Mixed-State Quantum Phases

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R. Ma, J.-H. Zhang, Z. Bi, MC, C. Wang, arXiv: 2305.16399
L. Lessa, MC, C. Wang, arXiv:2401.17357
L. Lessa, R. Ma, J.-H.Zhang, Z. Bi, MC, C. Wang, arXiv: 2405.03639
T. Ellison and MC, arXiv:2405.02390

Prospects in Theoretical Physics
July 10, 2024

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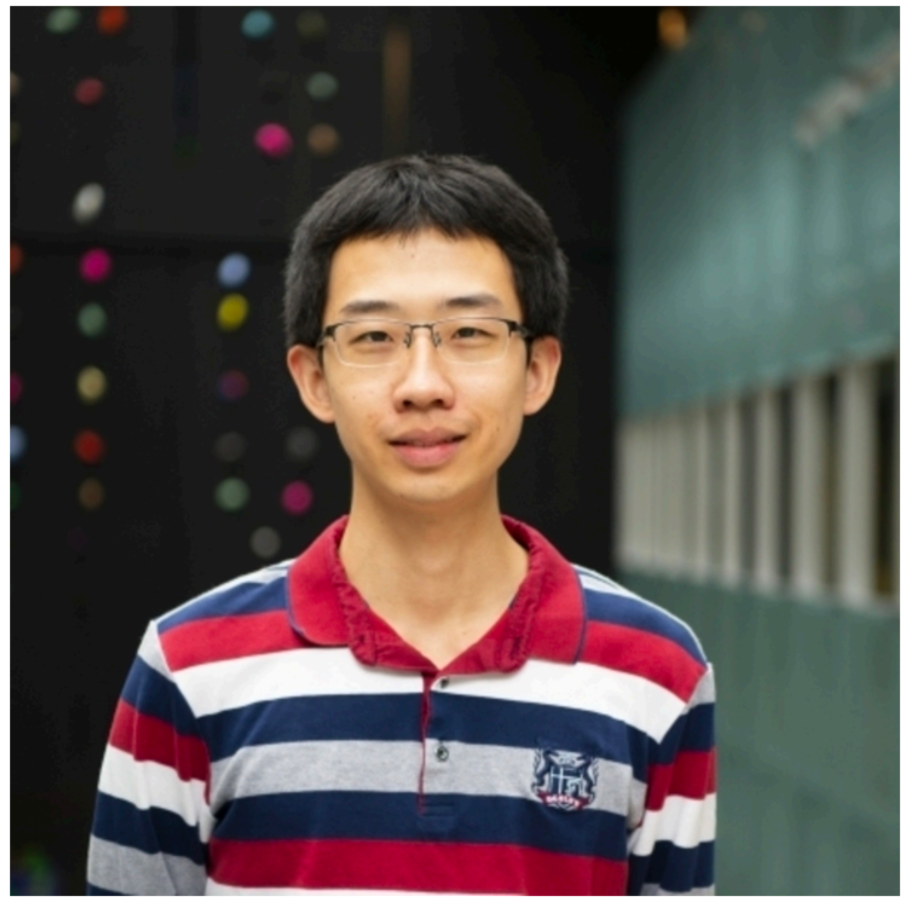

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Jianhao Zhang
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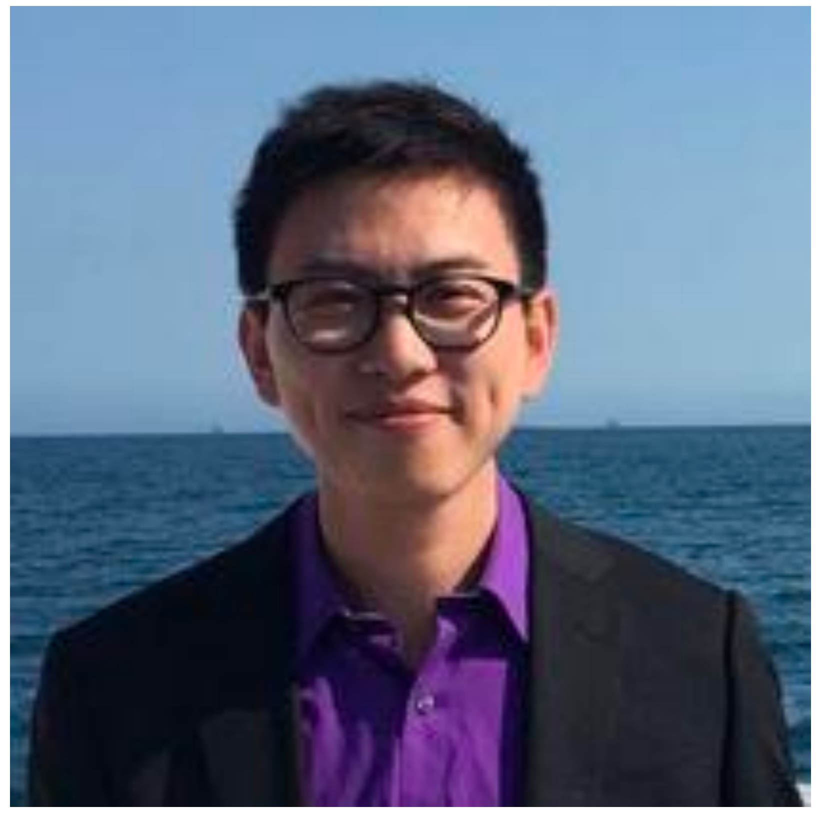
Yale → Perimeter → Purdue



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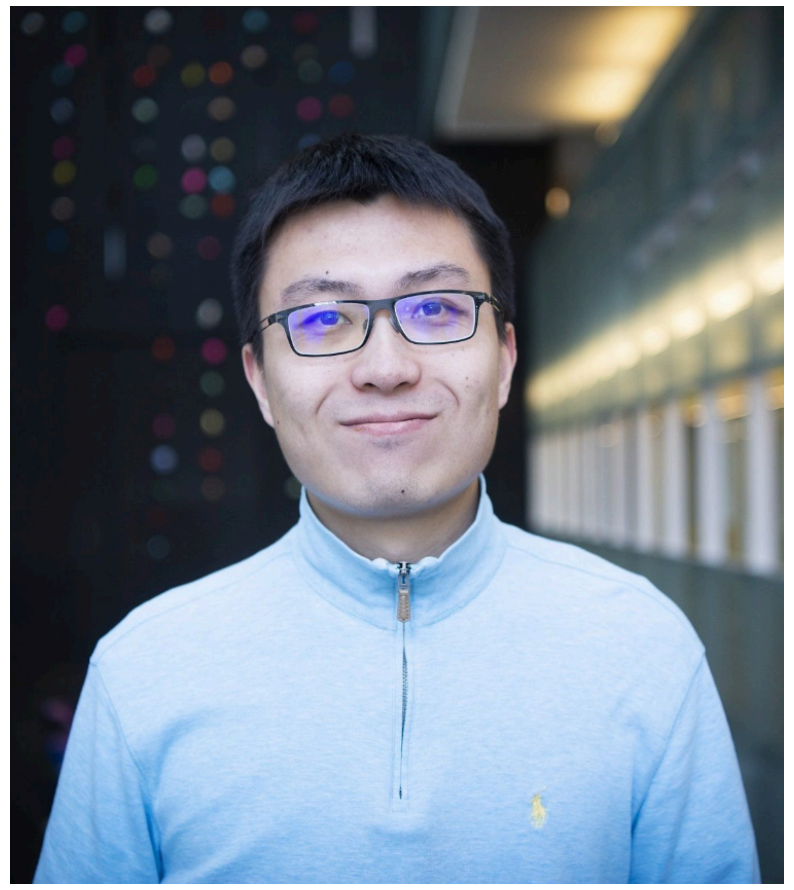

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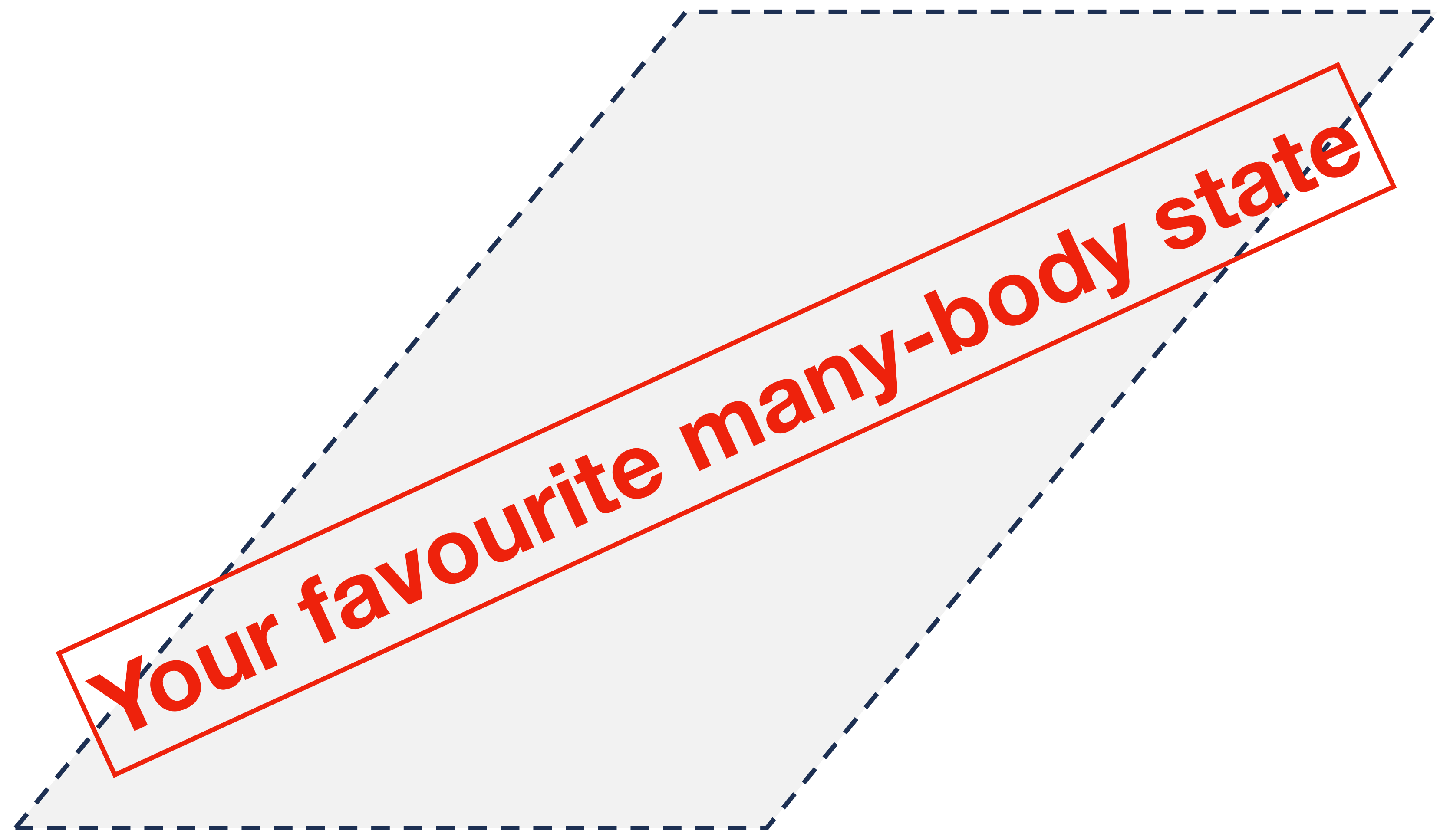
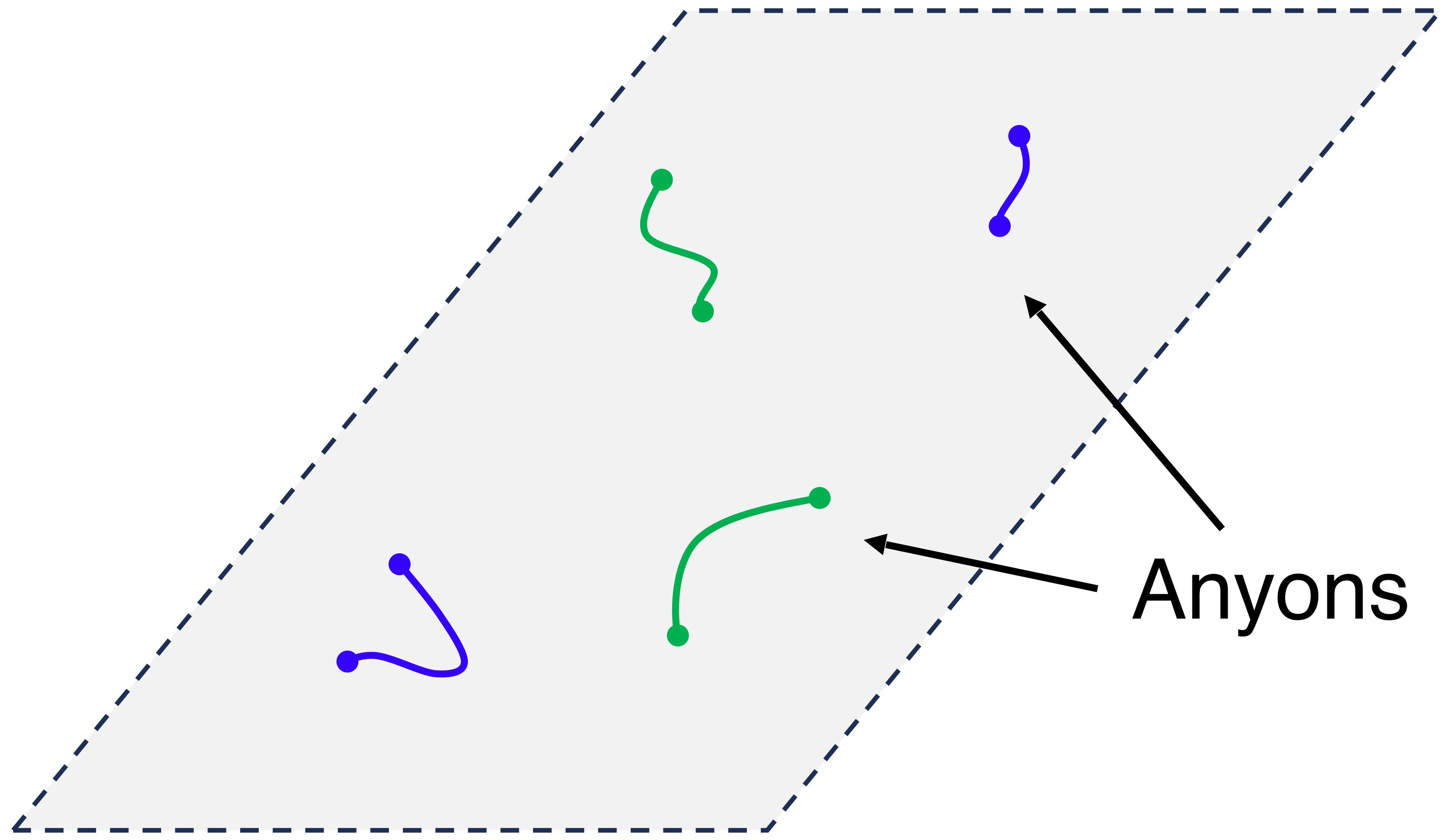
Outline

- Motivation and introduction
- Strong-to-weak symmetry breaking
- Two examples of “intrinsically” mixed-state topological phases

