Holonomy rank bounds for the unbounded denominators conjecture

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On our paper with Frank Calegari and Yunqing Tang

An overconvergence

Notation: $M_{\mathbb{Q}}$ will denote the set of *places* (= equivalence classes of absolute values) of the global field $\mathbb{Q}$.

Normalization of the p-adic absolute values: $|p|_p = 1/p$. Then we have the **product formula**: $\forall \alpha \in \mathbb{Q} \setminus \{0\}$, $\prod_{v \in M_{\mathbb{Q}}} |\alpha|_v = 1$.

We will have two stages of extending arithmetic algebraization past the Borel-Dwork criterion. The first was Pólya's theorem, which implies the following useful lemma exploited by Harbater in inverse Galois theory (*Galois covers of an arithmetic surface*, 1988):

Lemma (Harbater)

Suppose $f(x) \in \mathbb{Q}[[x]] \cap \overline{\mathbb{Q}(x)}$ is algebraic. If its v-adic convergence radii $R_v(f)$ fulfill

$$\prod_{v \in M_{\mathbb{Q}}} R_v(f) \geq 1,$$

then in fact $f(x) \in \mathbb{Q}[[x]] \cap \mathbb{Q}(x)$ is rational.

π_1: Harbater, Ihara, and Bost

Harbater's basic application (a result first discovered by Saito, by different means):

$$\pi_1\left(\mathbb{P}^1_{\mathbb{Z}} \smallsetminus \{0,1,\infty\}, *\right) = \{1\}.$$

Concretely: *no nonrational Belyĭ function can be in* $\mathbb{Z}[[x^{1/N}]]$.

E.g., $\sqrt[p]{1-x} \in \mathbb{Z}[1/p][[x]]$ with $R_p = p^{-p/(p-1)}$.

Passage: Suppose there were a finite covering $Y \to \mathbb{P}^1_{\mathbb{Z}}$ unramified away from the three sections $x = 0, 1, \infty$. If N is the order of the $x = 0$ local monodromy in this branched covering (N is finite by Puiseux's theorem), we may view $f(x^N) \in \mathbb{Z}[[x]]$ as an algebraic power series branched only along the multisection $\mu_N \cup \{\infty\}$ (but unramified away from 0). Then $R_v(f) = 1$ for all $v \in M_{\mathbb{Q}}$, and Harbater's lemma yields $f \in \mathbb{Q}(x)$.

Attachment: Saito's argument

[Appendix] T. Saito's original proof of Cor 1 of Th 2

It proceeds as follows. Let $L/k(t)$, $f : Y \to X = \mathbf{P}^1_{\mathfrak{O}}$ be as at the beginning of Theorem 2. Suppose that $f : Y \to X$ is etale outside $D_0 \cup D_1 \cup D_\infty$. Let $\mathfrak{p}$ be any prime ideal of $\mathfrak{O}$, and put $X_{\mathfrak{p}} = X \otimes_{\mathfrak{O}} (\mathfrak{O}/\mathfrak{p})$. Choose any cuspidal prime divisor D_i $(i = 0, 1, \infty)$ on X, and let P be the intersection of D_i with $X_{\mathfrak{p}}$, which is a closed point on $X_{\mathfrak{p}}$. Then the only prime divisor on X passing through P, along which f can possibly be ramified, is D_i. From this follows, by the generalized Abhyankar lemma ([G] Exp. XIII §5), that the ramification indices of $f_k = f \otimes k$ above $t = i$ cannot be divisible by the residue characteristic of $\mathfrak{p}$. Since $\mathfrak{p}$ and i are arbitrary, f must be etale

also above D_0, D_1, D_∞; hence $\pi_1(X - D_0 \cup D_1 \cup D_\infty) \simeq \pi_1(X) \simeq \pi_1(\operatorname{Spec} \mathfrak{O})$, as desired.

To address one of the questions asked: This is a snapshot of Ihara's appendix to *Horizontal divisors on arithmetic surfaces associated with Belyĭ uniformizations*, in the volume *The Grothendieck theory of Dessins d'Enfants (ed. Schneps, LNS 200).* It describes how Saito's used Abhyankar's lemma. The further references are in Ihara's paper.

