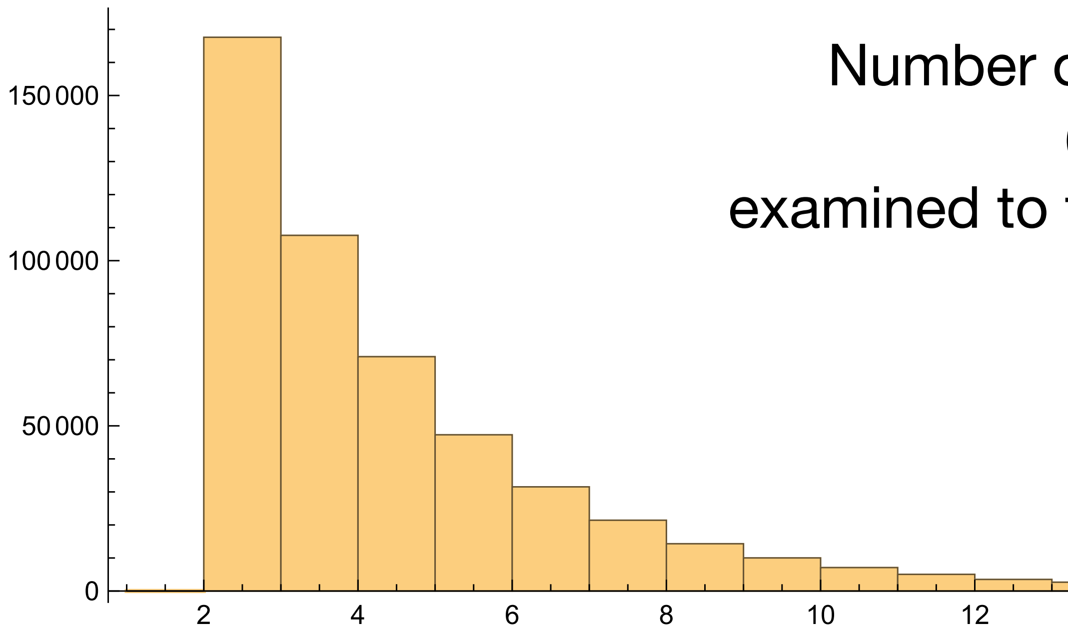
Problem

Are there more examples of totally hyperbolic triangle groups?

Experimental Evidence

There are no other purely hyperbolic triangle groups with (p,q,r) < 5000.

Recently verified using 5,000 cores running in parallel from 1-30 mins.



Number of Galois conjugates (out of 10^9) examined to find a spherical triangle.

Randomized

Total execution time 587 hours

The importance of being (14,21,42)

Motivation

Theorem (Veech)

Every geodesic curve $V \to \mathcal{M}_g$ *has a cusp.*

↑ **most cases**

Theorem

'Every' geodesic curve $V \to X_K$ *has a cusp, provided* $\dim(X_K)=2$

What happens when $\dim(X_K) > 2$*?*

What happens if dim $X_K > 2$?

Theorem

There exists a compact geodesic curve
V on a 6D Hilbert modular variety,

$$V = \mathbb{H}/\Delta' \to X_K,$$

such that there is no compact Shimura variety with
$V \subset S \subset X_K$.

(Δ' is Zariski dense in $\mathrm{SL}_2(\mathcal{O}_K)$)

Matrix models

$\Delta = \Delta(p,q,r) \subset SL_2(\mathbb{R})$

$K = \mathbb{Q}(\text{traces of elements in } \Delta)$

Δ can be realized as a subgroup of $SL_2(K)$

Fallacy *Correction*

$\Leftrightarrow$ quaternion algebra $B = \mathbb{Q}(\Delta)$ splits over K

$\Rightarrow$ Δ is totally hyperbolic (B splits at all $v|\infty$)

Theorem

Among the 11 known totally hyperbolic cocompact triangle groups, only

$$\Delta(14,21,42)$$

is also split at all finite places.

Corollary

$\Delta(14,21,42)$ *embeds in* $SL_2(K)$.