Motivation

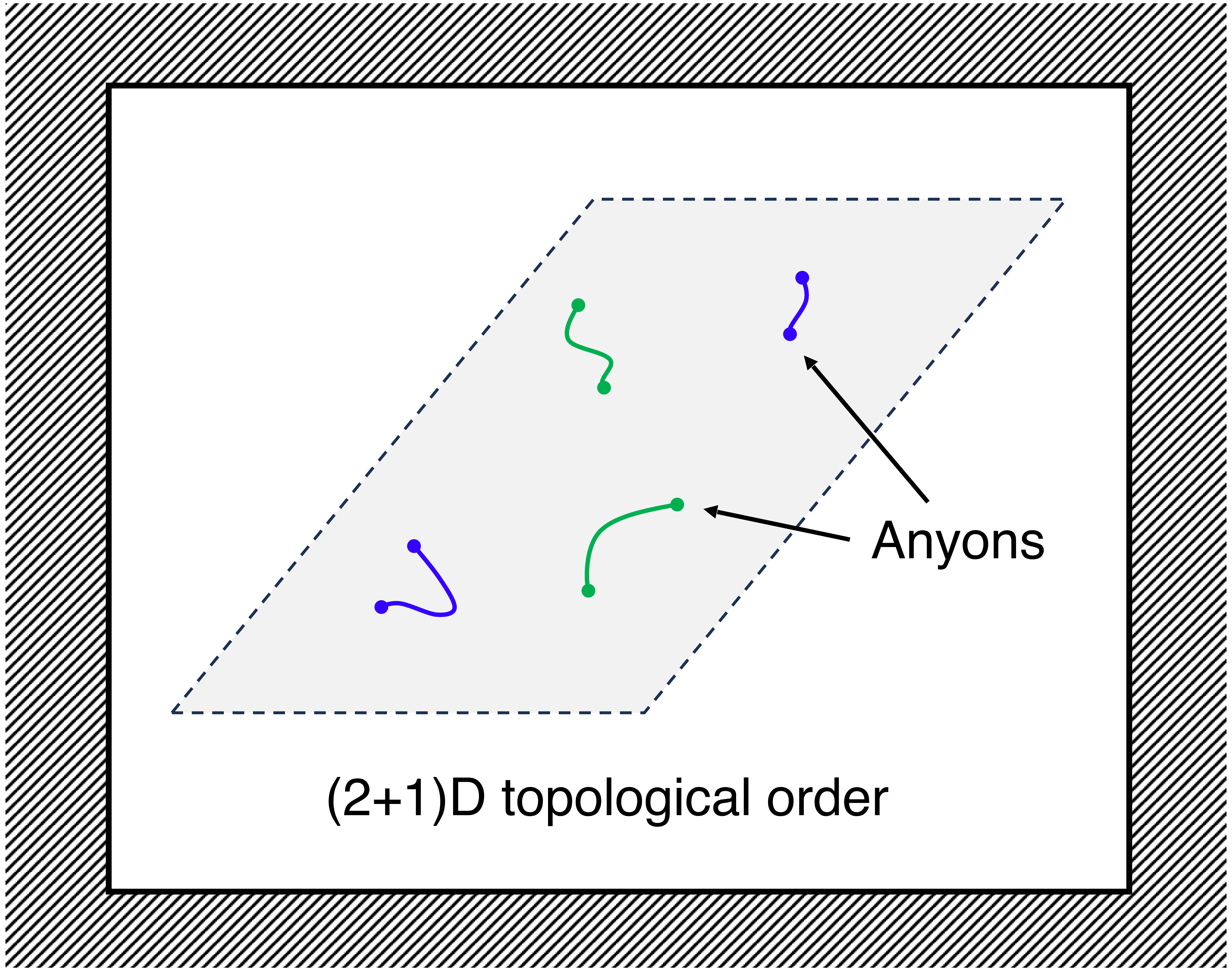


Motivation



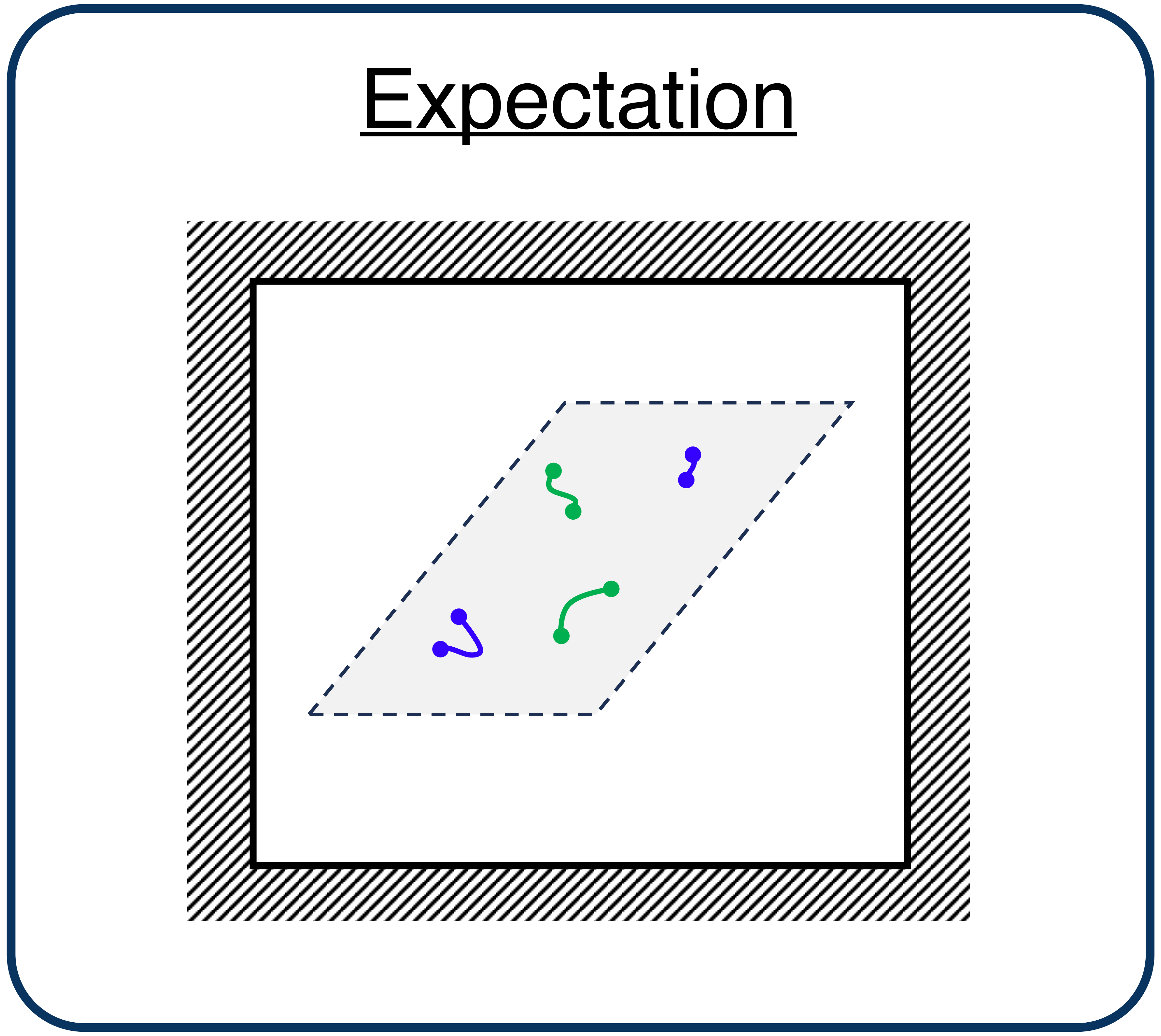
(2+1)D topological order

Motivation

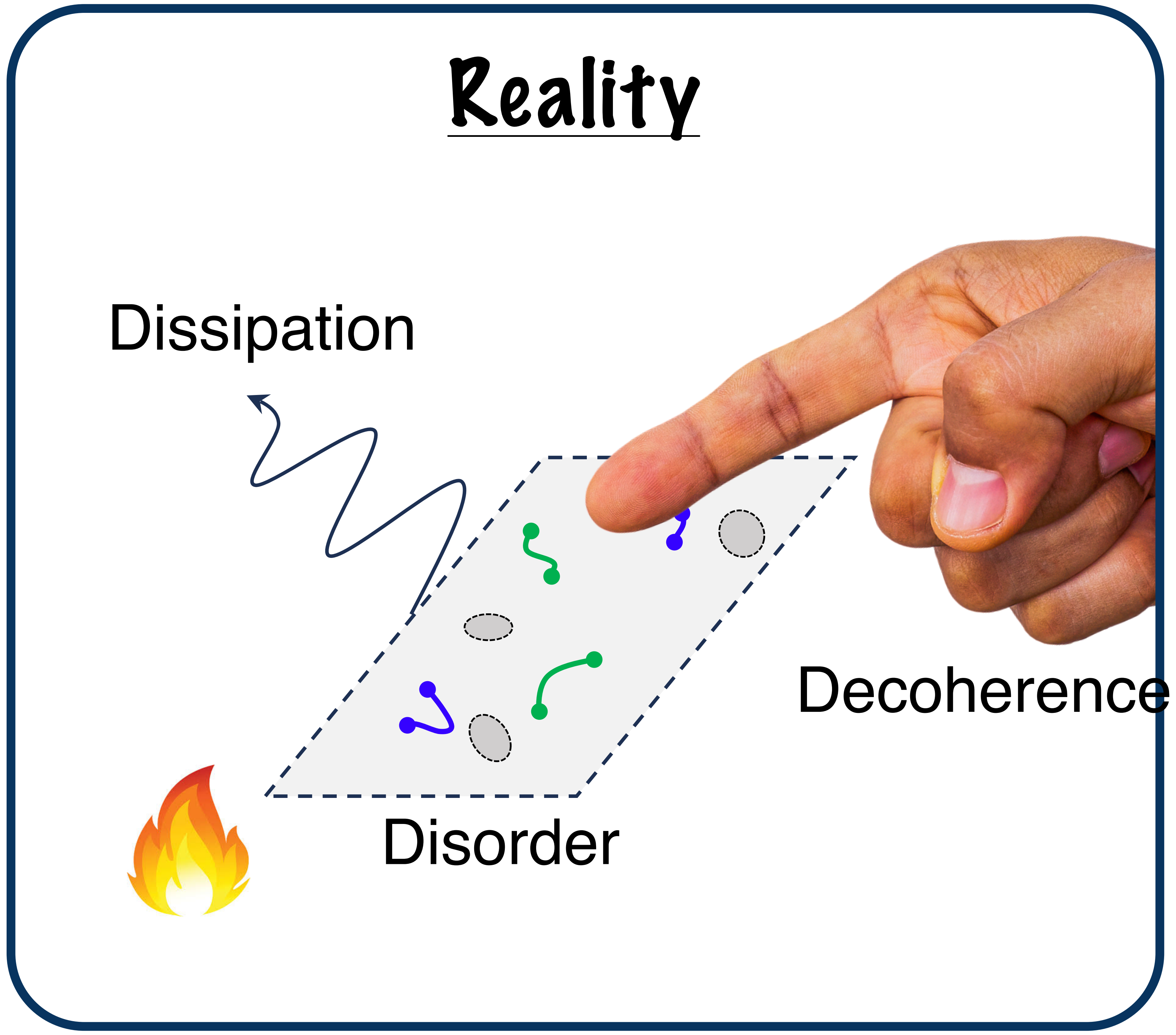


Assumption: well isolated from the environment

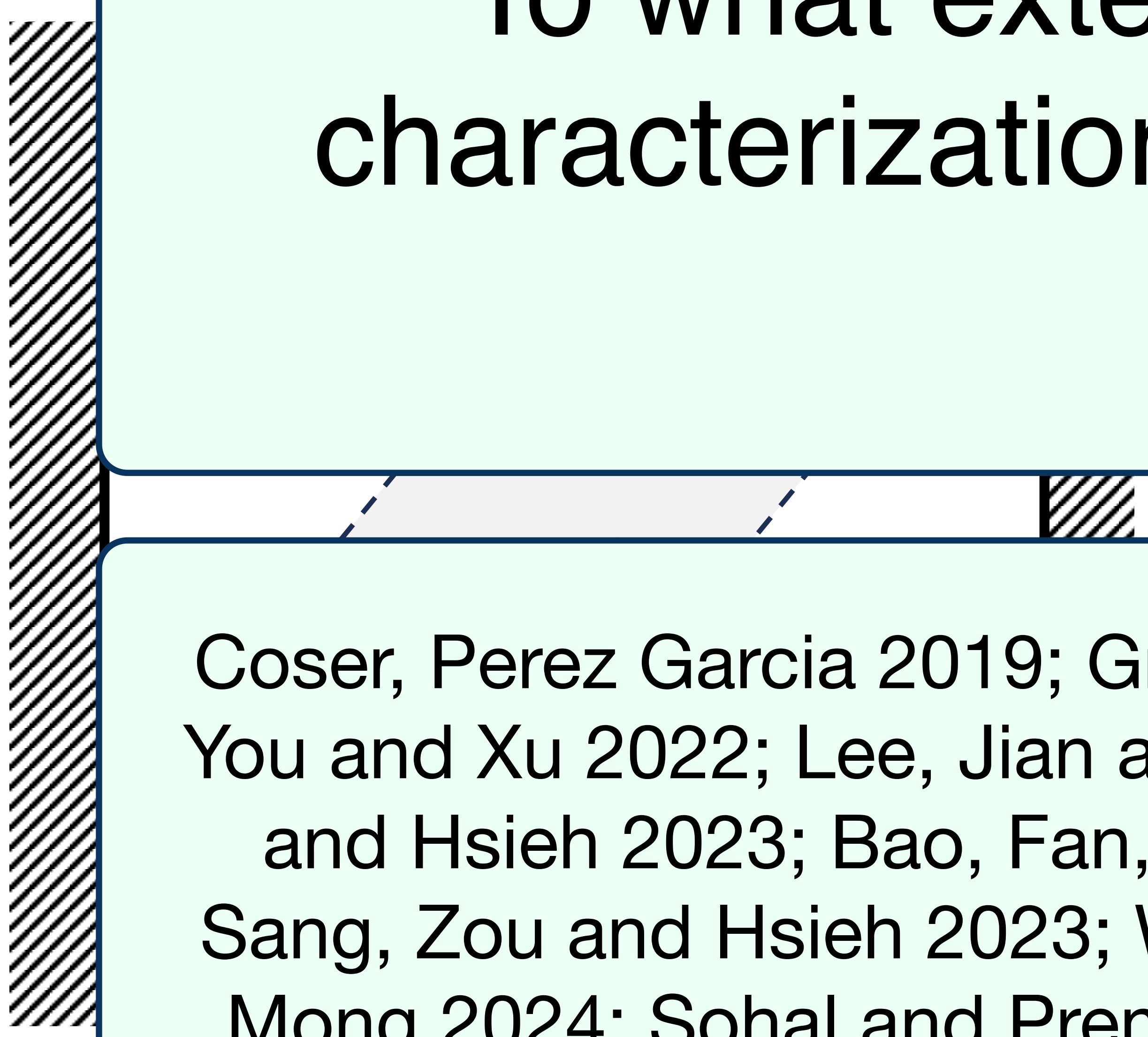
Motivation

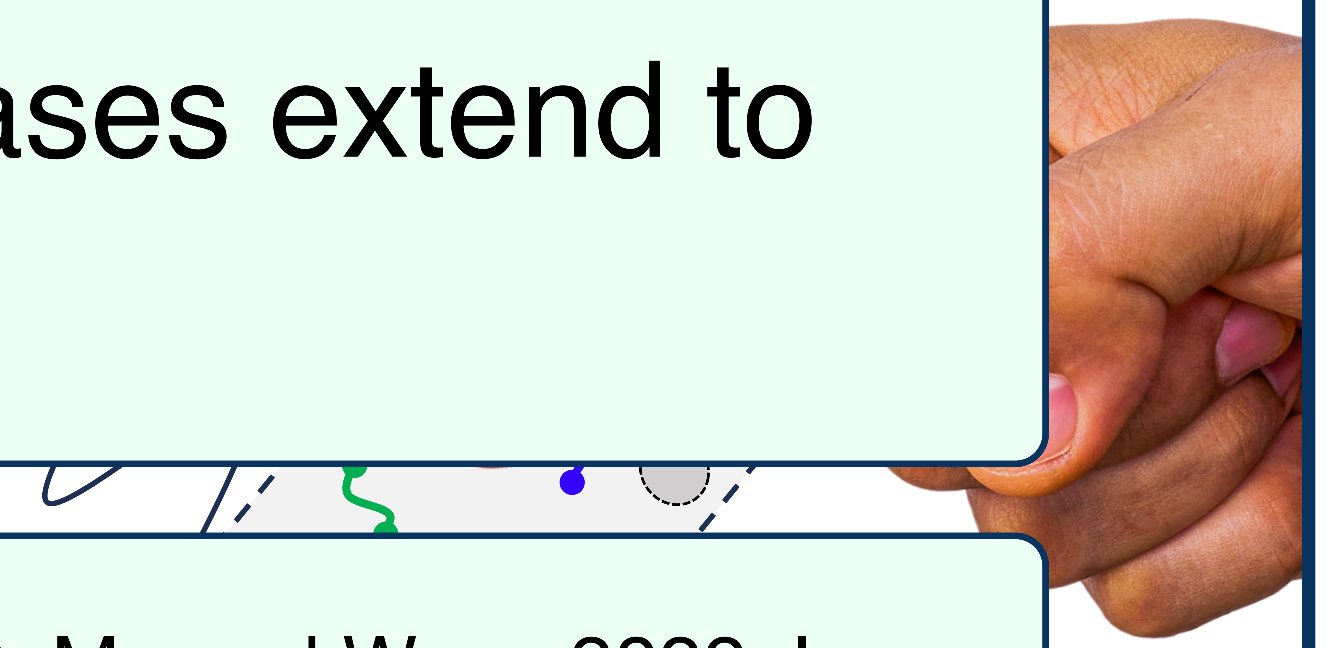


Vs.



Motivation





To what extent does the classification and characterization of pure-state phases extend to mixed states?

