From matrices to motivic homotopy theory II

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Review

Conjecture (M.P. Murthy '97)

Suppose k is an algebraically closed field and $d \geq 1$ is an integer. If X is a smooth affine k-variety of dimension $d+1$, and E is a rank d vector bundle on X, then $E \cong E' \oplus \mathbf{1}_X$ if and only if $0 = c_d(E) \in CH^d(X)$.

Theorem (A.–T. Bachmann–M.J. Hopkins '24)

If k is an algebraically closed field having characteristic 0, X is a smooth affine d-fold over k, then Murthy's conjecture holds.

(building on work of A.–J. Fasel)

Review: obstruction theory after Moore–Postnikov

- Fix X smooth affine of dimension $d+1$ over k alg. closed, $\xi : X \to Gr_d$.
- Affine representability $\Longrightarrow$ suffices to lift ξ to $X \to Gr_{d-1}$ in motivic homotopy
- Since $\dim X = d+1 \Longrightarrow$ lift exists if and only if
- **primary obstruction** in $H^d(X, \boldsymbol{\pi}_{d-1}(\mathbb{A}^d) \setminus 0)$ vanishes and
- assuming primary obstruction vanishes, obstruction theory $\Longrightarrow$ **secondary obstruction** in

$$o_2(\xi) \in H^{d+1}(X, \boldsymbol{\pi}_d(\mathbb{A}^d \setminus 0)(\det \xi))$$

vanishes.

Euler classes vs. top Chern classes

- Since $\boldsymbol{\pi}_{d-1}(\mathbb{A}^d \setminus 0) \cong \mathbf{K}_d^{MW}$ (F. Morel),
- universal example: $H^d(Gr_d, \mathbf{K}_d^{MW}(\det \gamma_d))$ free rank 1 $\mathbf{K}_0^{MW}(k)$-module generated by $e(\gamma_d)$
- the primary *Euler class* obstruction $e(\xi)$ lives in $H^d(X, \mathbf{K}_d^{MW}(\det \xi))$.
- There is an exact sequence:

$$0 \longrightarrow \mathbf{I}^{d+1}(\det \xi) \longrightarrow \mathbf{K}_d^{MW}(\det \xi) \longrightarrow \mathbf{K}_d^{M} \longrightarrow 0.$$

- the map $H^d(X, \mathbf{K}_d^{MW}(\det \xi)) \to H^d(X, \mathbf{K}_d^{M}) \cong CH^d(X)$ sends $e(\xi)$ to $c_d(\xi)$ (treat the universal example)
- kernel and cokernel $\Longrightarrow$ analyze $H^{d+j}(X, \mathbf{I}^{d+1}(\det \xi)), j = 0, 1$.

- $\dim X = d+1$, k alg. closed $\Longrightarrow$ the sheaf $\mathbf{I}^j(\det \xi) = 0$ for $j \geq d+2$ (Arason–Pfister)
- then,
  $\mathbf{I}^{d+1}(\det \xi) = \mathbf{I}^{d+1}(\det \xi)/\mathbf{I}^{d+2}(\det \xi) \cong \mathbf{K}^M_{d+1}/2 \cong \mathcal{H}^{d+1}_{\text{ét}}(\mu_2^{\otimes d+1})$
  (Milnor conjectures on quadratic forms by Orlov–Vishik–Voevodsky and *mod*2 norm residue homomorphism by Voevodsky–Rost)
- Bloch–Ogus s.s. analysis $\Longrightarrow H^{2d+1}_{\text{ét}}(X) \twoheadrightarrow H^d(X, \mathcal{H}^{d+1}_{\text{ét}}(\mu_2^{\otimes d+1}))$
- + X affine $\Longrightarrow H^d(X, \mathbf{I}^{d+1}(\det \xi)) = 0$ (Artin–Grothendieck vanishing)
- thus, $H^d(X, \mathbf{K}^{MW}_d(\det \xi)) \xrightarrow{\sim} CH^d(X)$ and this iso sends $e(\xi)$ to $c_d(\xi)$.