$K = \mathbb{Q}(\cos \pi/21)$ degree 6

Theorem (Cohen-Wolfart)

From the group theory:

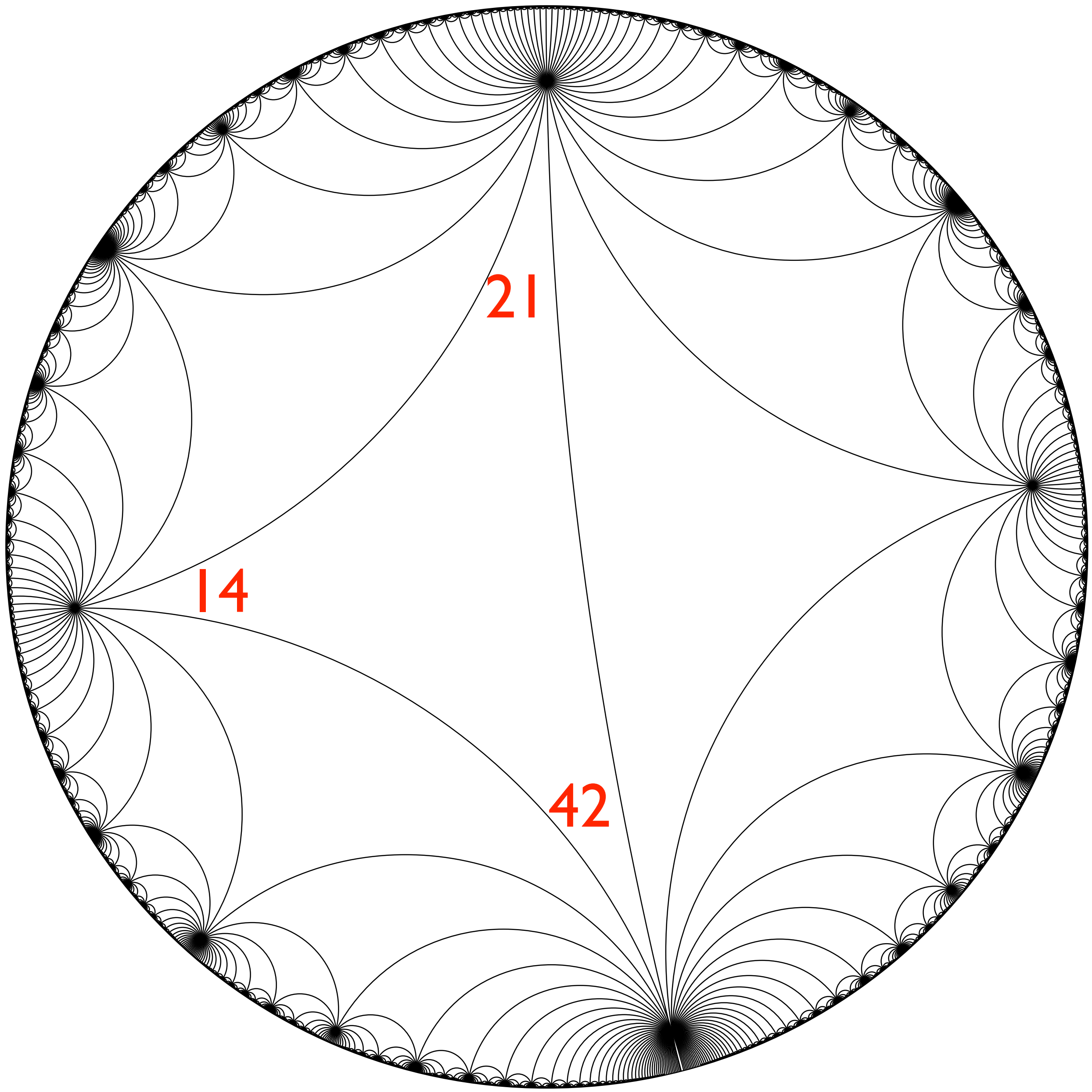
$$\Delta(14,21,42) \subset \mathbf{SL}_2(\mathcal{O}_K),$$