Coser, Perez Garcia 2019; Groot, Turzillo and Schuch 2022; Ma and Wang 2022; Lee, You and Xu 2022; Lee, Jian and Xu 2023; Zou, Sang and Hsieh 2023; Lu, Zhang, Vijay and Hsieh 2023; Bao, Fan, Altman and Vishwanath 2023; Chen and Grover 2023; Sang, Zou and Hsieh 2023; Wang, Wu and Wang 2023; Sang and Hsieh 2024; Li and Mong 2024; Sohal and Prem 2024; Sala et al 2024; Zhang, Agrawal and Vijay 2024; Gu, Wang and Wang 2024; Huang, Qi, Zhang and Lucas 2024; and more …

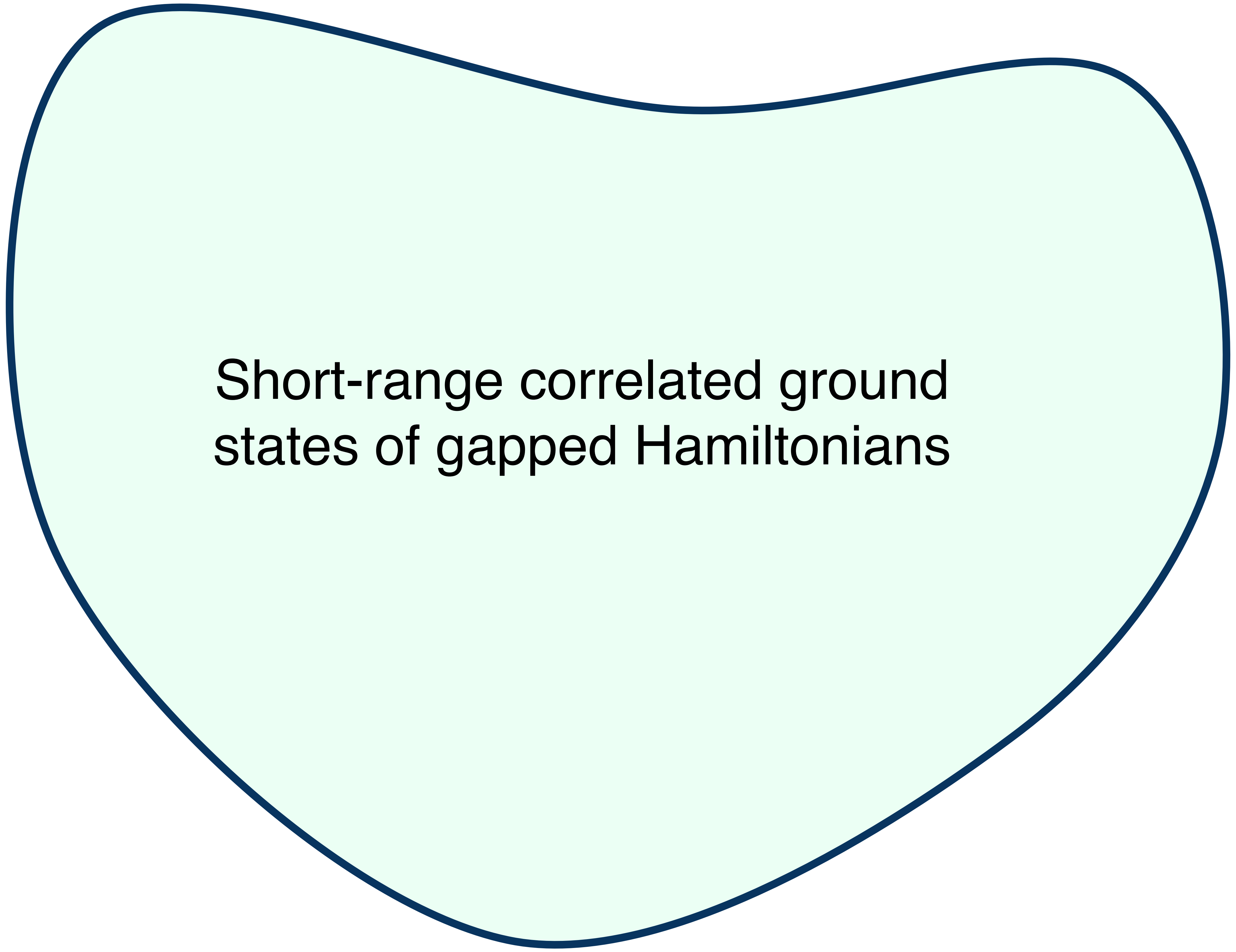
Pure-state topological phases

<u>Which states?</u>

1. Ground states of gapped local Hamiltonians
2. Short-range correlations

$$\langle O_i(x) O_j(y) \rangle - \langle O_i(x) \rangle \langle O_j(y) \rangle \sim e^{-|x-y|/\xi}$$

for all local O_i, O_j

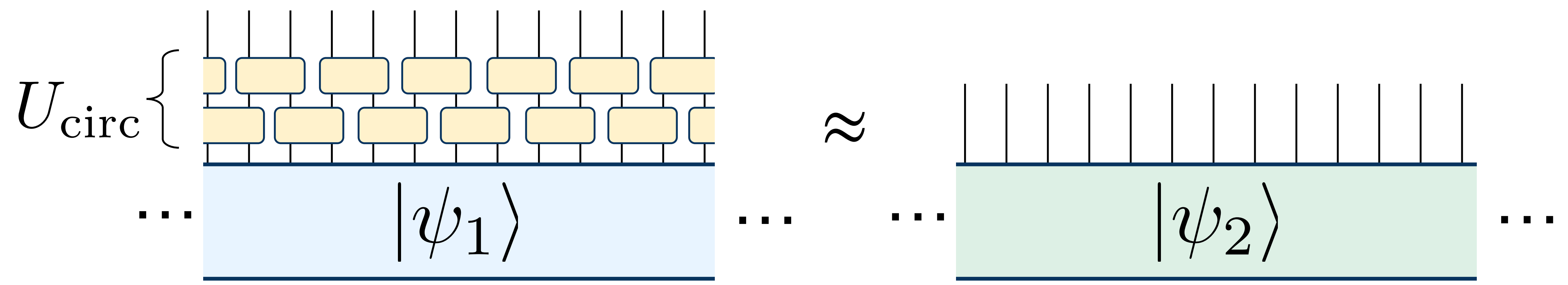


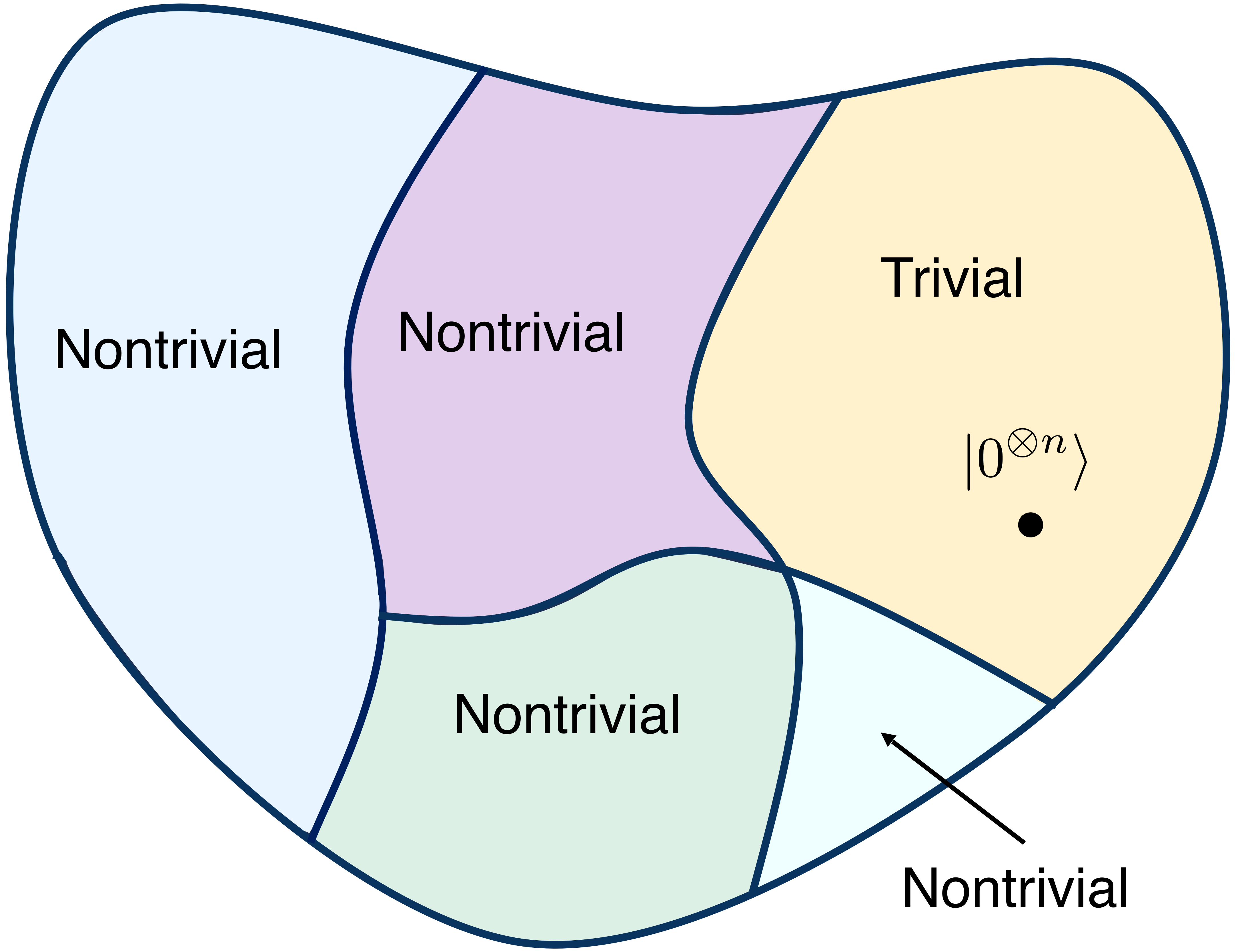
Pure-state topological phases

Equivalence relation:

$|\psi_1\rangle \sim |\psi_2\rangle$ if there exists a quasi-local circuit

$U = \mathcal{T}\left[e^{-i\int_0^1 dt H(t)}\right]$ such that $U|\psi_1\rangle = |\psi_2\rangle$





Mixed quantum states

Mixed states arise from the coupling between the system and the environment

- Gibbs states $\rho \propto e^{-\beta H}$ at finite temperature
- Quantum states under decoherence
- Steady states of dissipative (e.g. Lindbladian) evolution

Mixed quantum states

A quantum state is a hermitian, positive semi-definite operator ρ with $\operatorname{Tr}\rho = 1$

Pure state: $\rho = |\psi\rangle\langle\psi|$

Mixed state: $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, an ensemble of pure states

Any mixed state ρ has a purification, i.e. is a subsystem of a pure state

Any convex sum of density operators is still a density operator

$$\rho = \sum_i \lambda_i \rho_i, \sum_i \lambda_i = 1, \lambda_i \geq 0$$

Quasi-local quantum channels

- Quantum channels map mixed states to mixed states $\rho \xrightarrow{\mathcal{N}} \mathcal{N}(\rho)$

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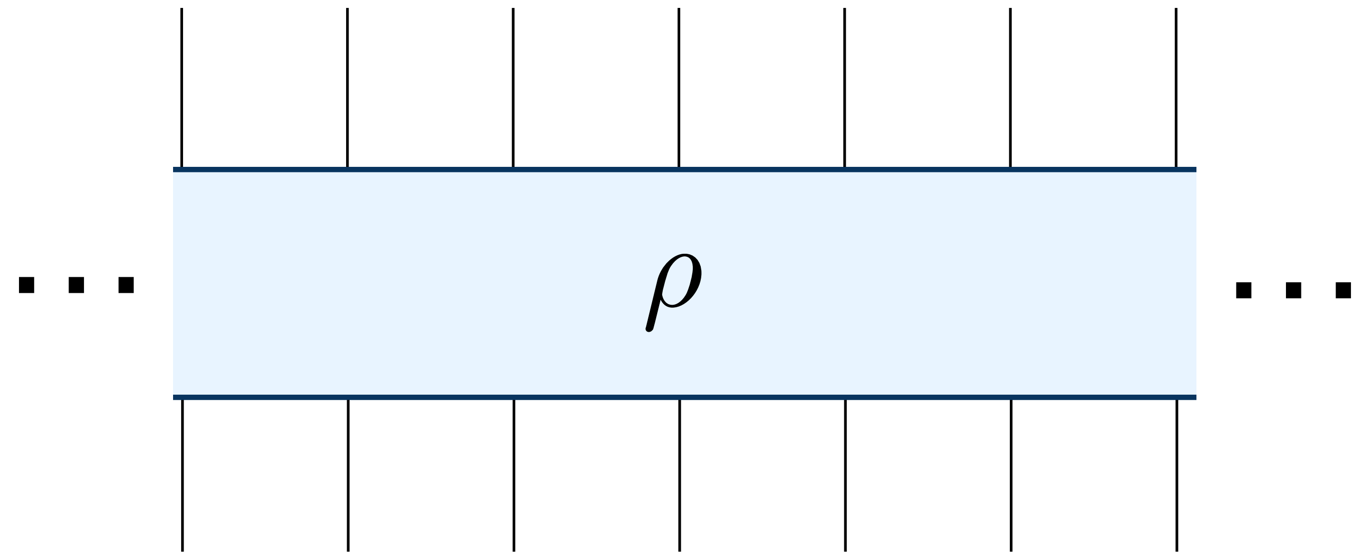
- Any quasi-local quantum channel is composed of three steps:



Quasi-local quantum channels

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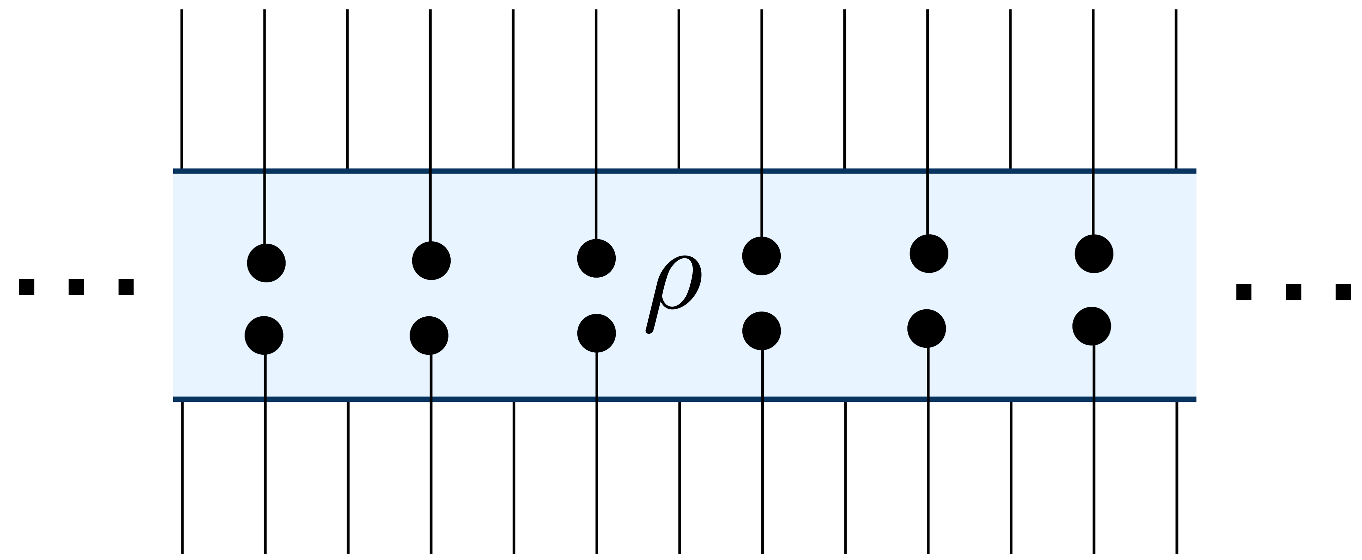
  1. Add ancilla



