IAS 2024 SUMMER COLLABORATION: NEARBY LAGRANGIANS AND FLOER HOMOTOPY THEORY

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Overview. During our two weeks at the Institute for Advanced Study, we studied foundations of Floer homotopy theory with applications to questions on nearby Lagrangians. Our work is motivated by the following conjecture and question:

Conjecture 1 (Arnold)**.** *If L is a closed exact Lagrangian in a cotangent bundle T^*Q, where Q is a closed manifold, then L is Hamiltonian isotopic to the zero section of T^*Q.*

Question 2. *If L is an exact Lagrangian filling of a Legendrian unknot in a subcritical Weinstein section X, then is L compactly Hamiltonian isotopic to the standard filling of the Legendrian unknot?*

During the collaboration, we primarily focused on establishing the following two results:

Theorem 3 (ADP)**.** *Let R be a commutative ring spectrum. If L is a closed Lagrangian R-brane in a cotangent bundle T^*Q, then L is equivalent in the wrapped Fukaya category with R-coefficients to the zero section equipped with some choice of R-brane structure.*

Theorem 4 (ADP)**.** *Let X be a subcritical Weinstein sector with $c_1(X) = 0 = c_2(X)$ and $\dim(X) \geq 8$ and let L be an exact Lagrangian filling of a Legendrian unknot. The map*

$$L/\partial L \longrightarrow X/\partial L \simeq X \longrightarrow X/X_{(n-2)} \cong \bigvee_i S^{n-1}$$

is null-homotopic, where $X_{(n-2)}$ denotes the $(n-2)$-skeleton of X.

The main technical details that were sorted during the collaboration include:

(1) Extending the argument for Theorem 3 from the case where R is connective to the case of general (not necessarily connective) R.
(2) Constructing and studying properties of the open-closed string map for the wrapped Fukaya category with R-coefficients.
(3) Establishing sufficient conditions for existence of MSpin-brane structures on Lagrangian submanifolds.

In the following subsections, we review our work on Floer homotopy theory and detail how it applies to Theorem 3 and Theorem 4.

Floer homotopy theory. Let R be a commutative ring spectrum. We fix Liouville sector X equipped with a stable polarization, i.e. a global Lagrangian section of $TX \oplus \underline{\mathbb{C}}^N$ for some $N \gg 0$. Given a Lagrangian submanifold $L \subset X$, the stable polarization on X determines a stable Lagrangian Gauss map $\mathcal{G}_L \colon L \to \mathrm{U/O}$.

Definition 5. A *Lagrangian R-brane* in X is an exact conical Lagrangian submanifold $L \subset X$ equipped with a *grading*, meaning a lift of $\mathcal{G}_L$ to the universal cover of U/O, and *R-orientation data*, meaning a lift of $\mathcal{G}_L$ to

$$(\mathrm{U/O})^{\#} := \operatorname{hofib}\left(\mathrm{U/O} \xrightarrow{\beta^{-1}} B^2\mathrm{O} \xrightarrow{B^2 J} B^2\,\mathrm{GL}_1(\mathbb{S}) \longrightarrow B^2\,\mathrm{GL}_1(R)\right),$$

where $\mathrm{GL}_1(R)$ and $\mathrm{GL}_1(\mathbb{S})$ denote the spaces of units associated to the ring spectra R and $\mathbb{S}$ and β^{-1} is given by (the inverse of) the Bott periodicity map.

Our main construction is that of an R-oriented flow category $\mathcal{CW}(L_0, L_1; R)$ associated to two Lagrangian R-branes L_0 and L_1. The objects of the flow category are transverse intersections between L_0 and L_1 and the morphisms between two objects are the compactified moduli space of J-holomorphic disks with boundary conditions on L_0 and L_1 and two boundary punctures mapping to the intersection points. R-orientations on these spaces are determined by the R-brane structures on the Lagrangians. Cohen–Jones–Segal systematically developed an approach to apply the Pontryagin–Thom construction to an R-oriented flow category to obtain an R-module spectrum. When applied to $\mathcal{CW}(L_0, L_1; R)$, we obtain the R-module spectrum $HW(L_0, L_1; R)$. In this manner, we can construct a wrapped Fukaya category with R-coefficients:

Definition 6. The *wrapped Donaldson–Fukaya category* of X is the category enriched over the homotopy category of R-module spectra defined as follows:

- The set of objects are Lagrangian R-branes.
- The morphisms from L to K is the homotopy class of the R-module spectrum $HW(L, K; R)$.
- The composition is given by a homotopy associative map

$$\mu^2 \colon HW(L_0, L_1; R) \wedge_R HW(L_1, L_2; R) \longrightarrow HW(L_0, L_2; R),$$

  which is defined via compactified moduli spaces of J-holomorphic disks with three boundary punctures and Lagrangian boundary conditions (the so-called "Floer triangles" in classical Floer theory).