The passage $1 \to 4 \to 16$

Let us now consider Belyĭ functions $f(x) \in \mathbb{Q}[[x]]$: algebraic formal power series with branching only over the three points $x = 0, 1, \infty$. As Frank explained:

- The Archimedean radius of convergence $= 1$, since that is the distance to the nearest singularity.
- In fact $f(x)$ is holomorphic on the larger domain $x \in \mathbb{C} \smallsetminus [1, \infty)$, of conformal radius $4 > 1$ at the origin. *Its*

  Riemann map is $\varphi(z) = 4z/(1+z)^2$, so we are simply saying that the pullback $(\varphi^* f)(z) = f(\varphi(z)) = f(4z/(1+z)^2)$ is holomorphic on the disc $|z| < 1$.
- 4 is the largest such radius for a *univalent* (injective) precomposition map $\varphi : D(0,1) \to \mathbb{C} \smallsetminus \{1\}$ omitting the singularities $\{1, \infty\}$.

  What is the largest conformal size $|\varphi'(0)|$ of some universal map $\varphi : D(0,1) \to \mathbb{C} \smallsetminus \{1\}$ for which $f(\varphi(z))$ is guaranteed to converge for every Belyĭ function $f(x)$ as above?

The passage $1 \to 4 \to 16$

$$g : z \mapsto \frac{4z}{(1+z)^2} : D(0,1) \to \mathbb{C} \setminus \{1\}.$$

- We have $1 - g(z) = \left(\frac{z-1}{z+1}\right)^2$, so g extends as a holomorphic map $\mathbb{C} \setminus \{\pm 1\} \to \mathbb{C} \setminus \{1\}$.
- Precompose g by the Riemann map $h : D(0,1) \to \mathbb{C} \setminus \big((-\infty,-1] \cup [1,\infty)\big)$. This is the conformally largest *univalent* map omitting the values $\{\pm 1\}$, and it is given by

$$h(z) = \sqrt{g(z^2)} = \frac{2z}{1+z^2} : D(0,1) \to \mathbb{C} \setminus \big((-\infty,-1] \cup [1,\infty)\big).$$

$$\varphi(z) = g(h(z)) = g\left(\sqrt{g(z^2)}\right) = \frac{8(z+z^3)}{(1+z)^4} : D(0,1) \to \mathbb{C} \setminus \{1\}.$$

The passage $1 \to 4 \to 16$

$$\varphi(z) = g(h(z)) = g\left(\sqrt{g(z^2)}\right) = \frac{8(z+z^3)}{(1+z)^4} \,:\, D(0,1) \to \mathbb{C} \setminus \{1\}.$$

This is now a generically bivalent map:

- $2:1$ over the complement of $\{0\} \cup (\mathbf{R} \setminus [-1,1])$
- $1:1$ over $\{0\}$ and $(-\infty,-1] \cup (1,\infty)$.

Yet, thanks to the singleton preimage property $\varphi^{-1}(0) = \{0\}$ over the only branch point $x = 0$ of $f(x)$ covered by the image of the multivalent φ, it still has

$$f(\varphi(z)) \textit{ convergent on } |z| < 1.$$

The passage $1 \to 4 \to 16$

Next stage after

$$g\left(\sqrt{g(z^2)}\right) = \frac{8(z+z^3)}{(1+z)^4} = 1 - \left(\frac{z-1}{z+1}\right)^4 ?$$

If we want to keep the $\varphi^{-1}(0) = \{0\}$ property, the next precomposition map will have to avoid not only the preimages $\{1\}$ and $\{-1\}$ of the branch points $x = 1$ and $x = \infty$, but also the "other" zeros $\{i, -i\}$ of the above (*preceding*) map. Again, the conformally maximal map with these requisite properties is essentially a Koebe map, renormalized:

$$\sqrt[4]{g(z^4)} = \frac{\sqrt{2}\, z}{\sqrt{1+z^4}} \;:\; D(0,1) \to \mathbb{C} \setminus \mu_4.$$