$\rho \otimes (|0^{\otimes n}\rangle\langle 0^{\otimes n}|)_A$

Quasi-local quantum channels

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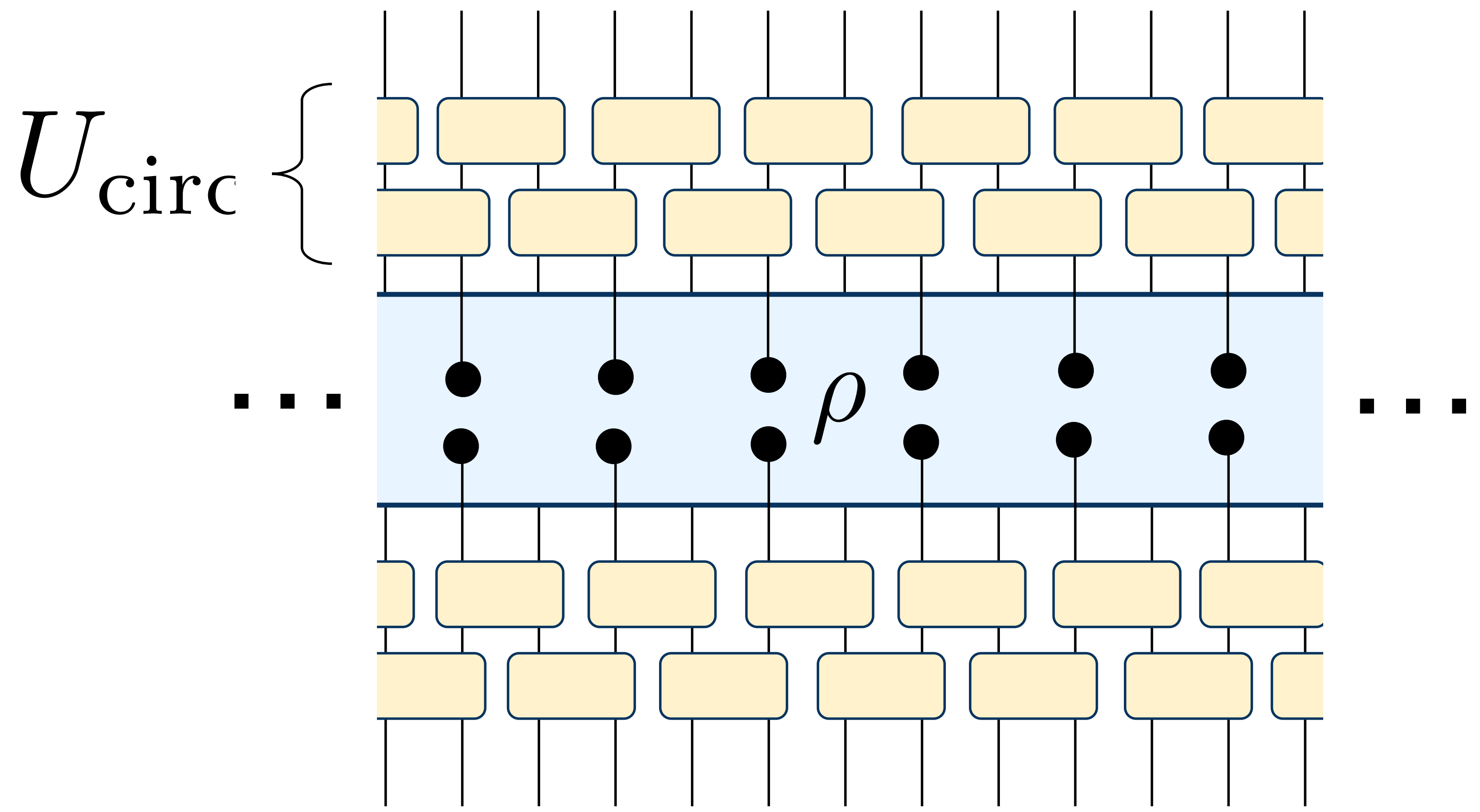
  1. Add ancilla

  2. Apply a quasi-local circuit

     $U = \mathcal{T}\left[e^{-i\int_0^1 dt H(t)}\right]$



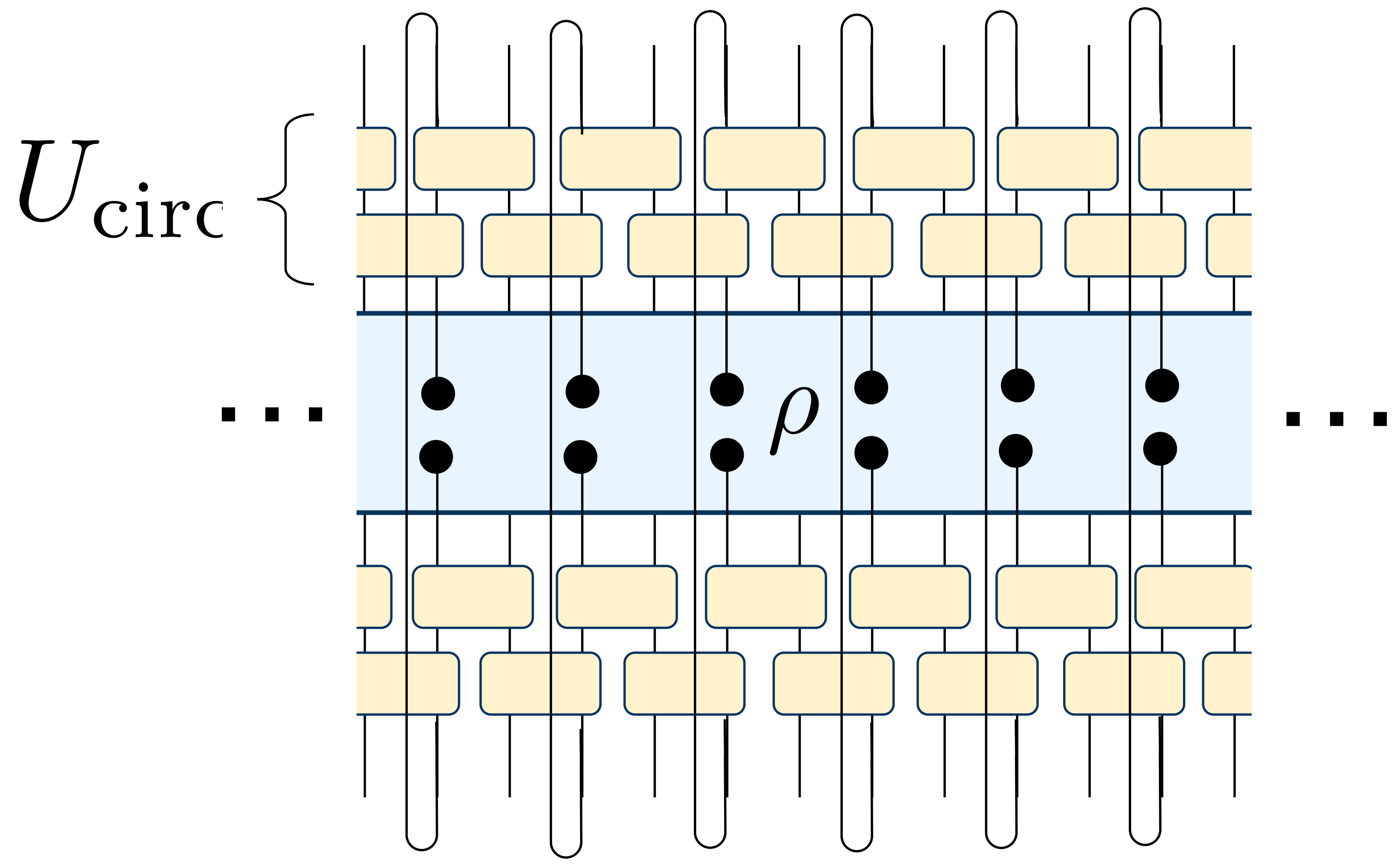
$U\rho \otimes (|0^{\otimes n}\rangle\langle 0^{\otimes n}|)_A U^\dagger$

Quasi-local quantum channels

- Quantum channels map mixed states to mixed states $\rho \xrightarrow{\mathcal{N}} \mathcal{N}(\rho)$

- Any quasi-local quantum channel is composed of three steps:

  1. Add ancilla
  2. Apply a quasi-local unitary
     $$U = \mathcal{T}\left[e^{-i\int_0^1 dt H(t)}\right]$$
  3. Trace out ancilla



$$\mathcal{N}(\rho) = \mathrm{Tr}_A\left[U\rho \otimes (|0^{\otimes n}\rangle\langle 0^{\otimes n}|)_A U^\dagger\right]$$

Decohering a quantum state

Consider a system of N qubits, with Pauli operators X_i, Z_i

Example of a local quantum channel

$$\mathcal{E}_X = \prod_{i=1}^{N} \mathcal{E}_i^X, \quad \mathcal{E}_i^X(\rho) = (1-p)\rho + pX_i\rho X_i$$

$$\mathcal{E}_X(\rho) = (1-p)^N\rho + p(1-p)^{N-1}\sum_i X_i\rho X_i + p^2(1-p)^{N-2}\sum_{i<j} X_iX_j\rho X_jX_i + \cdots$$

$\mathcal{E}_X(\rho)$ **is the convex sum of local unitary deformations of** ρ

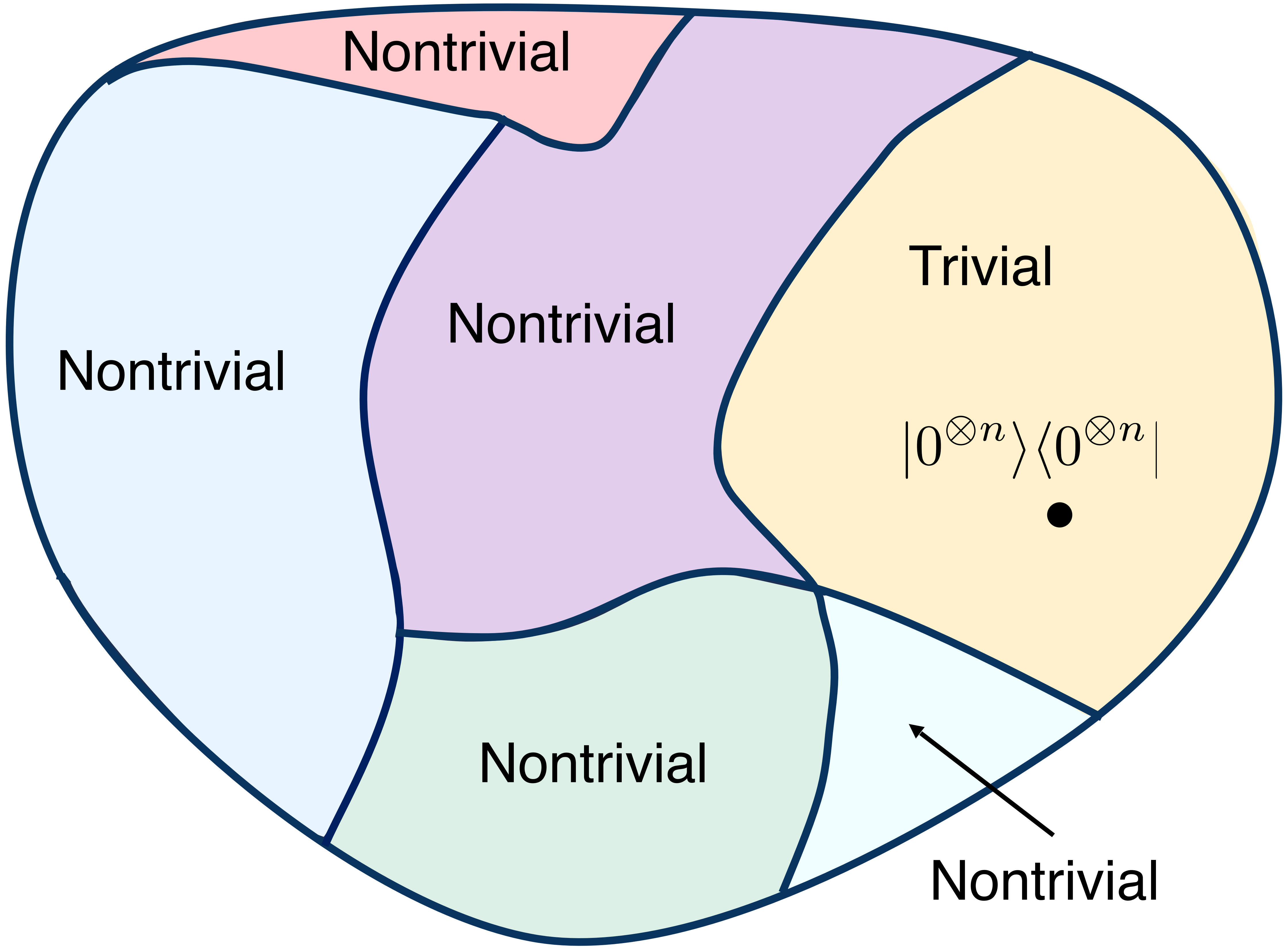
Mixed-state quantum phases

<u>Equivalence relation:</u>

$\rho_1 \sim \rho_2$ if there exists quasi-local quantum channels $\mathcal{N}_{12}$ and $\mathcal{N}_{21}$ such that

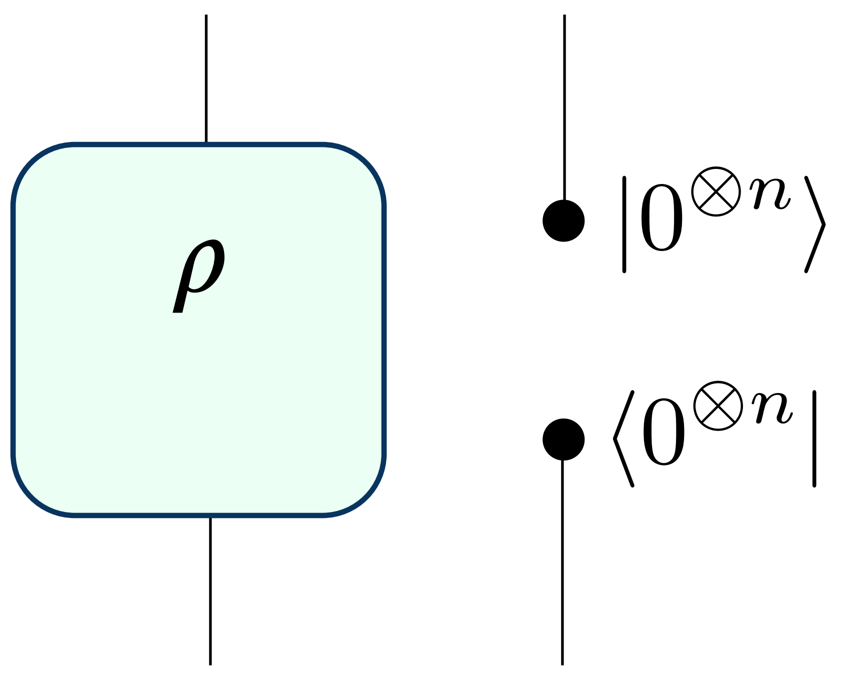
$$\mathcal{N}_{21}(\rho_1) = \rho_2\,, \quad \mathcal{N}_{12}(\rho_2) = \rho_1$$

Two-way connectivity by quasi-local quantum channels!