We established the following structural properties of the wrapped Fukaya category with R-coefficients:

Theorem 7 (ADP)**.** *The following structural properties hold:*

(1) An inclusion of Liouville sectors $X \hookrightarrow X'$ induces a pushforward functor $\mathcal{W}(X; R) \to \mathcal{W}(X'; R)$.
(2) If X is a stably polarized subcritical Weinstein sector, then for any $L_0, L_1 \in \mathrm{Ob}(\mathcal{W}(X; R))$ we have that $HW(L_0, L_1; R) \cong 0$ as R-module spectra.
(3) Let L and K are two closed Lagrangian R-branes which are isomorphic in $\mathcal{W}(X; R)$. If L is R-oriented, then K admits an R-orientation so that the R-homology classes represented by L and K in $H_n(X; R)$, determined by these R-orientations, coincide.

Item (3) of Theorem 7 was one of the main focuses of our efforts while at IAS. In brief, the technical aspect of item (3) revolves around determining how the Floer equivalence of L and K and the R-orientation on L determine a specific R-orientation on K. This amounted to studying R-orientation data on the flow categories associated to the relevant J-holomorphic curves and a reduction of the problem to Morse theory via considering C^2-small Hamiltonian perturbations.

Nearby Lagrangians. Let R be a commutative ring spectrum. In this section, we restrict to $X = T^*Q$, where Q is a closed smooth manifold. Note that X is polarized via the distribution of the cotangent fibers. A *nearby Lagrangian* is a closed exact Lagrangian submanifold $L \subset T^*Q$. The composition

$$L \longrightarrow \mathrm{U/O} \longrightarrow B^2\mathrm{O} \longrightarrow B^2\,\mathrm{GL}_1(\mathbb{S})$$

in Definition 5 is known to be null-homotopic (see [Jin20, Corollary 1.3] or [ACGK20, Theorem B]), which means that any nearby Lagrangian $L \subset T^*Q$ admits a choice of R-orientation data. With this in mind, we establish Theorem 3 as follows:

(1) Let $R_{\geq 0}$ denote the connective cover of R. By Definition 5, one can observe that an $R_{\geq 0}$-brane structure is equivalent to a R-brane structure. From this and functoriality properties of the Cohen–Jones–Segal construction, one obtains a natural essentially surjective functor $\mathcal{W}(X; R_{\geq 0}) \to \mathcal{W}(X; R)$. In particular, to prove Theorem 3, it suffices to prove the result for connective R.
(2) Letting $F \subset T^*Q$ denote a cotangent fiber, one has an equivalence of R-algebras $HW(F, F; R) \cong \Omega Q \wedge R$.
(3) One can show that two compact exact Lagrangian R-branes L and K in T^*Q are equivalent if and only if $HW(F, L; R) \cong HW(F, K; R)$ as $(\Omega Q \wedge R)$-modules. This is established by using the connectivity of R and a Whitehead argument to reduce proving the claim to prove the analogous result over discrete coefficients, which is already known classically.

(4) Finally, one shows that every free $(\mathit{\Omega}Q \wedge R)$-module of rank 1 that is obtained from a R-brane on a nearby Lagrangian can be obtained from a choice of R-brane structure on the zero section. This establishes the result.

Remark 8. Theorem 3 generalizes the well-known result that every nearby Lagrangian in T^*Q is Floer theoretically equivalent in the usual wrapped Fukaya category with integer coefficients to the zero section equipped with a particular choice of local system.

Fillings of unknots. Let X be a subcritical Weinstein sector. Let $\mathit{\Lambda} \subset \partial_\infty X$ be a Legendrian unknot in a Darboux ball in the contact boundary at infinity. Let $L \subset X$ be an exact conical Lagrangian submanifold, such that $L \cap \partial_\infty X = \mathit{\Lambda}$. Let C denote the standard Lagrangian filling of the unknot in the Darboux ball. We establish Theorem 4 as follows:

(1) By attaching a critical Weinstein handle to X along the boundary of L, we construct a Weinstein sector $\widehat{X}$ and Lagrangian submanifolds $\widehat{L}, \widehat{C} \subset \widehat{X}$ such that:
  - $\widehat{C}$ and $\widehat{L}$ are diffeomorphic to S^n.
  - $\widehat{X}$ is obtained from attaching subcritical handles to T^*S^n.
  - $\widehat{C}$ corresponds to the zero section in $T^*S^n \subset \widehat{X}$.

(2) Via some geometric arguments and algebraic topology, to prove Theorem 4, it suffices to show $\widehat{L}$ (with the standard spin structure) is equivalent to $\widehat{C}$ for some choice of MSpin-orientation on $\widehat{C}$.

(3) With this in mind and by item (3) of Theorem 7, we reduce our task to showing that $\widehat{L}$ is equivalent to $\widehat{C}$ for some choice of MSpin-brane structures. This is ultimately handled in the same manner as the proof of Theorem 3.

The technical part of Theorem 4 that we worked on during our time at IAS was establishing sufficient conditions for a Lagrangian to admit a MSpin-brane structure. First, in Definition 5, we assumed that X admits a stable polarization. The first aspect of study was showing that this can be weakened to X admitting a stable polarization over the 4-skeleton of X, which is ensured by $c_1(X) = 0 = c_2(X)$. This weakening is sufficient for the case of MSpin-branes structures. The second aspect of study was establishing sufficient conditions for the analogous stable Lagrangian Gauss map to be null-homotopic, which can be ensured by the vanishing of certain characteristic classes of the Lagrangian. These observations were established by studying the effect of the Bott map on cohomology and applying spectral sequence arguments.

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References

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