Continue by the tower (with k square root signs)

$$g(q) = \frac{4q}{(1+q)^2}, \quad g\left(\sqrt{g(\sqrt{\cdots\sqrt{g(q^{2^k})}})}\right)$$

The passage $1 \to 4 \to 16$

$$g(q) = \frac{4q}{(1+q)^2}, \quad g_k(q) := g\left(\sqrt{g\left(\sqrt{\cdots\sqrt{g(q^{2^k})}}\right)}\right)$$

Derivatives at the origin converging to $4^{1+1/2+1/4+1/8+\cdots} = 16$.

$$\varphi(q) := g_1(q) = \frac{8(q+q^3)}{(1+q)^4},$$

$$\varphi(2q) = 16q - 128q^2 + 704q^3 - 3072q^4$$

$$+11520q^5 - 38912q^6 + 121856q^7 - 360448q^8 + \cdots$$

$$\lambda(q) = 16q - 128q^2 + 704q^3 - 3072q^4$$

$$+11488q^5 - 38400q^6 + 117632q^7 - 335872q^8 + \cdots.$$

The modular lambda map

$$g_k(q) \to \lambda(q) := \frac{\left(\sum_{n\in\mathbb{Z}} q^{(n+1/2)^2}\right)^4}{\left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^4} = 16q \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^8.$$

Indeed,

$$L(q) := \sqrt[4]{\lambda(q^4)} = \frac{\sum_{n\,\mathrm{odd}} q^{n^2}}{\sum_{n\,\mathrm{even}} q^{n^2}}$$

clearly fulfills the defining equation (*Landen's transform*)

$$L(q^{1/4})^4 = \lambda(q) = \frac{8L(q)(1+L(q)^2)}{(1+L(q))^4}$$

of the limiting map $L(q)$.

André's algebraicity criterion

Theorem (André)

Let $f(x) \in \mathbb{Z}[[x]]$, and consider a holomorphic mapping $\varphi : D(0,1) \to \mathbb{C}$ taking $\varphi(0) = 0$ with derivative $|\varphi'(0)| > 1$, and such that the germ $f(\varphi(z)) \in \mathbb{C}[[z]]$ is analytic on $|z| < 1$.
Then $f(x)$ is algebraic.
(If furthermore $\varphi : D(0,1) \hookrightarrow \mathbb{C}$ is injective, then in fact $f(x)$ is rational: this was Pólya's theorem.)

A basic example: $\sqrt[4]{1-8x} \in \mathbb{Z}[[x]]$ and it meets the criterion with $\varphi(z) := \lambda(z)/8$, of conformal size $|\varphi'(0)| = 2 > 1$.

An interesting boundary case

Since $\lambda(q)$ has conformal size $\lambda'(0) = 16$, we shall scale our branch values by that factor to keep up with $\mathbb{Z}[[x]]$ expansions: $\{0, 1, \infty\} \mapsto \{0, 1/16, \infty\}$.
Let $x := \lambda(q)/16 = q - 8q^2 + \cdots \in q + q^2\mathbb{Z}[[q]]$. We may formally invert that expansion, using the equality of completed rings $\mathbb{Z}[[q]] = \mathbb{Z}[[x]]$, and write

$$q = x + 8x^2 + 91x^3 + \cdots \in x + \mathbb{Z}[[x]].$$

There are infinitely many $\mathbb{Q}(x)$-linearly independent algebraic functions $f(x) \in \mathbb{Z}[[x]]$ such that $f\big(\lambda(q)/16\big) \in \mathbb{Z}[[q]]$ is convergent on the open unit q-disc $|q| < 1$.

They come from congruence modular functions! For each $N = 1, 2, 3, \ldots$, take $\lambda(q^N) \in \mathbb{Z}[[q]] = \mathbb{Z}[[x]]$ written out in terms of $x = \lambda(q)/16$.