Two-way connectivity

- Two-way connectivity is essential
- Every mixed state ρ can be mapped to $|0^{\otimes n}\rangle\langle 0^{\otimes n}|$ by local channel



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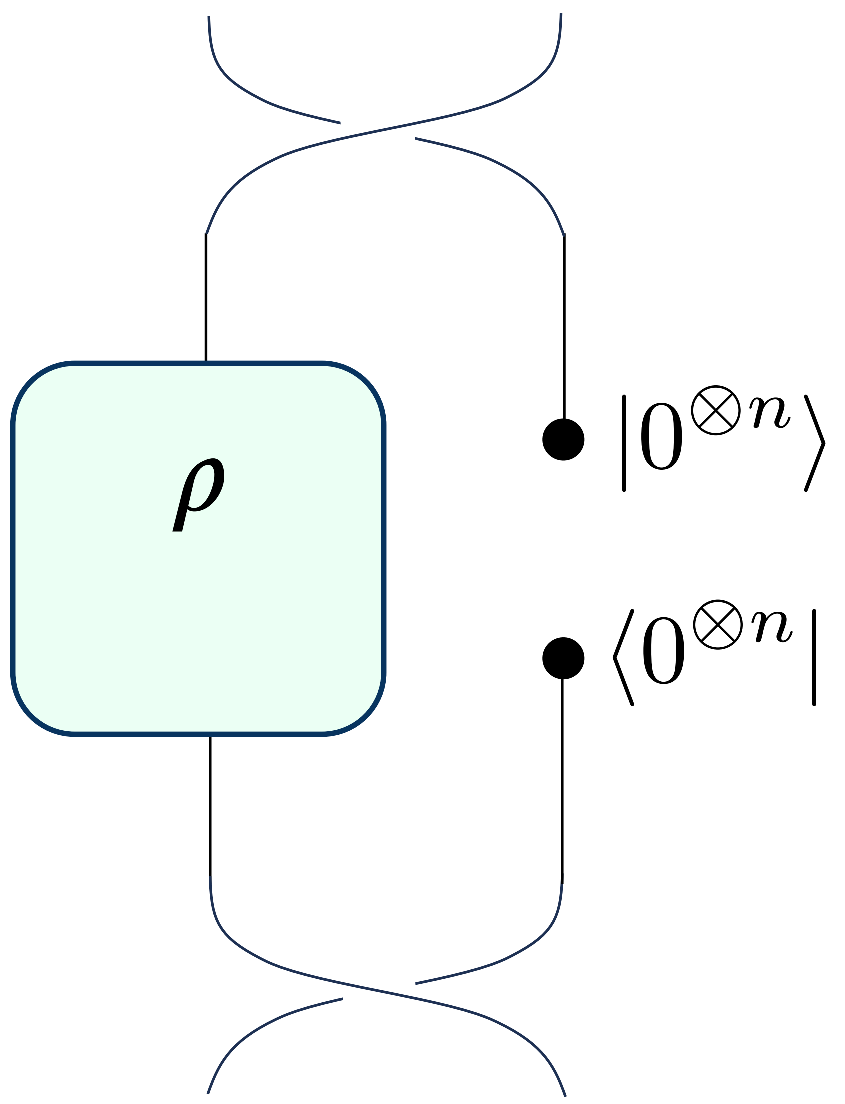

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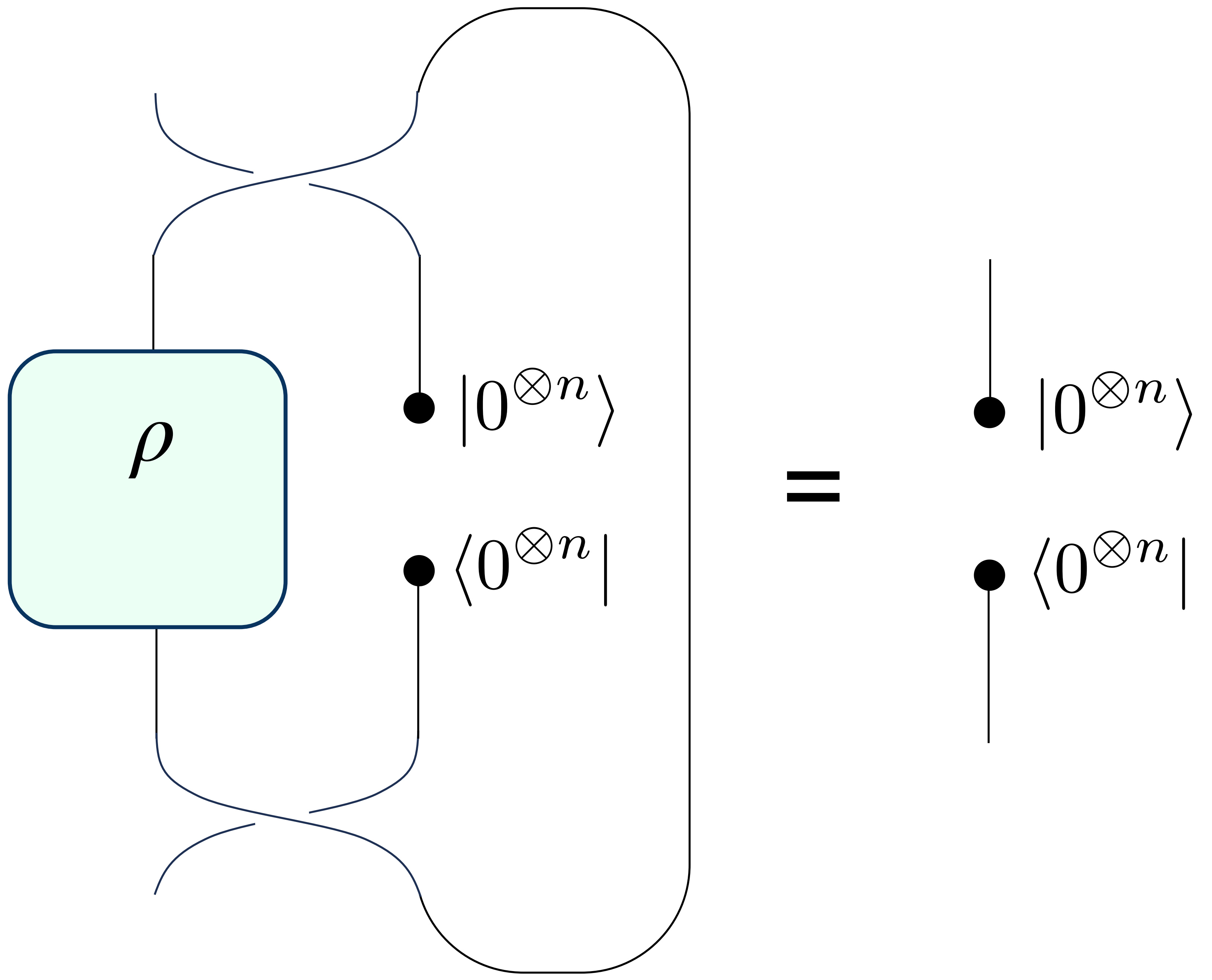


Symmetries of mixed states

Weak symmetry:

$$W\rho W^\dagger = \rho\,, \qquad \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|\,, \qquad W|\psi_i\rangle = e^{i\phi_i}|\psi_i\rangle$$

Example: grand canonical ensemble (system + bath symmetric)

Strong symmetry:

$$W\rho = e^{i\phi}\rho\,, \qquad \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|\,, \qquad W|\psi_i\rangle = e^{i\phi}|\psi_i\rangle$$

Independent of i

Example: canonical ensemble (symmetry preserved on the system)

Strong symmetry is strong

- It knows about the projective phase factors, i.e. anomalies
- If a convex sum of density matrices has strong symmetry, so is each one of them.
- Strong symmetry can be “commuted” through a quantum channel.

The (un-generalized) Landau paradigm

Phases of matter are classified by
patterns of spontaneous symmetry breaking

Classical Ginzburg-Landau:

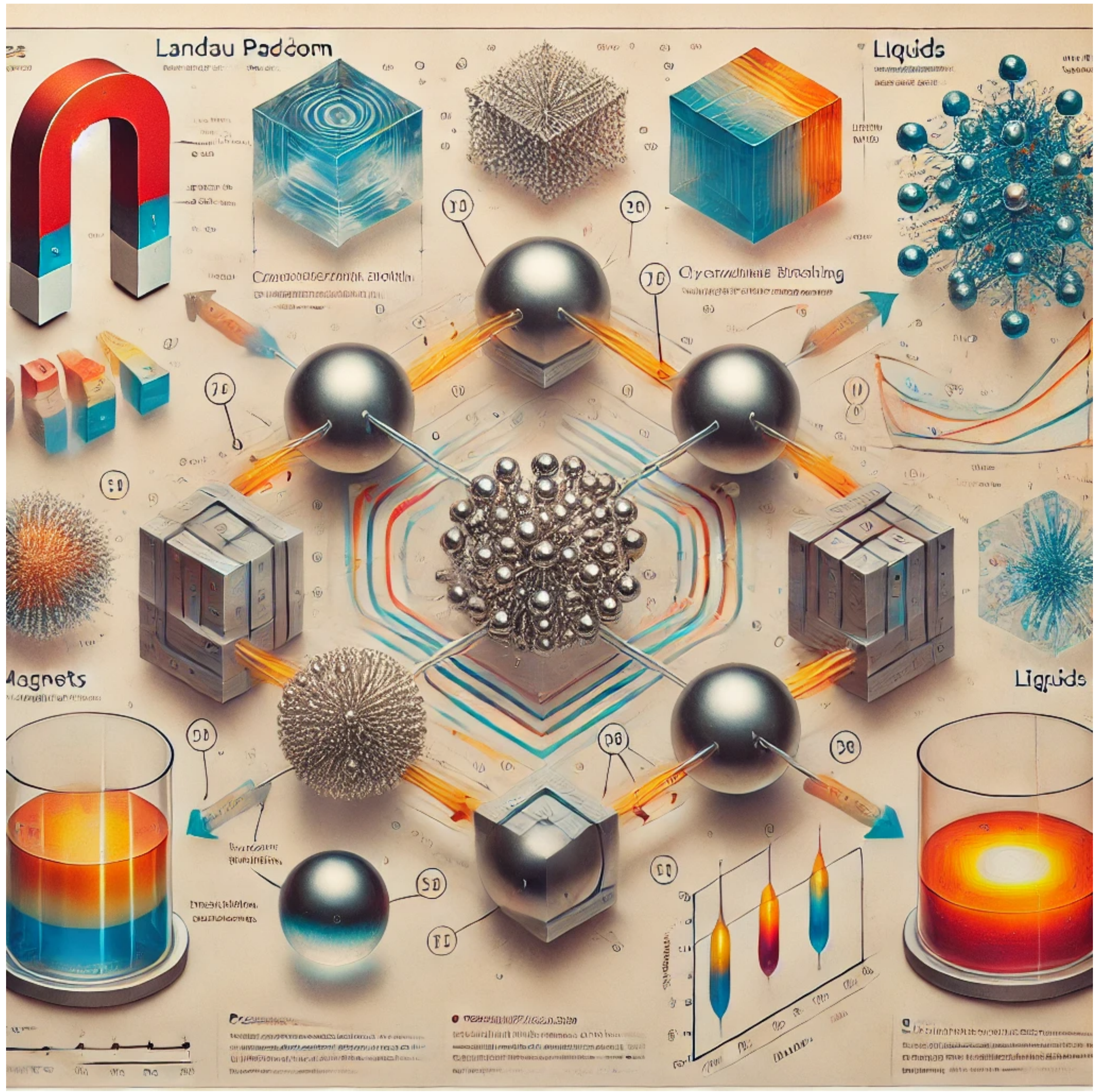
Weak symmetry → nothing (or a subgroup)

Quantum Ginzburg-Landau:

Strong symmetry → nothing (or a subgroup)

A new possibility in generic mixed states:

Strong symmetry → Weak (SW SSB)



Illustrated by ChatGPT 4o

Pure-state SSB

A system of qubits, $\mathbb{Z}_2$ symmetry $X = \prod_i X_i$

GHZ state: $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|\uparrow \cdots \uparrow\rangle + |\downarrow \cdots \downarrow\rangle)$

In the X eigenbasis $|\psi_+\rangle \propto \sum_{\prod_i x_i = 1} |\{x_i\}\rangle$

Long-range order: $\langle \psi | Z_i Z_j | \psi \rangle = 1$

Coherent "condensation" of symmetry charges

Strong-to-weak SSB

$$\rho \propto 1 + X \propto \sum_{\prod_i x_i = 1} |\{x_i\}\rangle\langle\{x_i\}|$$

"Incoherent" condensation of symmetry charges

ρ spontaneously breaks the strong X to a weak X

Only short-range correlation for any local operator $\mathrm{Tr}\,(Z_i Z_j \rho) = 0$

Definition of LRO: ρ and $Z_i Z_j \rho Z_j Z_i$ are very "similar" states

Lee et al, 2301.05238; Ma et al 2305.16399v2

Lee et al, 2301.05238; Ma et al 2305.16399v2

Defining SW SSB: fidelity correlator

Fidelity between two states ρ and σ: $F(\rho, \sigma) = \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$

Definition: ρ with strong symmetry G has SW-SSB if there exists a local order parameter $O(x)$ such that

$$\lim_{|x-y|\to\infty} F\left(\rho, O(x)O^\dagger(y)\rho O(y)O^\dagger(x)\right) \to O(1)$$

Stability theorem: If ρ has SW SSB, and $\mathscr{E}$ is a symmetric finite-depth channel, then $\mathscr{E}[\rho]$ also has SW SSB.

The theorem establishes SW SSB as a robust indicator of mixed-state phase

SW SSB in thermal states

Canonical ensemble ⟷ **Local equivalence** ⟷ Grand Canonical ensemble

$$\mathrm{Tr}\left(\rho_C M_k\right) = \mathrm{Tr}\left(\rho_G M_k\right)$$

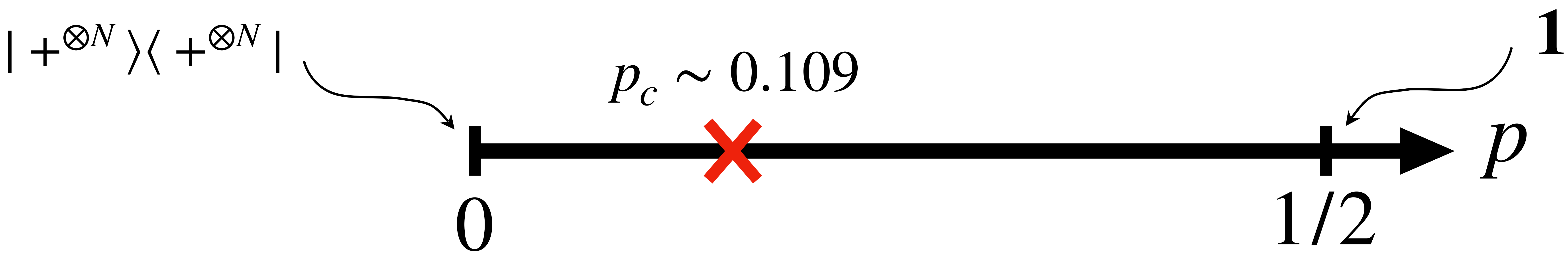
ρ_C : Strongly symmetric	ρ_G : Weakly symmetric

Conjecture: a canonical ensemble of a local Hamiltonian H spontaneously breaks any (0-form) strong symmetry

Decohered quantum Ising model

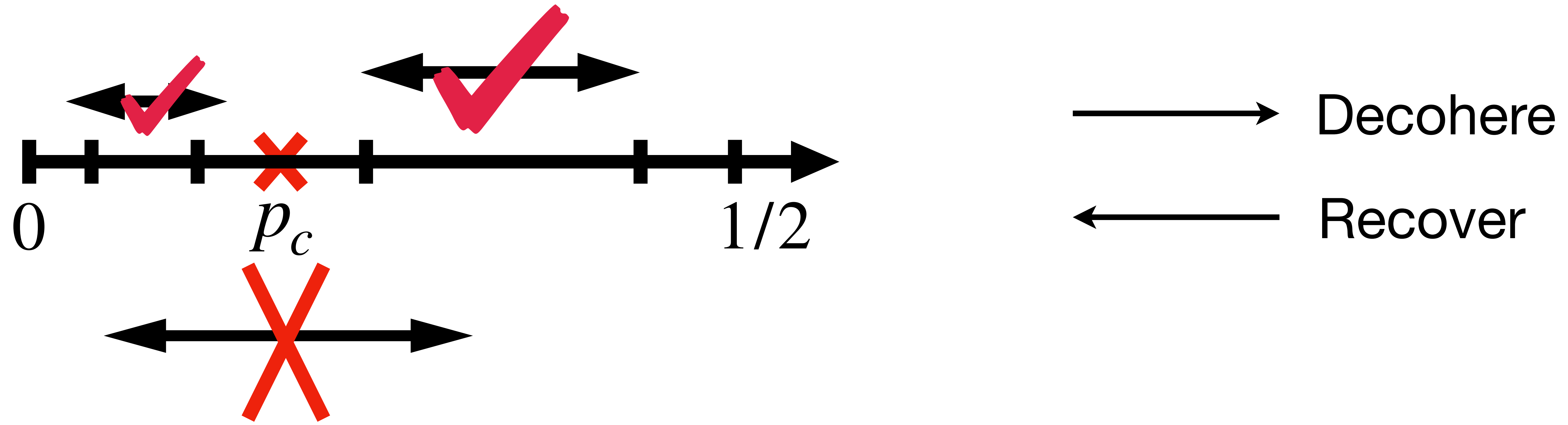
2D Ising model $H = -\sum_j X_j$, with strong $\mathbb{Z}_2$ symmetry $X = \prod_j X_j$

$$\mathcal{E}_{ZZ} = \prod_{\langle i,j \rangle} \mathcal{E}_{ij}^{ZZ}, \;\; \mathcal{E}_{ij}^{ZZ}[\rho] = (1-p)\rho + pZ_iZ_j\rho Z_iZ_j$$



$$F_Z(x,y) = F(\rho, Z_xZ_y\rho Z_yZ_x) \sim \begin{cases} e^{-|x-y|/\xi} & p < p_c \\ O(1) & p > p_c \end{cases}$$

Distinguishing the two phases



NO channel to recover $p < p_c$ from $p > p_c$: stability theorem!