Low-dimensional exceptional isomorphisms

- Cartesian square:

$$\begin{array}{ccc} Sp_{2n-2} & \longrightarrow & Sp_{2n} \\ \downarrow & & \downarrow \\ SL_{2n-1} & \longrightarrow & SL_{2n} \end{array}$$

- $Sp_{2n}/Sp_{2n-2} \cong SL_{2n}/SL_{2n-1} \sim \mathbb{A}^{2n} \setminus 0$
- $SL_{2n-1}/Sp_{2n-2} \cong SL_{2n}/Sp_{2n} \qquad n = 2; SL_4/Sp_4 \sim \mathbb{A}^3 \setminus 0$
- analyze motivic homotopy by stabilizing: standard inclusions

$$Sp_2 \subset Sp_4 \subset \cdots \subset Sp$$

$$SL_4/Sp_4 \subset GL_4/Sp_4 \subset GL_6/Sp_6 \subset \cdots \subset GL/Sp$$

- $\mathbb{Z} \times BSp$ represents symplectic K-theory (Schlichting–Tripathi)
- Connectivity estimate $\Longrightarrow \boldsymbol{\pi}_i(Sp_2) \longrightarrow \boldsymbol{\pi}_i(Sp) \cong \boldsymbol{\pi}_{i+1}(BSp)$ is

$$\begin{cases} \boldsymbol{\pi}_1(\mathbb{A}^2 \setminus 0) \cong \mathbf{K}_2^{MW} \xrightarrow{\sim} \mathbf{K}_2^{Sp} & \text{(Suslin's theorem)} \\ \boldsymbol{\pi}_2(\mathbb{A}^2 \setminus 0) \twoheadrightarrow \mathbf{K}_3^{Sp} \end{cases}$$

- stabilization fiber sequences $\Longrightarrow$ exact sequence

$$0 \longrightarrow \operatorname{coker}(\mathbf{K}_4^{Sp} \longrightarrow \mathbf{K}_4^{MW}) \longrightarrow \boldsymbol{\pi}_2(\mathbb{A}^2 \setminus 0) \longrightarrow \mathbf{K}_3^{Sp} \longrightarrow 0$$

- $\operatorname{coker}(\mathbf{K}_4^{Sp} \to \mathbf{K}_4^{MW}) \cong$ "$\mathbf{K}_4^{M}/12 \times_{\mathbf{K}_4^M/2} \mathbf{I}^4$"
- Borel–Serre in classical topology: $\pi_6(S^3) = \pi_7(BSp_2) = \mathbb{Z}/12$

- GL/Sp represents a another Hermitian K-theory (Schlichting–Tripathi):

$$\Omega^{2,1} GL/Sp \sim Sp$$

- Connectivity estimates $\Longrightarrow$
$\pi_i(SL_4/Sp_4) \longrightarrow \pi_i(SL/Sp) \cong \pi_i(GL/Sp), i \geq 2$ is

$$\begin{cases} \pi_2(\mathbb{A}^3 \setminus 0) \cong \mathbf{K}_3^{MW} \xrightarrow{\sim} \mathbf{GW}_3^3 \\ \pi_3(\mathbb{A}^3 \setminus 0) \twoheadrightarrow \mathbf{GW}_4^3 \end{cases}$$

- stabilization fiber sequences $\Longrightarrow$ exact sequence

$$0 \longrightarrow \mathrm{coker}(\mathbf{GW}_5^3 \longrightarrow \mathbf{K}_5^{MW}) \longrightarrow \pi_3(\mathbb{A}^3 \setminus 0) \longrightarrow \mathbf{GW}_4^3 \longrightarrow 0$$

- $\mathrm{coker}(\mathbf{GW}_5^3 \to \mathbf{K}_5^{MW}) \cong$ "$\mathbf{K}_5^M/24 \times_{\mathbf{K}_5^M/2} \mathbf{I}^5$"
- Classical topology $\pi_8(S^5) \cong \mathbb{Z}/24$ gen. by suspension of Hopf map ν

Homotopy of $\mathbb{A}^n \setminus 0$, $n \geq 4$

The notion of "stable range" in topology comes from:

Theorem (Freudenthal)

If $n \geq 2$, X is a pointed $(n-1)$-connected space, then

$$X \longrightarrow \Omega\Sigma X$$

is $2n-1$-connected.

- Morel established an exact analog of this statement for S^1-suspension.
- Since $\mathbb{A}^{n+1} \setminus 0 \sim \mathbb{P}^1 \wedge \mathbb{A}^n \setminus 0$, we need to understand how homotopy sheaves behave with respect to $\mathbb{P}^1$-suspension.
- Morel's calculations $\Longrightarrow$ Freudenthal suspension is *false* for $\mathbb{P}^1$-suspension: $\boldsymbol{\pi}_n(S^n) = \mathbb{Z}$, while $\boldsymbol{\pi}_n(\Omega^{2,1}S^{n+2,1}) = \mathbf{K}_0^{MW}$ (the source is not "$\mathbb{G}_m$-connected")

Weak cellular classes

The collection of $n-1$-connected spaces is the smallest class of spaces containing S^n, stable under (homotopy) colimits and (homotopy) cofiber extensions, i.e., if

$$A \longrightarrow B \longrightarrow C$$

is a cofiber sequence and A and C are $(n-1)$-connected, then so is B.