A holonomy rank bound

The following will be applied with $t = q^{1/N}$, $p(x) := x^N$, $x = x(t) := \sqrt[N]{\lambda(t^N)/16} : D(0,1) \to U := \mathbb{C} \smallsetminus 16^{-1/N}\mu_N$, *and* $\varphi : D(0,1) \to \mathbb{C} \smallsetminus (16)^{-1/N}\mu_N$ *the universal covering map restricted disc.*

Theorem

Let $p(x) \in \mathbb{Q}[x] \smallsetminus \mathbb{Q}$ and $x(t) = t + \cdots \in \mathbb{Q}[[t]]$ be such that $p(x(t)) \in \mathbb{Z}[[t]]$. Fix the holomorphic mapping $\varphi : \overline{D(0,1)} \to U$ with $\varphi(0) = 0$ and $|\varphi'(0)| > 1$. Then, the totality of formal functions $f(x) \in \mathbb{Q}[[x]]$ that

- fulfill a linear ODE over $\mathbb{Q}(x)$ without singularities on U, and
- have a t-expansion $f(x(t)) \in \mathbb{Z}[[t]]$,

span over $\mathbb{Q}(p(x))$ a finite-dimensional vector space of dimension at most

$$e \cdot \frac{\int_{|z|=1} \log^+ |p \circ \varphi| \, \mu_{\mathrm{Haar}}}{\log |\varphi'(0)|}.$$

($e = 2.71\ldots$ is Euler's constant)

The proof of the holonomy theorem

It follows a method of André, itself going back to D. & G. Chudnovsky in their Diophantine approximations proof of the Faltings isogeny theorem for elliptic curves over $\mathbb{Q}$. A crucial new twist (obviously inspired by Thue–Siegel–Schneider–Roth) is to let the number of auxiliary variables $\mathbf{x} := (x_1, \ldots, x_d)$ to $d \to \infty$.

Lemma (Siegel's lemma)

Let A be an $L \times M$-matrix whose entries are rational integers bounded in absolute value by B. Then, if $L > M$, the linear system $A \cdot \mathbf{x} = \mathbf{0}$ of M equations in L variables $x_1, \ldots, x_L$ has a nontrivial integral solution $\mathbf{x} \in \mathbb{Z}^L \smallsetminus \{\mathbf{0}\}$ with

$$\max_{1 \leq i \leq L} |x_i| \leq (LB)^{\frac{M}{L-M}}.$$

A Minkowski argument: pigeonholing a solution.

The proof of the holonomy theorem

Suppose there are m such functions $f_1(x), \ldots, f_m(x) \in \mathbb{Q}[[x]]$ linearly independent over $\mathbb{Q}(p(x))$. We use the m^d split variables univariate products $\prod_{s=1}^d f_{i_s}(x_s)$ and Siegel's lemma to create an auxiliary function of the form:

$$F(\mathbf{x}) = \sum_{\substack{\mathbf{i}\in\{1,\ldots,m\}^d \\ \mathbf{k}\in\{0,\ldots,D-1\}^d}} a_{\mathbf{i},\mathbf{k}}\, p(\mathbf{x})^{\mathbf{k}} \prod_{s=1}^{d} f_{i_s}(x_s) \in (\mathbf{x})^{\alpha}\, \mathbb{Q}[[\mathbf{x}]] \setminus \{0\},$$

with sub-exponentially small coefficients $a_{\mathbf{i},\mathbf{k}} = \exp(o(\alpha))$ as firstly $\alpha \to \infty$ and secondly $d \to \infty$. *With a degree D as low as possible.*

Siegel's lemma: the parameter count

$$F(\mathbf{x}) = \sum_{\substack{\mathbf{i}\in\{1,\ldots,m\}^d \\ \mathbf{k}\in\{0,\ldots,D-1\}^d}} a_{\mathbf{i},\mathbf{k}}\, p(\mathbf{x})^{\mathbf{k}} \prod_{s=1}^{d} f_{i_s}(x_s) \in (\mathbf{x})^{\alpha}\, \mathbb{Q}[[\mathbf{x}]] \setminus \{0\},$$