Strongly symmetric recovery channel inside $p < p_c$ and $p > p_c$

Explicit construction using Petz map

Sketch: perform Petz recovery locally, and check no global issue (it's subtle)

SW SSB phase transition

“Observables” (like fidelity correlator) governed by the 2D random-bond Ising model along the Nishimori line

The same “critical theory” in certain deformed models, e.g.

$$\mathscr{E}_{ij}^{ZZ}(\rho) = (1-p)\rho + p e^{i\theta Z_i Z_j} \rho e^{-i\theta Z_i Z_j}$$

Hint at universality? Field theory?…

The postmodern Landau paradigm

Many generalizations of symmetry: higher-form, non-invertible, subsystem, …

See Shu-Heng and Xie's lectures

Much more to symmetry than the group structure: "higher" structure in defects

Global symmetry is associated with **'t Hooft anomaly**

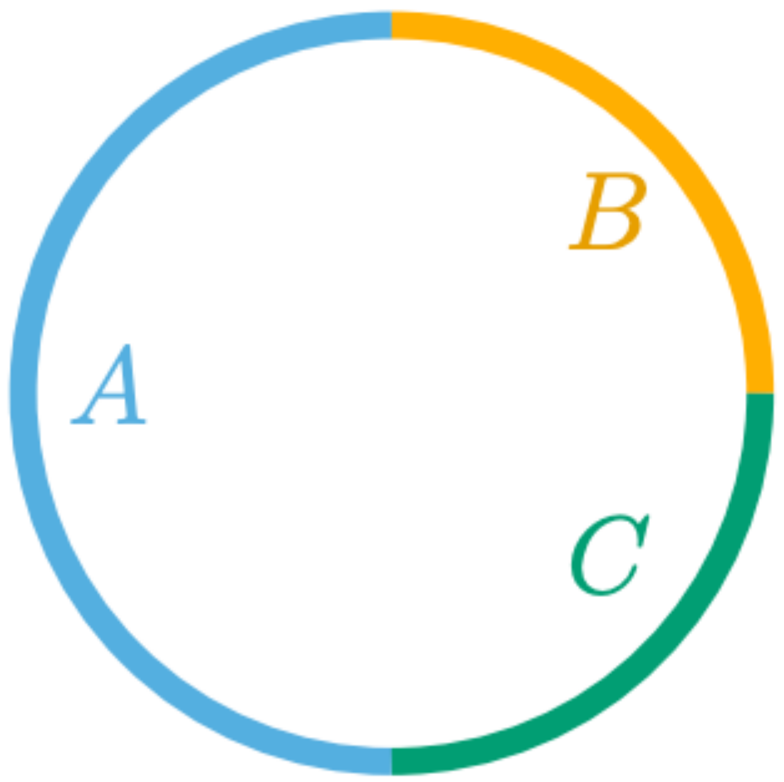
~ Obstruction to having a "trivial" symmetric state

SW SSB of anomalous symmetry leads to interesting mixed state phases

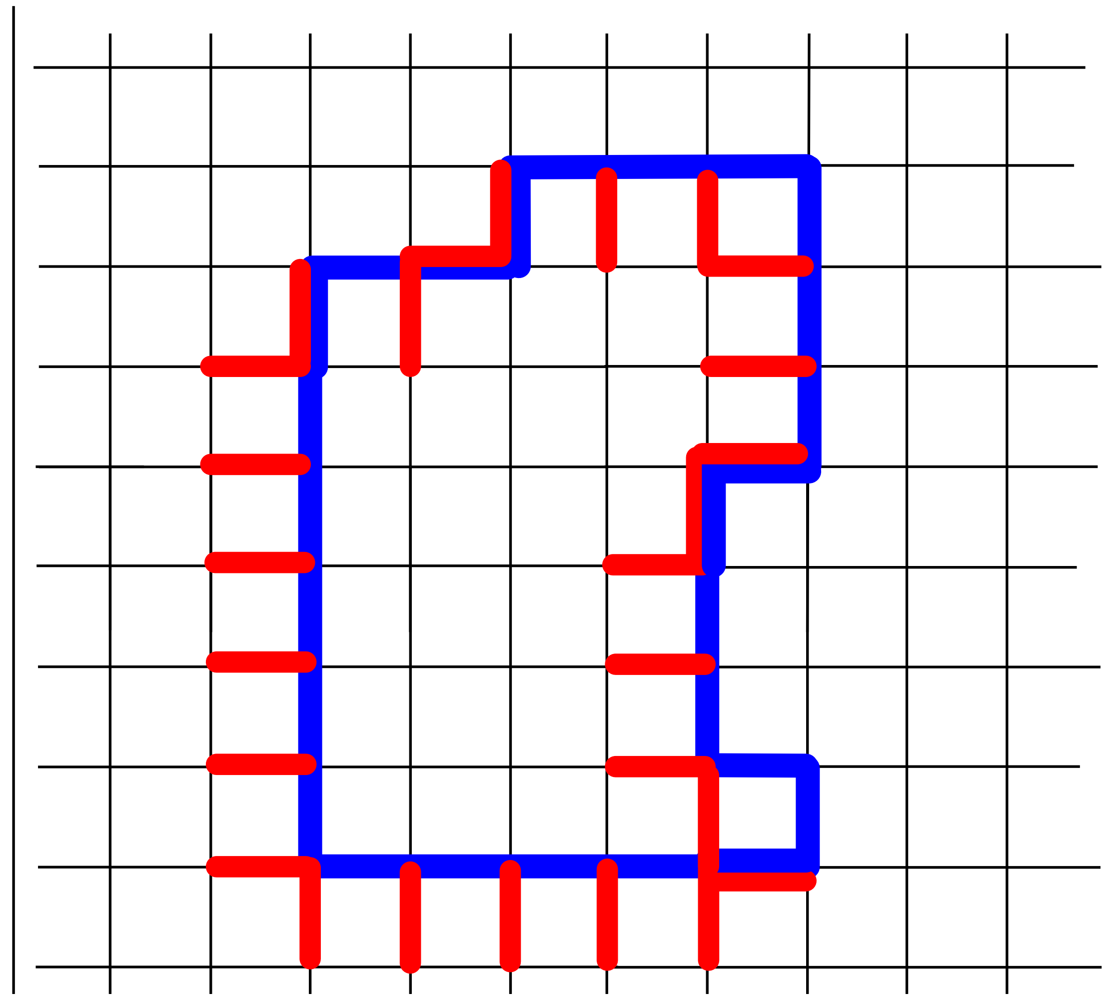
Two examples of “intrinsically” mixed-state phases

Both from SW SSB of anomalous symmetry

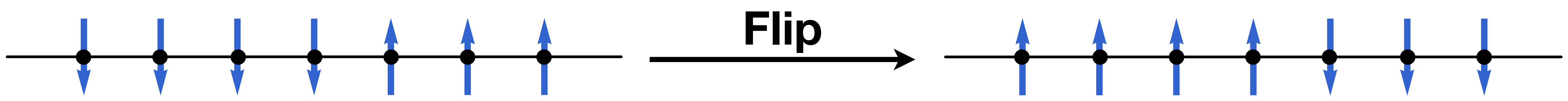
Anomalous $\mathbb{Z}_2$ symmetry in 1D



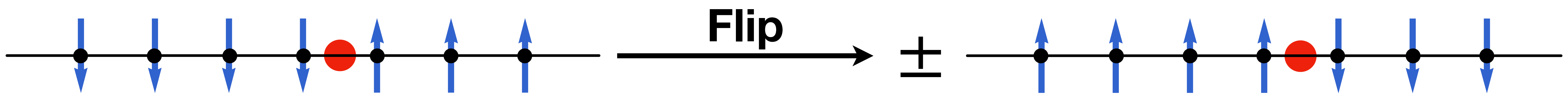
Anomalous $\mathbb{Z}_2^{(1)}$ 1-form symmetry in 2D



Anomalous $\mathbb{Z}_2$ symmetry in 1D



Non-anomalous $\mathbb{Z}_2$ symmetry: $X = \prod_n X_n$



Anomalous $\mathbb{Z}_2$ symmetry: $U = \prod_n X_n \prod_n \mathrm{CZ}_{n,n+1}$

SW SSB of the anomalous $\mathbb{Z}_2$ symmetry

"Fixed-point" SW SSB state: $\rho_+ \propto 1 + U$

Completely featureless, e.g. trivial correlation functions of local op's

But ρ_+ is inequivalent to a trivial state!

Proof sketch: if $\rho_+ = \mathcal{N}(|\psi\rangle\langle\psi|)$, then $\tilde{U}|\psi\rangle = |\psi\rangle$

$\tilde{U}$ is also an anomalous $\mathbb{Z}_2$ symmetry.

So $|\psi\rangle$ can not be trivial (e.g. could be SSB or CFT state)

In fact, it can be made even stronger:

ρ_+ is inequivalent to any pure state, "intrinsically mixed"

Multipartite Separability

Bipartite (A and B) separable: $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i$

Tripartite (A, B and C) separable: $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i \otimes \rho_C^i$

Theorem: ρ_+ is bipartite separable, but not tripartite separable!

$$\rho_+ \sim \sum \mathrm{GHZ}_A \otimes \mathrm{GHZ}_B$$

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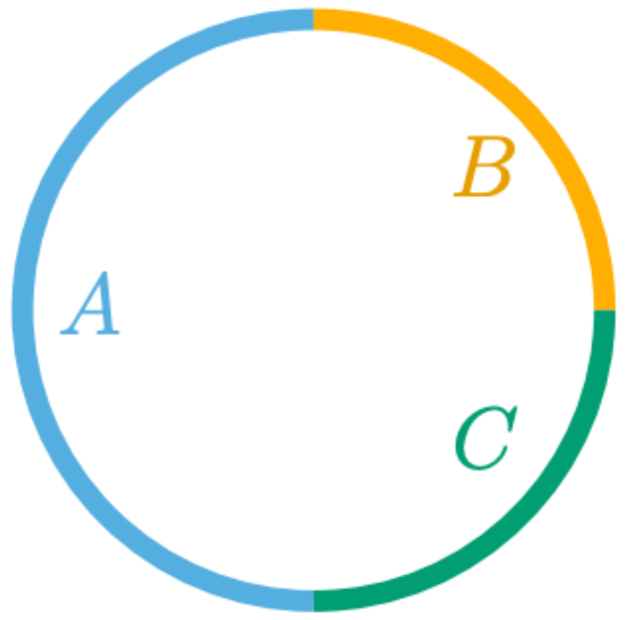


Theorem: ρ_+ can not be prepared from a tri-separable state via local channel

Topological order in (2+1)d

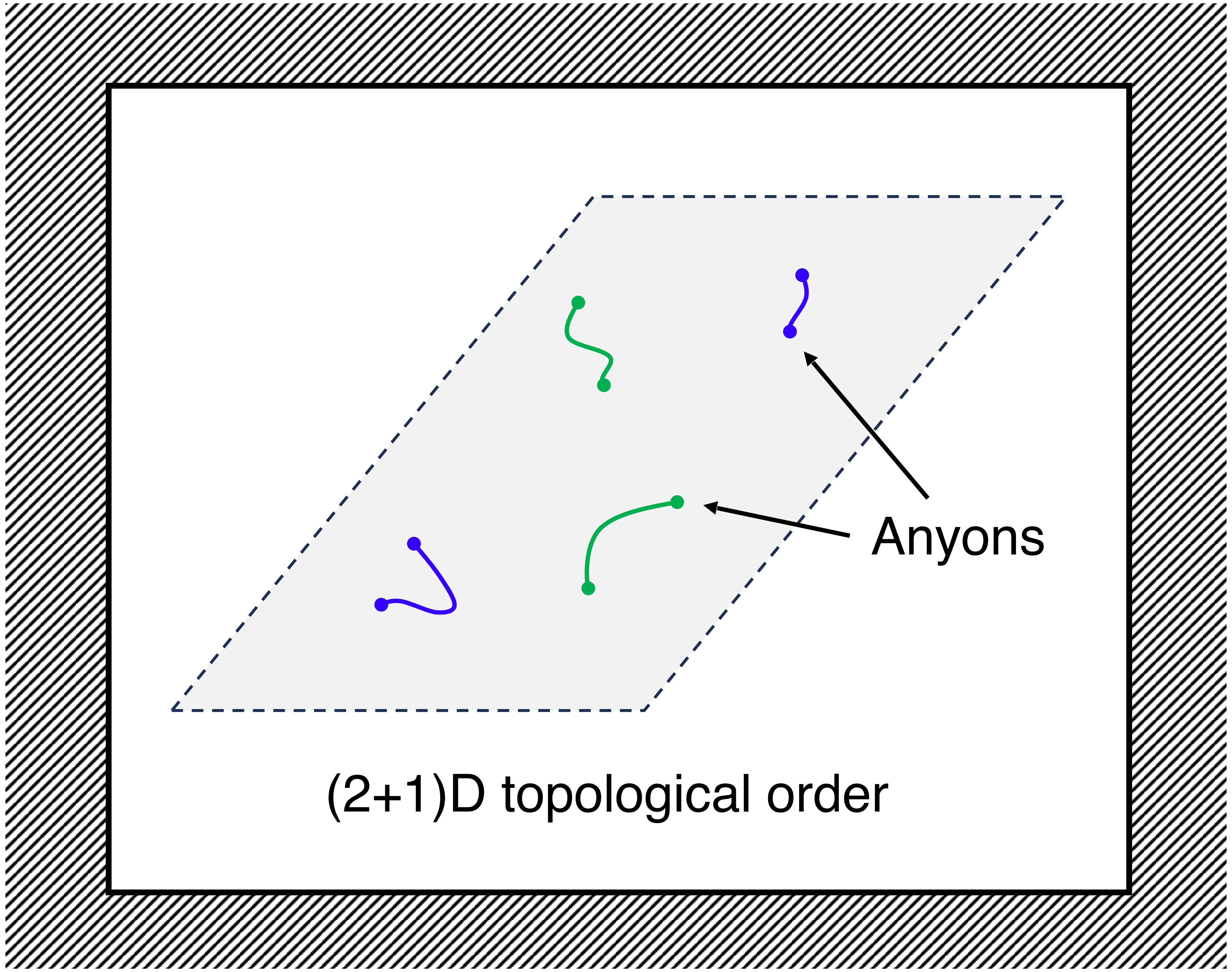
<u>Classification (bosonic):</u>

- **Modular** tensor categories ~ anyon theory
  (up to stacking with E_8 states)
- Captures universal properties of quasiparticle excitations (anyons)

(2+1)D topological orders are widely believed to be completely classified

<u>Symmetry perspective:</u>

- (Spontaneously broken) generalized 1-form symmetries

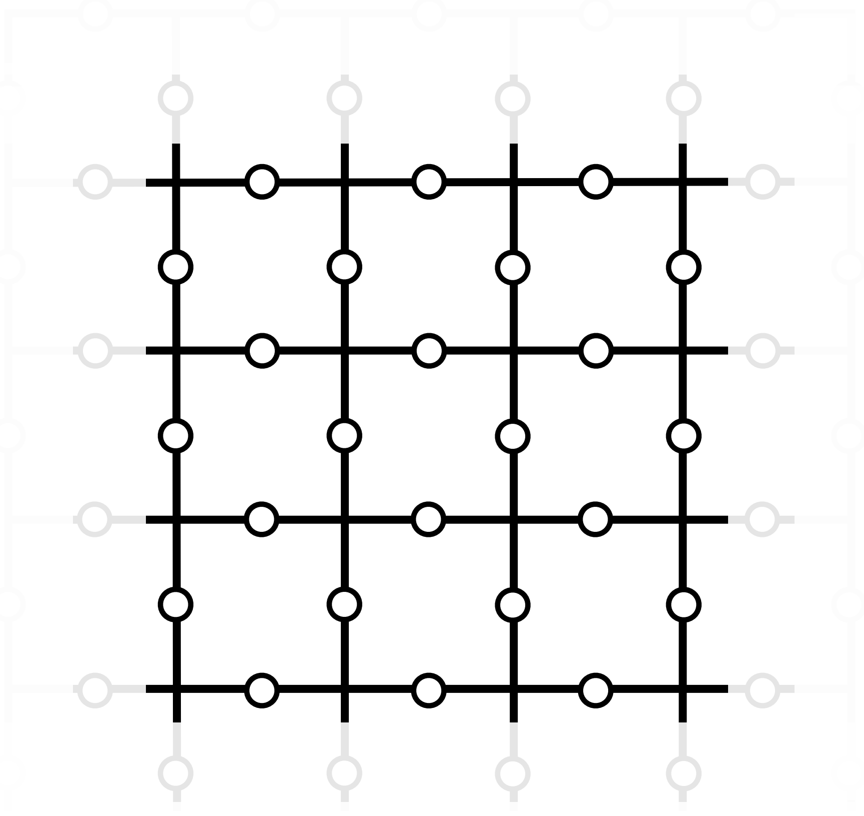


Toric code

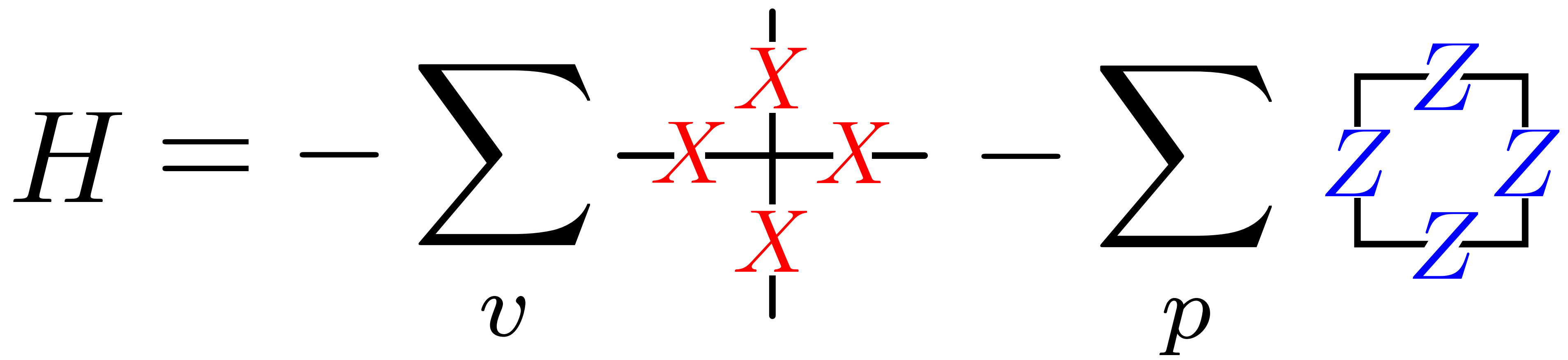
Hilbert space and Hamiltonian:

$X^2 = Z^2 = 1$

$ZX = -XZ$

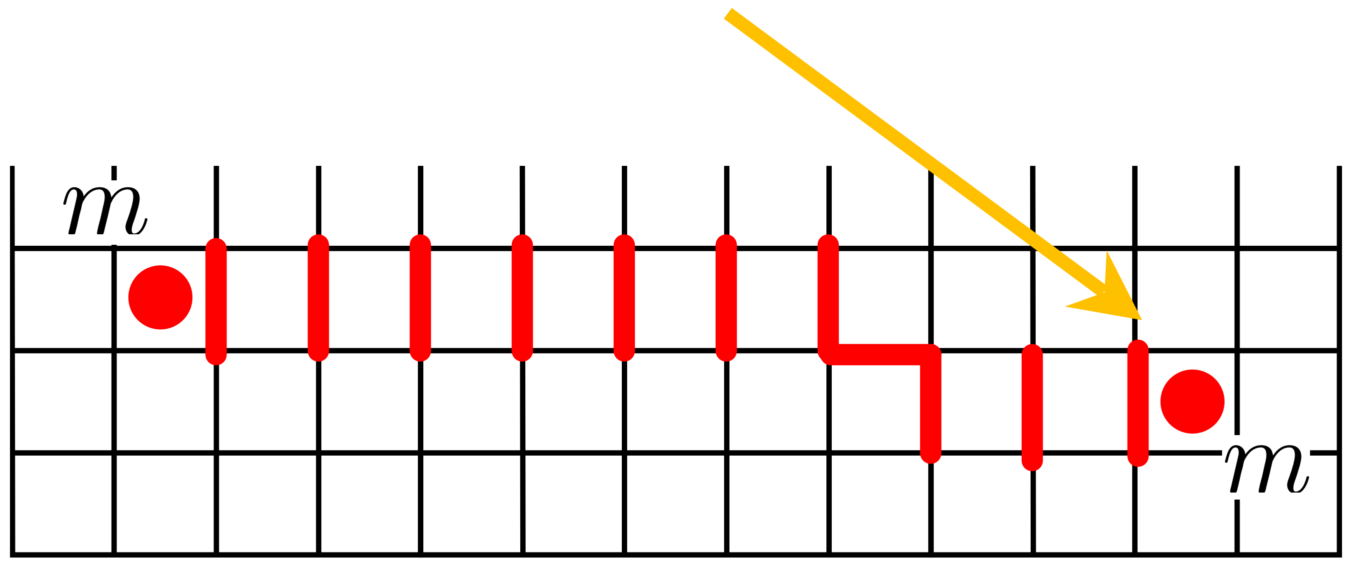


$\circ = \mathbb{C}^2$



Anyons:

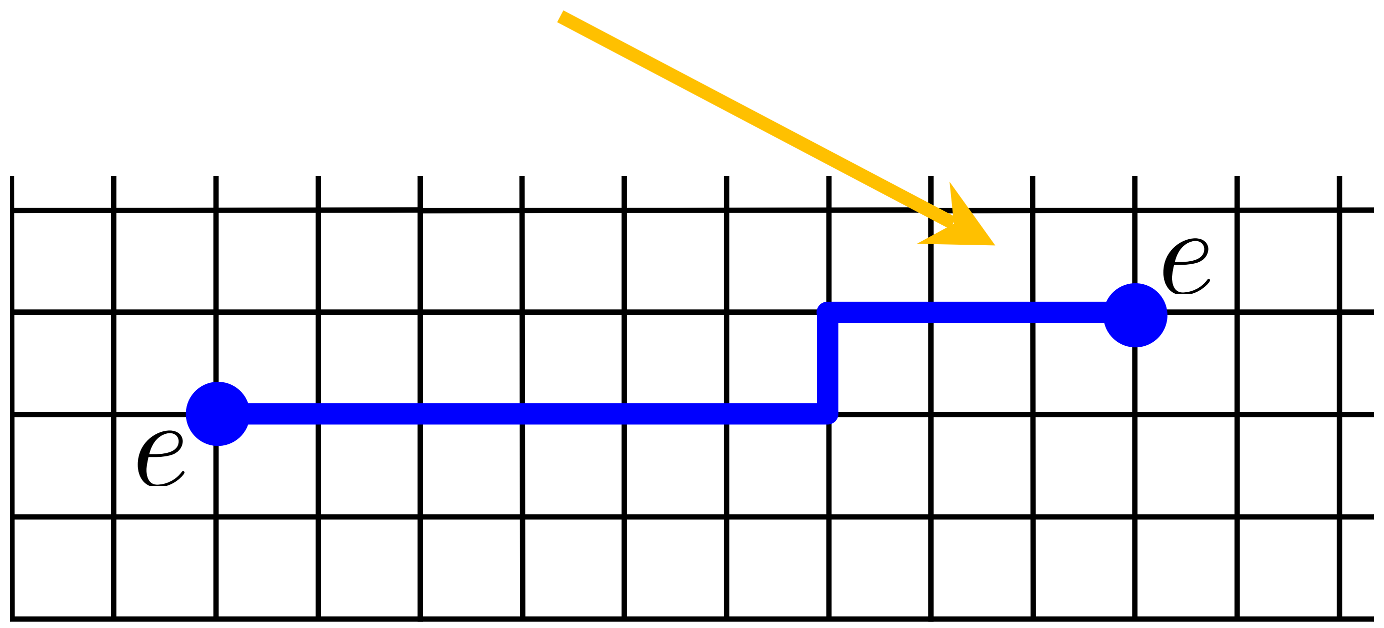
Violates plaquette term



= X

(π flux)

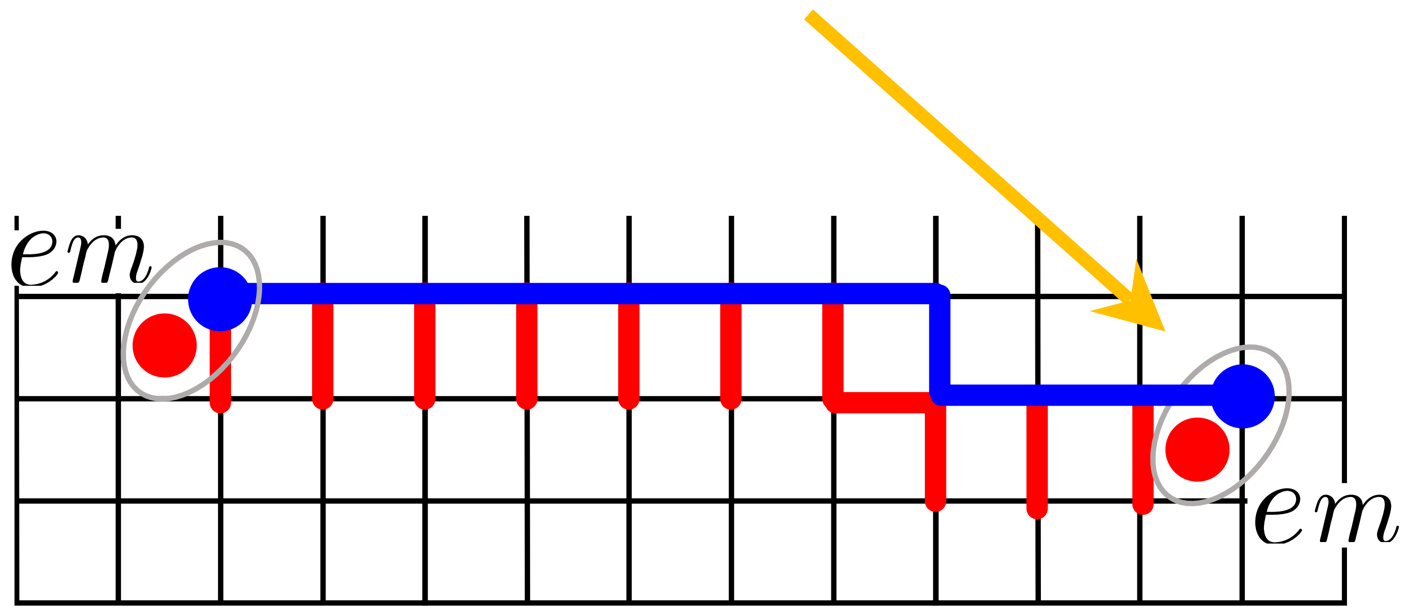
Violates vertex term



= Z

(Gauge charge)

Violates both vertex and plaquette term



= Z = X

(Dyon)

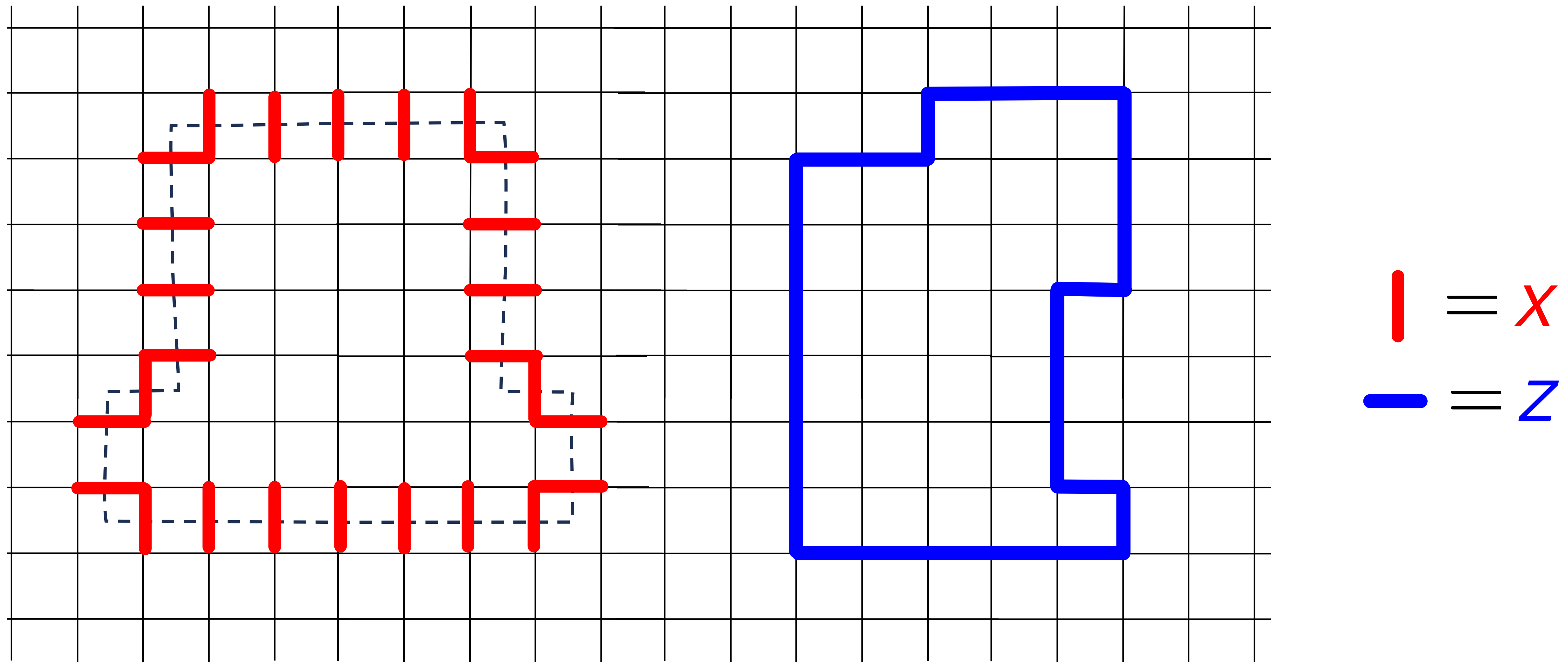
Toric code 1-form symmetries

$\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry

(Not topological operators)

$$H = -\sum_v \prod X - \sum_p \prod Z$$

Generators of 1-form symmetry



X-decohered toric code

Bit-flip channel:

$$\mathcal{N}^X_{e,p}(\rho) = (1-p)\rho + pX_e\rho X_e$$

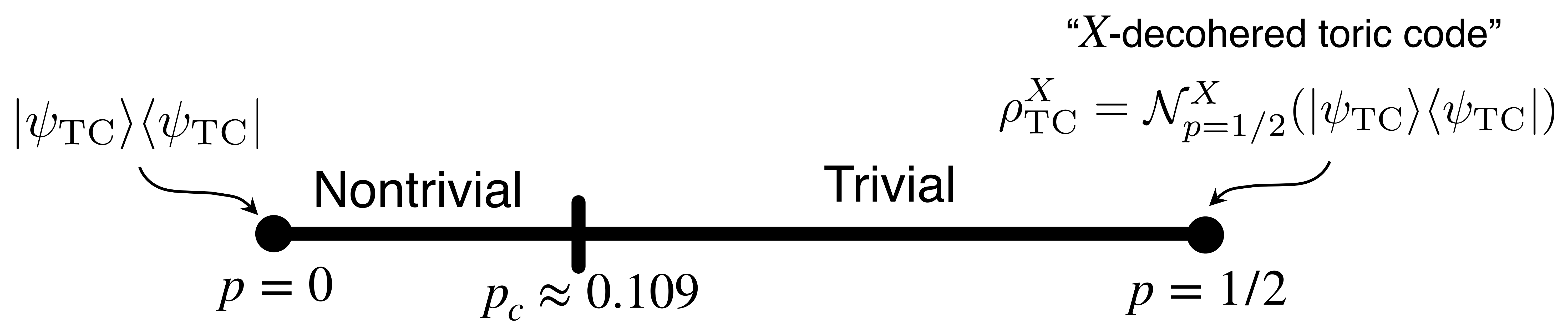
Global bit-flip channel:

$$\mathcal{N}^X_p(\rho) = \bigotimes_e \mathcal{N}^X_{e,p}(\rho)$$

Maximum strength:

$$\mathcal{N}^X_{p=1/2}(\rho) = \sum_{\mathbf{e}\subset E}\left(\prod_{e\in\mathbf{e}} X_e\right)\rho\left(\prod_{e\in\mathbf{e}} X_e\right)$$

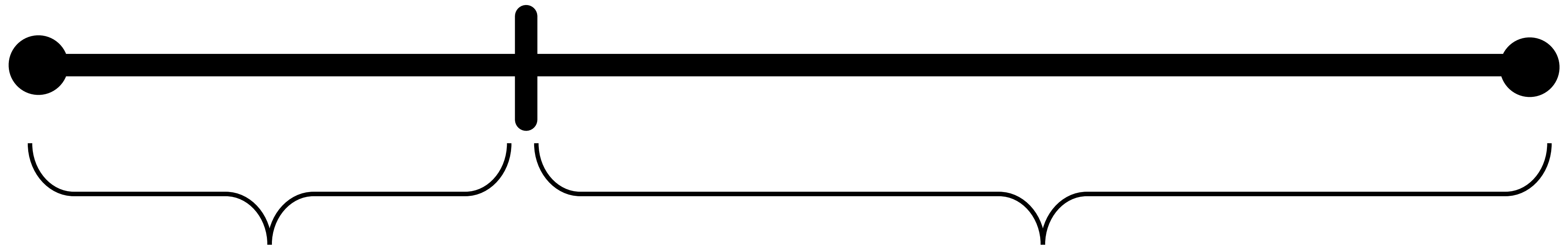
<u>X-decohered toric code phase diagram:</u>



X-decohered toric code

Explicitly breaking <u>strong</u> e 1-form symmetry →

$p = 0$ $\quad p_c \approx 0.109$ $\quad p = 1/2$



Emergent $\mathbb{Z}_2 \times \mathbb{Z}_2$
1-form symmetry
(anomalous)

SSB

Emergent $\mathbb{Z}_2$
m 1-form symmetry
(anomaly free)

ŚW SSB

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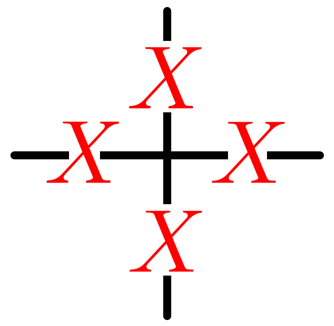