Definition

Say $\mathscr{X} \in O(S^{p,q})$ (weakly $S^{p,q}$-cellular) if it is contained in the smallest subcategory of pointed motivic spaces containing all spaces of the form $S^{p,q} \wedge X_+$, $X \in \mathrm{Sm}_k$ that is closed under formation of (homotopy) colimits and (homotopy) cofiber extensions.

If $\mathscr{X} \in O(S^{p,q})$, then $\mathscr{X}$ is $(p-q-1)$-connected.

Example

The space $\mathbb{A}^n \setminus 0$ *lies in* $O(S^{2n-1,n})$.

Lemma

If $\mathscr{F} \to \mathscr{E} \to \mathscr{B}$ *is a motivic fiber sequence,* $\mathscr{F}, \mathscr{B} \in O(S^{p,q}) \Longrightarrow \mathscr{E} \in O(S^{p,q})$ *(assuming all spaces are connected).*

Example

The space $\mathrm{fib}(BSp_2 \to BSp) \sim Sp/Sp_2 \in O(S^{7,4})$:

- $Sp/Sp_2 = \mathrm{colim}_n\, Sp_{2n}/Sp_2$, *and*
- $n \geq 2$, *there are fiber sequences*

$$Sp_{2n-2}/Sp_2 \longrightarrow Sp_{2n}/Sp_2 \longrightarrow Sp_{2n}/Sp_{2n-2} \sim S^{4n-1,2n}$$

Example

If $\mathscr{X} \in O(S^{p,q})$, $p - q \geq 0$, *then* $J(\mathscr{X}) \sim \Omega\Sigma\mathscr{X} \in O(S^{p,q})$ *as well.*

Example

The space $\mathrm{fib}(SL_4/Sp_4 \to SL/Sp) \in O(S^{8,5})$*:*

- $SL/Sp = \mathrm{colim}_n\, SL_{2n}/Sp_{2n}$, *and*
- *fibers of composites*
- $n \geq 3$, *there are fiber sequences*

$$\Omega S^{4n-3,2n-1} \longrightarrow SL_{2n-2}/Sp_{2n-2} \longrightarrow SL_{2n-1}/Sp_{2n-2} \cong SL_{2n}/Sp_{2n}$$

Example

The space $K(\mathbb{Z}(n), 2n)$ *(i.e., Voevodsky's motivic Eilenberg Mac Lane space) lies in* $O(S^{2n,n})$*: use Voevodsky's motivic Dold–Thom theorem.*

Calculus of cellular classes

For "sufficiently nice" motivic spaces (e.g., 1-connected):

- We can build a functorial weakly $S^{p,q}$-cellular cover: $\tau_{\geq(p,q)}$ by analogy with connective covers.
- If $\mathscr{X} \in O(S^{p,q})$, then taking Postnikov layers need not preserve weak cellular class:
  - one can refine the Postnikov tower to preserve cellularity.
  - if $q \geq 2$, layers of refined tower are $\mathbb{P}^1$-infinite loop spaces (Bachmann–Yakerson, which uses motivic infinite loop space technology of EHKSY)
- For p, q large enough, if $\mathscr{X} \in O(S^{p,q})$, then $\Omega^{2,1}\mathscr{X} \in O(S^{p-2,q-1})$; similar statements can be made about fibers (Levine's results on the slice/homotopy coniveau tower)

The motivic Freudenthal theorem

Theorem (A.-Bachmann-Hopkins)

Assume k is a field having characteristic 0*. If* $\mathscr{X} \in O(S^{p,q})$ *with* $p - q \geq 2, q \geq 2$*, then*

$$\mathrm{fib}(\mathscr{X} \longrightarrow \Omega^{2,1}\Sigma^{2,1}\mathscr{X}) \in O(S^{a,2q}),$$

where $a = \min(2p-1, p+2q-1)$ *(these agree when* $p = 2q$*).*

Corollary

If k has characteristic 0*, then*

$$\mathrm{fib}(\mathbb{A}^3 \setminus 0 \longrightarrow \Omega^{2,1}\mathbb{A}^4 \setminus 0) \in O(S^{9,6});$$

in particular, $\boldsymbol{\pi}_3(\mathbb{A}^3 \setminus 0) \to \boldsymbol{\pi}_3(\Omega^{2,1}\mathbb{A}^4 \setminus 0)$ *is surjective.*

- Step 1. Reduce to the case of $\mathbb{P}^1$-infinite loop spaces using the weakly-cellular refinement of the Postnikov tower.
- Step 2. Devissage, using ideas of Levine's slice convergence results as extended by Bachmann–Elmanto–Østvær (uses the Voevodsky–Rost verification of the Bloch–Kato conjecture!), to the case of motives of smooth varieties.
- Step 3. Treat the case of motives of smooth varieties by an explicit argument using the geometry of symmetric powers.