- $(mD)^d$ free parameters $a_{\mathbf{i},\mathbf{k}}$
- $\binom{\alpha+d}{d} \sim \alpha^d/d! \approx (e\alpha/d)^d$ equations to solve
- $\#\mathtt{parameters} > \#\mathtt{equations}$ if $dD > e(1+o(1))\frac{\alpha}{m}$ asymptotically
- by letting also $d \to \infty$, we can also make sure the Dirichlet exponent $\to 0$, and the coefficients a are $\exp(o(\alpha))$
- then we can asymptotically take the degree parameter $dD \sim e\,\frac{\alpha}{m}$.

The extrapolation

The idea is that the function $G(\mathbf{z}) := F(\varphi(\mathbf{z})) \in \mathbb{C}[[\mathbf{z}]]$ is analytic on $\overline{D(0,1)}$ (by Cauchy's theorem), and yet since $\varphi(z) = \varphi'(0)z + \cdots$ and $x(t) = t + \cdots$, it also inherits from $F(x(\mathbf{t})) \in \mathbb{Z}[[\mathbf{t}]]$ an integrality property of its *lexicographically lowest* term $c\,\mathbf{z}^{\boldsymbol{\beta}}$:

- $c \in \varphi'(0)^{|\boldsymbol{\beta}|}\,\mathbb{Z} \setminus \{0\}$, with total degree $|\boldsymbol{\beta}| \geq \alpha$
- hence the Liouville lower bound for that coefficient: $\log|c| \geq \alpha \log|\varphi'(0)|$
- (A simplification step pointed out to us by André) We can use the plurisubharmonic property of $\log|\mathrm{holomorphic\ function}|$ together with an easy induction scheme on d to prove that, *for our lexicographically lowest monomial $c\,\mathbf{z}^{\boldsymbol{\beta}}$*, we have a bound in the other direction:

$$\log|c| \leq \int_{\mathbf{T}^d} \log|G|\,\mu_{\mathrm{Haar}}.$$

The base case $d = 1$ is simply the subharmonic property of $\log|z^{-\beta}G(z)|$.

The holonomy rank bound: proof completion

- $\alpha \log |\varphi'(0)| \leq \int_{\mathbf{T}^d} \log |F| \, \mu_{\mathrm{Haar}}$
- the RHS is upper estimated by our arithmetic information from the shape of F and the asymptotically subexponential coefficients bound in Siegel's lemma:

$$\alpha \log |\varphi'(0)| \leq \int_{\mathbf{T}^d} \log |G| \, \mu_{\mathrm{Haar}} \leq dD \int_{\mathbf{T}} \log^{+} |p \circ \varphi| \, \mu_{\mathrm{Haar}} + o(\alpha)$$

- With the degree parameter asymptotic estimate $dD \sim e\alpha/m$, the last inequality amounts in the $\alpha \to \infty$, $d \to \infty$ limit to

$$m \leq e \frac{\int_{\mathbf{T}} \log^{+} |p \circ \varphi| \, \mu_{\mathrm{Haar}}}{\log |\varphi'(0)|},$$

that is precisely what we aimed to prove.

What have we got so far

In Lecture 2, we will be using the preceding with the choices

$$t := q^{1/N} = e^{\pi i \tau / N}, p(x) = x^N,$$
$$x(t) := \sqrt[N]{\lambda(t^N)/16}, \quad U := \mathbb{C} \setminus 16^{-1/N} \mu_N,$$
$$\varphi(z) := 16^{-1/N} F_N(rz) \quad : \quad \overline{D(0,1)} \to U$$

for $r := 1 - 1/(2N^3)$.

$$e \cdot \frac{\int_{|z|=1} \log^+ |p \circ \varphi| \, \mu_{\mathrm{Haar}}}{\log |\varphi'(0)|}.$$

We will see in Lecture 2 that this dimension bound is an $O(N^3 \log N)$ in the above situation, and derive from this the Unbounde Denominators conjecture.