X-decohered toric code

$$\rho_{\rm TC}^X = \mathcal{N}_{p=1/2}^X(|\psi_{\rm TC}\rangle\langle\psi_{\rm TC}|)$$

Coherent proliferation m anyons

$$|\psi_{\rm TC}\rangle \longrightarrow |\psi_{\rm TC}^X\rangle \propto \sum_{\mathbf{m}} |\mathbf{m}\rangle$$

Satisfies: $X_e|\psi_{\rm TC}^X\rangle = |\psi_{\rm TC}^X\rangle$

Pair creates m anyons

Incoherent proliferation m anyons

$$|\psi_{\rm TC}\rangle\langle\psi_{\rm TC}| \longrightarrow \rho_{\rm TC}^X \propto \sum_{\mathbf{m}} |\mathbf{m}\rangle\langle\mathbf{m}|$$

Satisfies: $X_e \rho_{\rm TC}^X X_e = \rho_{\rm TC}^X$

Fermion-decohered toric code

<u>Coherent proliferation of *em* anyons</u>

$$|\psi_{\mathrm{TC}}\rangle \not\to |\psi_{\mathrm{TC}}^{em}\rangle \propto \sum_{\mathbf{em}} |\mathbf{em}\rangle$$

Want to satisfy: $S_e|\psi_{\mathrm{TC}}^{em}\rangle = |\psi_{\mathrm{TC}}^{em}\rangle$

Pair creates *em* anyons

Exchange statistics are an obstruction!
('t Hooft anomaly)

<u>Incoherent proliferation *em* anyons</u>

$$|\psi_{\mathrm{TC}}\rangle\langle\psi_{\mathrm{TC}}| \longrightarrow \rho_{\mathrm{TC}}^{em} \propto \sum_{\mathbf{em}} |\mathbf{em}\rangle\langle\mathbf{em}|$$

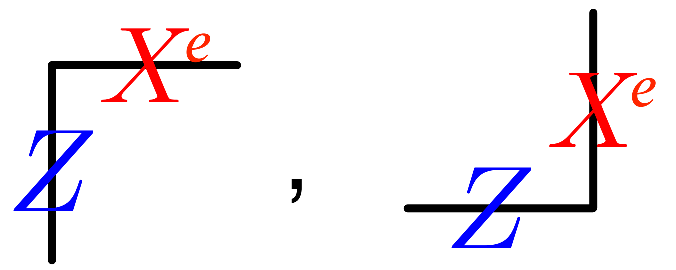
Satisfies: $S_e \rho_{\mathrm{TC}}^{em} S_e = \rho_{\mathrm{TC}}^{em}$

No obstruction to incoherent proliferation!

Fermion-decohered toric code

Fermion hopping operators:

$S_e =$



Fermion-noise channel:

$$\mathcal{N}^{em}_{e,p}(\rho) = (1-p)\rho + pS_e\rho S_e$$

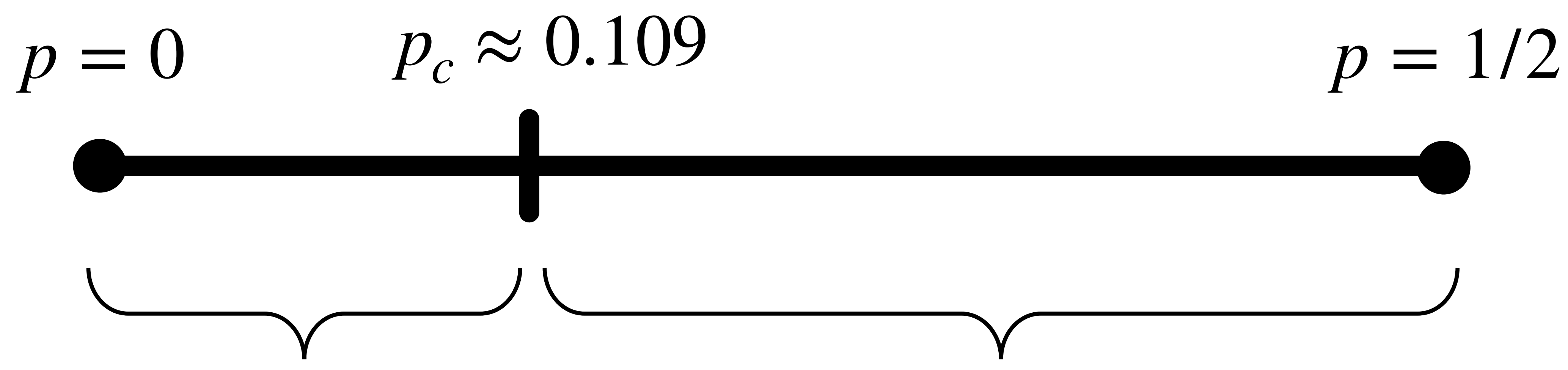
Global fermion-noise channel:

$$\mathcal{N}^{em}_{p}(\rho) = \bigotimes_e \mathcal{N}^{em}_{e,p}(\rho)$$

Fermion-decohered toric code phase diagram:

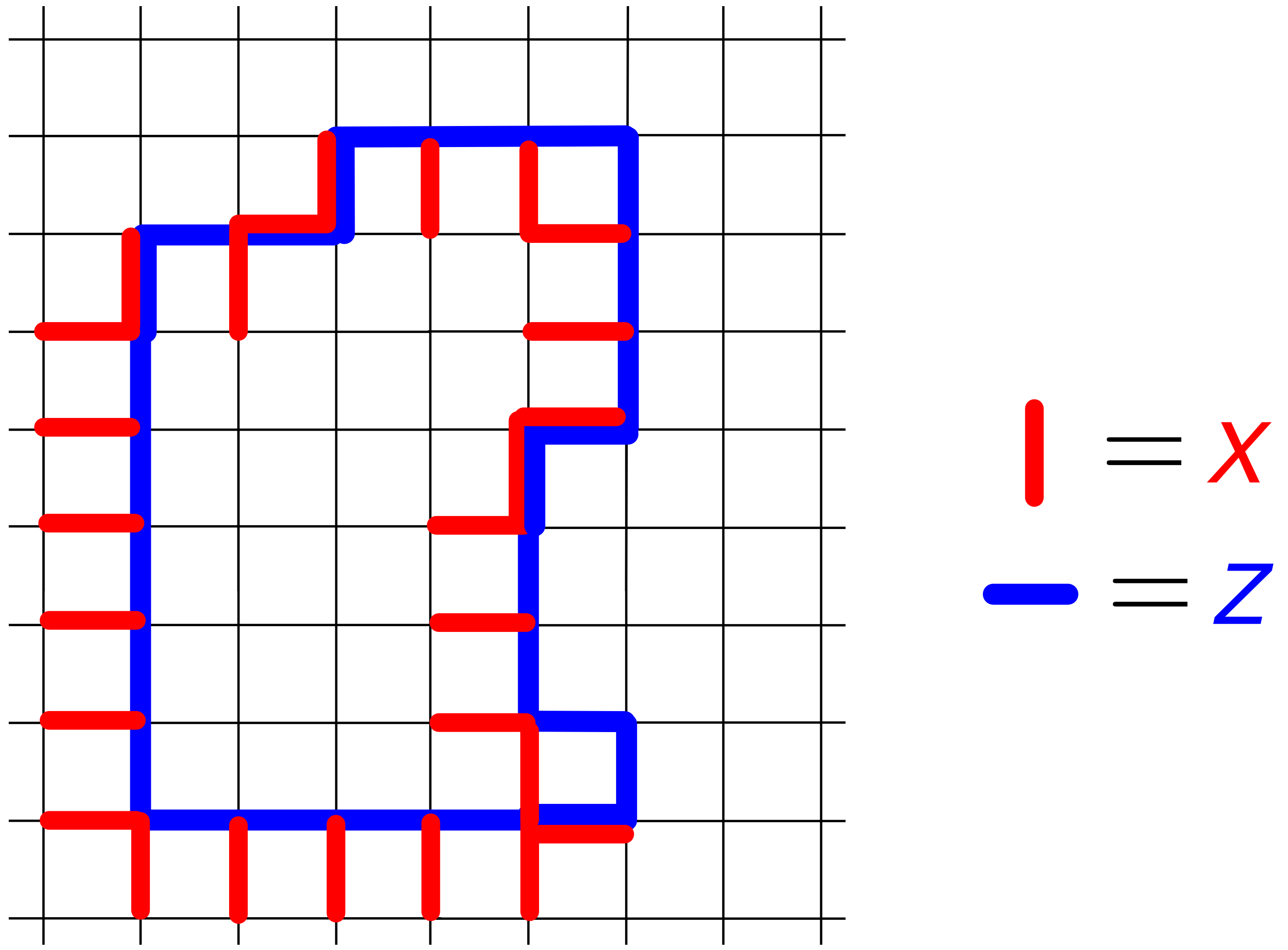
Fermion-decohered toric code

Fermion-decohered toric code phase diagram:



Emergent $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry (anomalous)	Emergent $\mathbb{Z}_2$ 1-form symmetry (anomalous)
SSB	ŚW SSB

Strong 1-form symmetry:



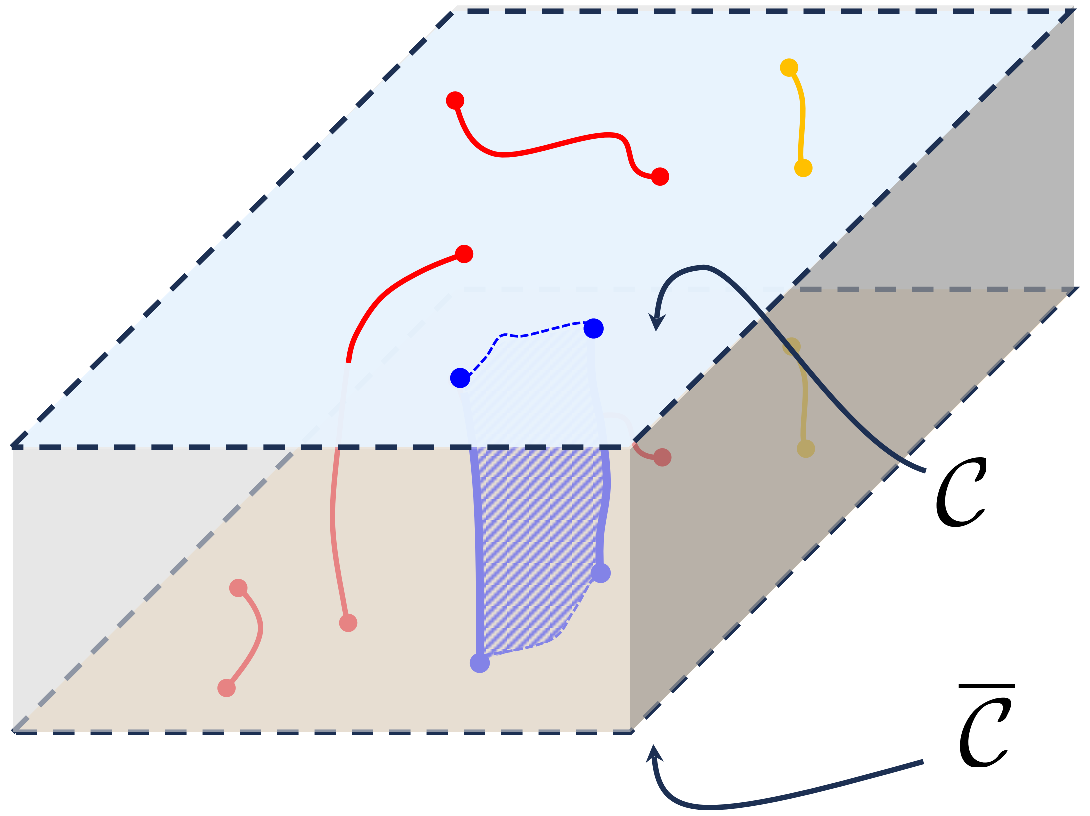
(Bosonic) intrinsically mixed topological order!

$$\mathbb{Z}_2^{(1)} = \{1, em\}$$

Premodular!

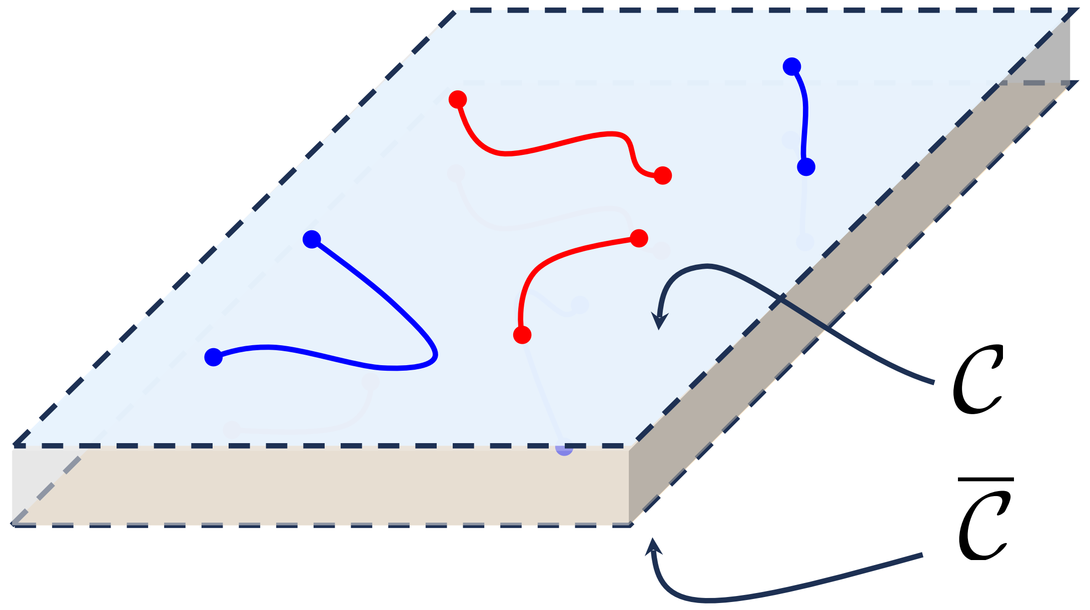
Walker-Wang construction of mixed states

1.



(3+1)D lattice model

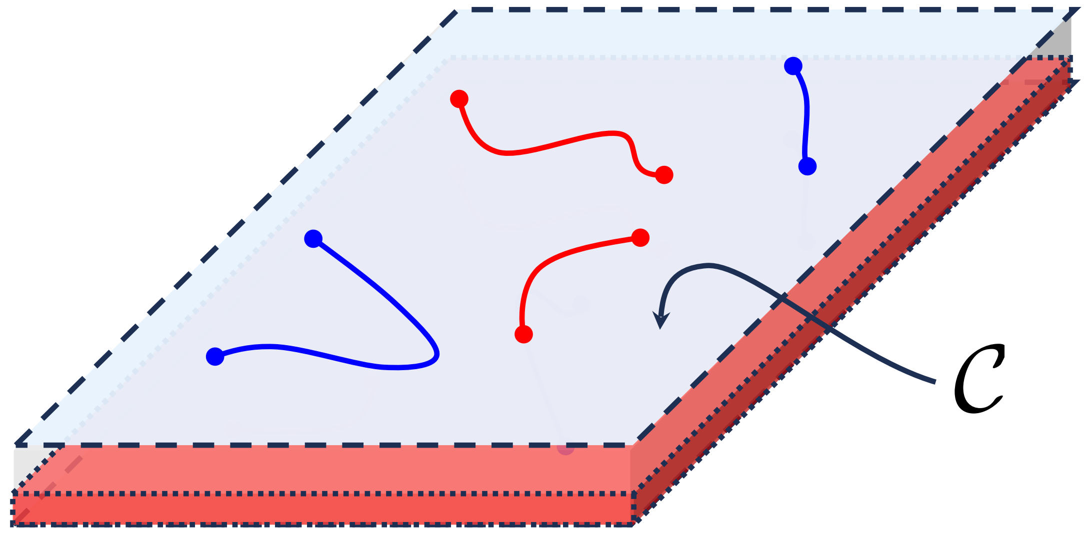
2.



Quasi-(2+1)D lattice model

(Quantum double of $\mathcal{C}$)

3.



Break $\overline{\mathcal{C}}$ symmetry by adding noise*
to the bottom surface

*Fully depolarizing noise

Mixed-state TOs corresponding to arbitrary
pre-modular tensor category $\mathcal{C}$!

SW SSB of the transparent subcategory of $\mathcal{C}$

Summary

- Strong and weak global symmetries help anchor our understanding of mixed-state phases

- SW SSB is a robust many-body phenomenon.

- SW SSB of anomalous symmetry leads to many examples of intrinsically mixed topological phases. Maybe “mixed” Landau paradigm?