we obtain a geodesic curve

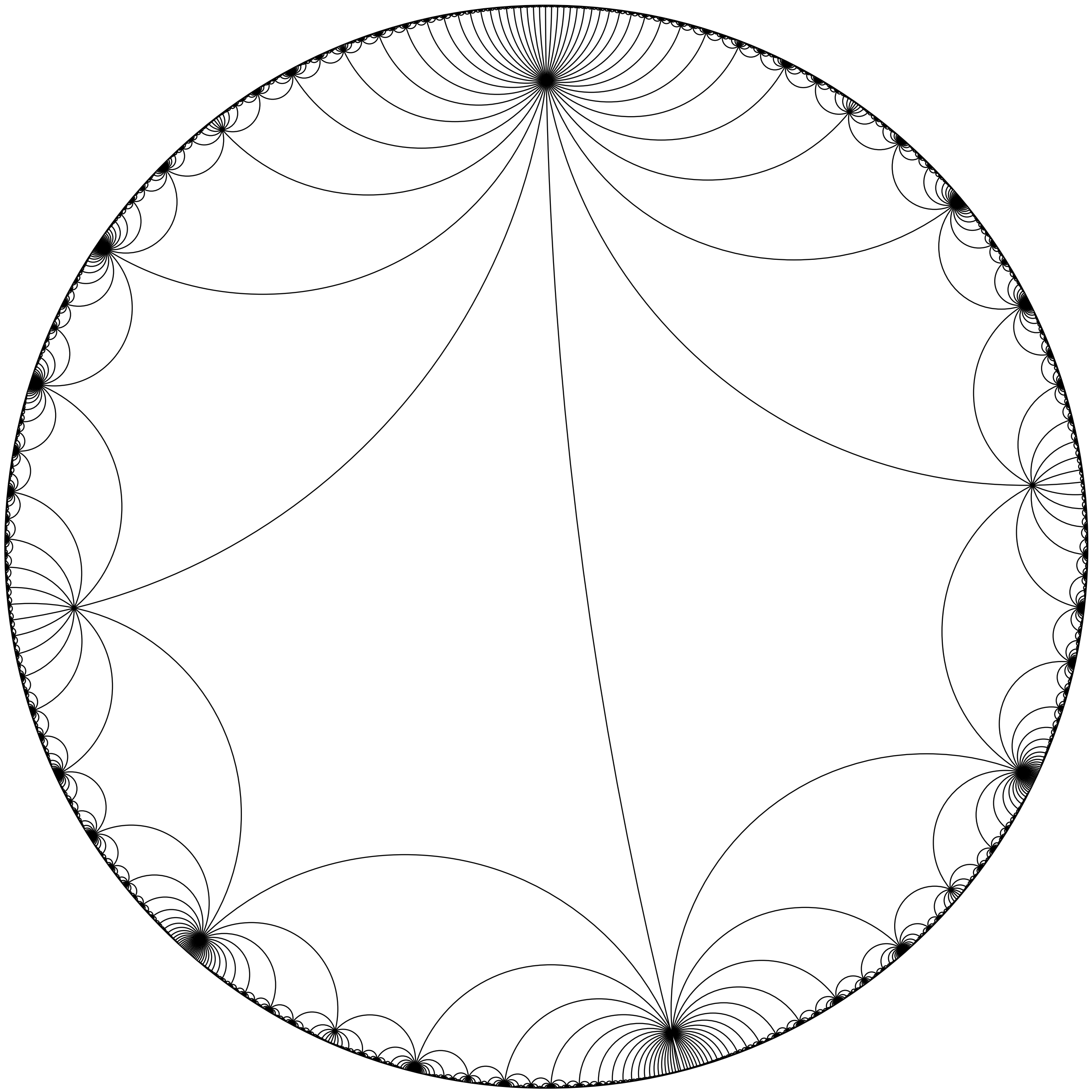
$$V = \mathbb{H}/\Delta' \to X_K.$$

Special to triangles!

