Torsors over affine curves

First problem session

Exercise 1. Let R be a commutative ring and let M be a R–module of finite presentation. Let $\mathbb{W}(M)$ be the R–functor defined by $S \mapsto M \otimes_R S$. Show that $\mathbb{W}(M)$ is representable if and only if M is a locally free R–module of finite type.

[Hint: To show that if $\mathbb{W}(M)$ is representable by a G–scheme then M is a locally free R–module of finite type, one can show that G is smooth and then consider the tangent vector bundle.]

Exercise 2. Let R be an unitary commutative ring.

Show that $\mathrm{Pic}(R[t]) = \mathrm{Pic}(R)$ when R is a normal ring of finite type over a field k of dimension n.

[Hint: One can use that, for every X–scheme and every effective Cartier divisor $\mathcal{L}$ of X, there is a Gysin-map $\mathcal{L}.$ from $CH_k(X)$ to $CH_{k-1}(|\mathcal{L}|)$ define on cycle $[Y]$ by $\mathcal{L}.[Y] = [i^*\mathcal{L}]$ where $i : Y \hookrightarrow X$ is an irreducible subscheme of X of dimension k. (See [Ful84].)]

Remark 1. See [Wei94] for examples in which $\mathrm{Pic}(R[t]) \neq \mathrm{Pic}(R)$.

Exercise 3. Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_0$ be three categories and $\alpha_i : \mathcal{C}_i \to \mathcal{C}_0$ for $i = 1, 2$. We define the category $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ whose objects are the triple $(\mathcal{C}_1, \mathcal{C}_2, \phi)$ where $\phi : \alpha_1(\mathcal{C}_1) \to \alpha_2(\mathcal{C}_2)$ is an isomorphism and a morphism from $(\mathcal{C}_1, \mathcal{C}_2, \phi)$ to $(\mathcal{C}_1', \mathcal{C}_2', \phi')$ consists of two morphisms $f_i : \mathcal{C}_i \to \mathcal{C}_i'$ for $i = 1, 2$ such that $\phi' \circ \alpha_1(f_1) = \alpha_2(f_2) \circ \phi$.

If F is a field, we denote by $\mathrm{Vect}_n(F)$ the category of F–vector spaces of dimension n and if R is a ring, we denote by $\mathrm{Mod}_n(R)$ the category of R–modules locally free of rank n.

1. Let F_1, F_2 be two fields, F_0 be an extension of this two fields, and let $F = F_1 \cap F_2$.

   Show that if for every n, $\mathrm{GL}_n(F_0) = \mathrm{GL}_n(F_1)\mathrm{GL}_n(F_2)$, then the map:

$$\begin{array}{cccc} \beta : & \mathrm{Vect}_n(F) & \to & \mathrm{Vect}_n(F_1) \times_{\mathrm{Vect}_n(F_0)} \mathrm{Vect}_n(F_2) \\ & V & \mapsto & (V \otimes_F F_1, V \otimes_F F_2, \phi : V \otimes_F F_1 \otimes_{F_1} F_0 \xrightarrow{\sim} V \otimes_F F_2 \otimes_{F_2} F_0) \end{array}$$

   is an equivalence of categories.

2. If R is a DVR and K is its field of fractions, one can define in the same way an application

$$\gamma : \mathrm{Mod}_n(R) \to \mathrm{Vect}_n(K) \times_{\mathrm{Vect}_n(\hat{K})} \mathrm{Mod}_n(\hat{R}).$$

   First show that $\mathrm{GL}_n(\hat{K}) = \mathrm{GL}_n(K)\mathrm{GL}_n(\hat{R})$ and then that γ is an equivalence of categories.

References

[Ful84] W. Fulton. *Intersection Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete : a series of modern surveys in mathematics. Folge 3. Springer-Verlag, 1984.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*. Vol. 38. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 1994.