- The identity map on $K(\mathbb{Z}(n), 2n) \sim \Omega^{2,1} K(\mathbb{Z}(n+1), 2n+2)$ factors as

$$K(\mathbb{Z}(n), 2n) \longrightarrow \Omega^{2,1}\Sigma^{2,1} K(\mathbb{Z}(n), 2n) \xrightarrow{\Omega^{2,1} a_n} \Omega^{2,1} K(\mathbb{Z}(n+1), 2n+2),$$

  and it suffices to establish a suitable cellularity estimate for $\mathrm{cof}(a_n)$.
- The space $K(\mathbb{Z}(n), 2n) = \mathrm{colim}_r \, \mathrm{Sym}^r(\mathbb{P}^{1^{\wedge n}})$ (motivic Dold–Thom).
- The assembly map a_n arises from the Σ_r-equivariant map $\mathbb{A}^1 \times (\mathbb{A}^n)^{\times r} \longrightarrow \mathbb{A}^n \times (\mathbb{A}^n)^{\times r} \cong (\mathbb{A}^{n+1})^{\times r}$ by taking quotients and applying Thom spaces and taking a colimit.
- It suffices to understand the cofiber of the map just described, which following Nakaoka and Voevodsky can be described explicitly in terms of representation theory of the symmetric group and is built iteratively out of Thom complexes where weak cellularity class are easier to understand.

Back to Murthy's conjecture

- The maps

$$\begin{cases} S^{3,2} \sim SL_2 = Sp_2 & \longrightarrow Sp \\ S^{5,3} \sim SL_4/Sp_4 \subset GL_4/Sp_4 & \longrightarrow GL/Sp \end{cases}$$

  described earlier, fit into a sequence.
- Via the geometric form of Bott periodicity for Hermitian K-theory, the unit map $\mathbf{S}^0 \to \mathbf{KO}$ is stabilization of sequence of maps

$$S^{2n-1,n} \longrightarrow \Omega^{-2n-1,-n} B_{\text{ét}} O;$$

  realized explicitly as (hyperbolic quadric) Q_{2n-1} to spaces in geometric form of Bott periodicity via "Suslin matrices" (Asok–Fasel).
- Factors through the weakly-cellular cover $\tau_{\geq 2n-1,n} \Omega^{-2n-1,-n} B_{\text{ét}} O$; induced map is an iso on bottom degree homotopy by Morel's calculations.

For $n \geq 4$, the motivic Hopf map $\nu : S^{2n+2,n+2} \to S^{2n-1,n}$ covers the fiber of the above map; this can be accomplished by comparison with the stable situation (Röndigs–Spitzweck–Østvær).

Theorem

If k has characteristic 0, and $n \geq 4$, then there is an exact sequence

$$0 \longrightarrow \mathbf{K}^M_{n+2}/24 \longrightarrow \pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \longrightarrow \boldsymbol{\pi}_n(\tau_{\geq 2n-1,n}\Omega^{-2n-1,-n}B_{\acute{e}t}O) \longrightarrow 0.$$

The explcit description of $\boldsymbol{\pi}_d(\mathbb{A}^d \setminus 0)$ allows us to analyze the secondary obstruction:

Part coming from ν: for X of dimension $d+1$ over a field k:

- $H^{d+1}(X, \mathbf{K}^M_{d+2}/24) \cong H^{2d+3,d+2}(X, \mathbb{Z}/24)$
- also a quotient of $\bigoplus_{x \in X^{(d+1)}} \mathbf{K}^M_1(\kappa_x)/24$ (Gersten)

If k algebraically closed, $\mathbf{K}^M_1(\kappa_x)$ is divisible so $\mathbf{K}^M_1(\kappa_x)/24$ vanishes.

Part coming from the destablized unit map to **KO**:

- $H^{d+1}(X, \tau_{\geq 2n-1,n}\Omega^{-2n-1,-n}B_{\text{ét}}O) \cong H^{d+1}(X, \mathbf{GW}^{d}_{d+1})$

Theorem (A., Fasel)

If X is a smooth k-scheme of dimension $d+1$ over a field k (char. $k \neq 2$), then

$$H^{d+1}(X, \mathbf{GW}^{d}_{d+1}(L)) \cong H^{2d+2,d+1}(X, \mathbb{Z}/2)/(Sq^2 + c_1(L)\cup)H^{2d,d}(X, \mathbb{Z}/2).$$

If X affine, and k alg. closed, then $H^{2d+2,d+1}(X, \mathbb{Z})$ is divisible (Roitman), hence $H^{2d+2,d+1}(X, \mathbb{Z}/2)$ vanishes as well.

Thank you!