PROJECT REPORT FOR THE SUMMER COLLABORATORS PROGRAM AT IAS:

DETECTING NON-FILLABILITY OF BOUNDARIES OF CONCAVE PLUMBINGS VIA PSEUDOHOLOMORPHIC CURVES

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Together we have been studying plumbings of disc bundles over spheres, that provide compact 4-manifolds endowed with a symplectic form. The boundary of these manifolds is a closed contact 3-manifold, that is in some cases convex and in other cases concave. Only the convex case gives rise to a symplectic filling of the contact 3-manifold and was already studied by Gay-Stipsicz, Li-Mak [GS09; LM19]. Our project is to study the concave case and determine when the contact 3-manifold is symplectically fillable. We completed this classification in the case of linear plumbings. Then, we used this case to develop techniques that can then be applied to more general types of plumbings, towards a classification of contact structures in these manifolds. Our work involves the study of dynamical properties of these contact manifolds, as there is subtle interplay relating to the connection between symplectic and complex manifolds, and is motivated in part by complex algebraic geometry and singularity theory.

Our approach has two main directions: linear plumbings are toric manifolds providing a first way to study them, and as contact manifolds the structure is simple enough for computing parts of the Embedded Contact Homology (ECH) quantities that have not been computed outside simpler examples. Embedded contact homology is a three dimensional gauge theory which is equivalent to monopole Seiberg-Witten Floer cohomology, but defined in terms of contact dynamics and holomorphic curves. In the non-linear case, the obtained 3-manifolds are not longer toric but still have toric pieces to which we want to export our techniques.

We started to work together almost 2 years ago and the IAS summer program gave us the opportunity of being the six at the same place (in the same time zone) for the first time. Five of us had met in September 2023 at Banff (Canada). We advanced significantly in our project in two directions. We are nearly complete in having prepared a first publication with our results for which we believe that all the arguments are complete after the two weeks of work at the IAS.

In particular, our paper and work together at the IAS conclude the first part of the project by articulating all the arguments we have been discussing and learning from each other since the project began. We can now determine both from the toric perspective and from the ECH perspective, when the contact structure is overtwisted and this can be read from the plumbing graph. From the toric perspective we can actually construct an overtwisted disc. From the ECH perspective we are able to decide when algebraic torsion is zero or not. The ECH algebraic torsion has not been proven to be a contact invariant and is defined using the collections of closed periodic orbits that are in the same ECH class as the empty set. A contact structure whose ECH algebraic torsion is zero for every contact form (and quasi-holomorphic structure) is conjectured to be overtwisted. Furthermore, overtwisted contact manifolds are known to be nonfillable.

Secondly, we understood how some of our techniques apply to other plumbings. We conjecture and expect that we can prove that if a plumbing has a certain type of very simple subplumbing, then ECH algebraic torsion is zero, proving that a big family of concave boundaries of plumbings are overtwisted. The way these arguments apply is now very clear – we carried several different computations before understanding a simpler statement about them. Moreover, in some of these cases we are able to build explicitly an overtwisted disc.

The results we obtained so far confirm known results on non-fillability of these contact structure, hence we provide with new proofs. We were also able to elucidate the ECH part of the story. We did not succeed so far to apply our techniques to examples whose fillability has not been established. The progress we made at IAS was unexpected and very motivating.

References

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