Start with Δ(14,21,42)



Pass to Δ' of index 2

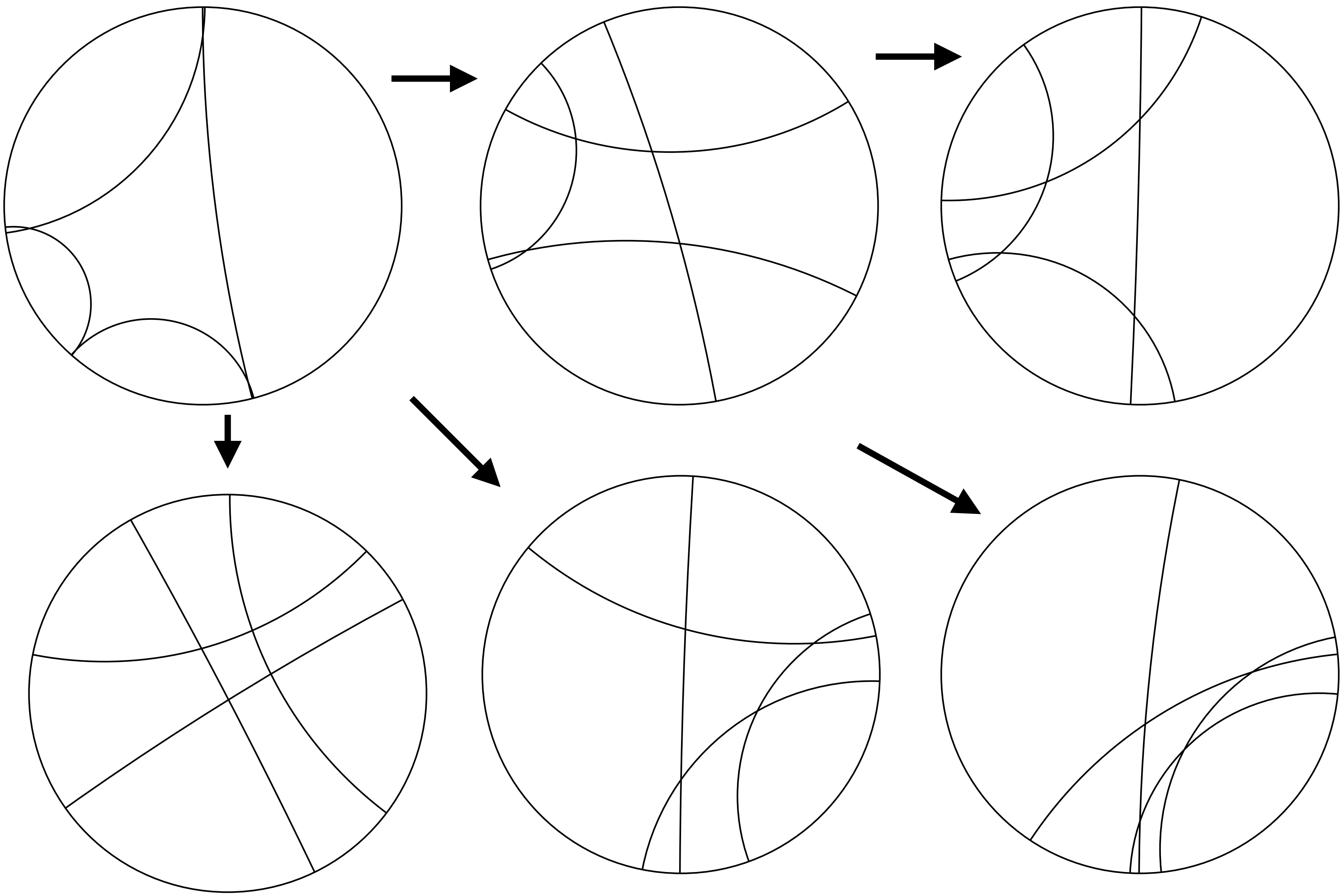


$V = \mathbb{H}/\Delta'$

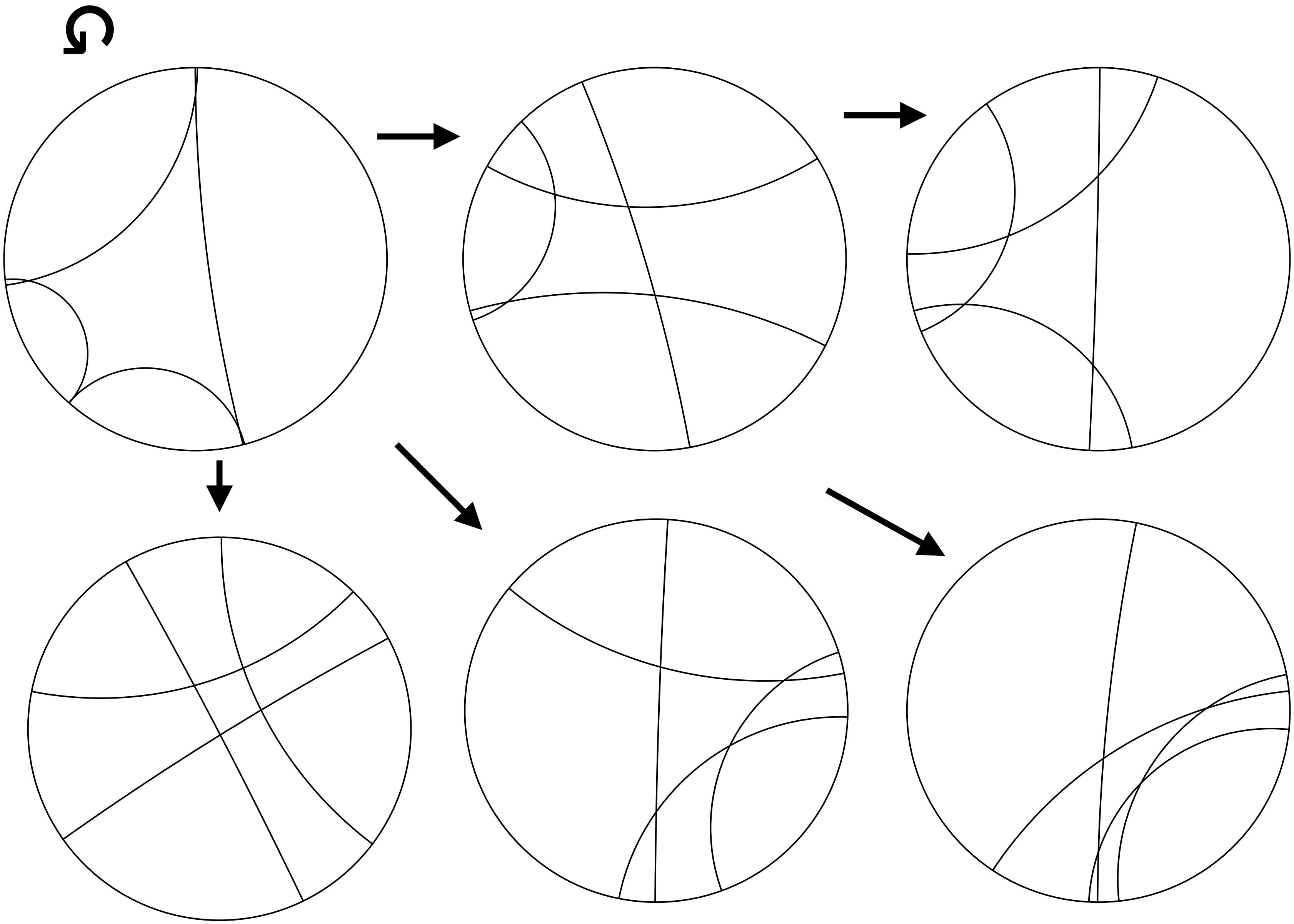
Construct 6 maps $\mathbb{H}$ to $\mathbb{H}$ from 6 real places of K

(uses Riemann mapping theorem)

$\Delta' \circlearrowleft$



Resulting map $\mathbb{H} \to \mathbb{H}^6$ covers exotic $V \to X_K$



Conclusion

V gives an exotic compact geodesic curve on X_K, dim=6.

(exotic because $\Delta' \subset \mathbf{SL}_2(\mathcal{O}_K)$ is Zariski dense,
so V is contained in no Shimura subvariety)

Conjecture

$\Delta(14,21,42)$ is the only triangle group whose invariant quaternion algebra splits.

Problem

Are there more examples of exotic curves?
For example, with $\dim X_K = 3$?

How to construct more geodesic curves?

References

Teichmüller dynamics and unique ergodicity …

Modular symbols for Teichmüller curves

Billiards and the arithmetic of non-arithmetic groups

Galois orbits in the moduli space of all triangles

Triangle groups and Hilbert modular varieties

Triangle groups: Cusps, congruence and chaos

Billiards in regular polygons

www.math.harvard.edu/~ctm/papers