# From matrices to motivic homotopy theory I

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# Splitting problems in topology

# Homotopy classification of vector bundles

- $M$ - a "nice" space (manifold, CW complex) of dimension $d$
- $\mathscr{V}_r(M)$ - set of isomorphism classes of rank $r$ (real) vector bundles on $M$
- $\mathscr{V}_r(M) \to \mathscr{V}_r(M \times I)$ is a bijection (homotopy invariance)
- $Gr_r^{top}$ - Grassmannian of $r$-dimensional subspaces of an infinite dimensional real vector space
- $Gr_r^{top}$ carries a rank $r$ "tautological" vector bundle
- Continuous maps $M \to Gr_r^{top}$ yield rank $r$ vector bundles on $M$ by pullback; homotopic maps yield isomorphic bundles

**Theorem (Pontryagin–Steenrod)**

*Pulling back the tautological bundle determines a bijection:*

$$[M, Gr_r^{top}] \xrightarrow{\sim} \mathscr{V}_r(M).$$

# The topological splitting problem

- There is a map $s_r : Gr_{r-1}^{top} \to Gr_r^{top}$ classifying the sum of the tautological bundle of rank $r-1$ and a trivial rank 1 bundle.
- A rank $r$ bundle has a nowhere vanishing section if and only if it splits off a trivial rank 1 summand, this yields a lifting problem: given $\xi : M \to Gr_r^{top}$, can we complete the following diagram:

$$\begin{array}{ccc} & & Gr_{r-1}^{top} \\ & \overset{\exists ?}{\nearrow} & \downarrow{\scriptstyle s_r} \\ M & \xrightarrow{\xi} & Gr_r^{top}. \end{array}$$

# Negative corank

Take $M$ a closed manifold of dimension $d$ and fix $\xi : M \to Gr_r^{top}$.

- By the corank of $\xi$ we will mean the integer $d - r$.
- If $r > d$, then a generic section of $\xi$ is nowhere vanishing (general position).
- Equivalently, a lift exists as long as $r > d$.

# Non-negative corank: the primary obstruction

Take $M$ a closed manifold of dimension $d$ and fix $\xi : M \to Gr_d^{top}$.

- Evident obstruction to existence of a nowhere vanishing section: the cohomology class Poincare dual to the vanishing locus of a generic section, a.k.a., **(twisted) Euler class**.
- Since $\pi_1(Gr_r^{top}) \cong \mathbb{Z}/2$ (via the determinant), $\xi$ yields an orientation character $\omega_\xi : \pi_1(M) \to \mathbb{Z}/2 \Longleftrightarrow$ orientation local system on $M$.

**Theorem**

*Given a topological space having the homotopy type of CW complex of dimension d and $\xi$ as above, then the vanishing of the (twisted) Euler class*

$$e(\xi) \in H^d(M, \mathbb{Z}[\omega_\xi])$$

*is the* only *obstruction to splitting off a free rank* 1 *summand.*

# Corank 0: the proof

- The failure of $s_r$ to be a weak homotopy equivalence is measured by the fiber sequence:

$$S^{r-1} \longrightarrow Gr_{r-1}^{top} \xrightarrow{s_r} Gr_r^{top}.$$

- Obstruction theory: there is an inductive procedure to decide whether lifts exist using *unstable* homotopy of $S^{r-1}$.
- The Euler class arises from:

$$deg : \pi_{r-1}(S^{r-1}) \xrightarrow{\sim} \mathbb{Z}$$

for $r \geq 2$; dimension hypothesis implies only obstruction, and this group is *stable*, i.e., the answer is independent of $r$.

# Corank 1: ideas

**Theorem (S.D. Liao '54)**

*Assume $M$ is a manifold of dimension $d+1$, $d \geq 4$. If $\xi$ is an oriented rank $d$ vector bundle on $M$, then $\xi$ splits off a trivial rank 1 summand and if and only if $0 = e(\xi) \in H^d(M,\mathbb{Z})$ and*
*$0 = o_2(\xi) \in H^{d+1}(M,\mathbb{Z}/2)/(Sq^2 + w_2(\xi)\cup)H^{d-1}(M,\mathbb{Z}/2)$.*

- Dimension hypothesis implies there are two obstructions to analyze; primary obstruction is the Euler class.
- If the primary obstruction vanishes, there is a well-defined secondary obstruction arising from:

$$\pi_d(S^{d-1}) = \begin{cases} \mathbb{Z} & \text{if } d = 3 \\ \mathbb{Z}/2 & \text{if } d \geq 4 \end{cases};$$

- if $d \geq 4$ the computation is "stable", while $d = 3$ is "unstable".

# Splitting algebraic vector bundles I: a classical story

# Projective modules

Throughout the talk: $R$ is a commutative (unital) ring.

**Definition**

An $R$-module $P$ is called **projective** if it is a direct summand of a free $R$-module.

Equivalently, $P$ is projective if:

- (lifting property) given an $R$-module map $f : P \to M$, and a surjective $R$-module map $N \twoheadrightarrow M$, we may always find $\tilde{f} : P \to N$.
- (linear algebraic) if $P$ is also finitely generated, then there exist an integer $n$, and $\epsilon \in End_R(R^{\oplus n})$ such that $\epsilon^2 = \epsilon$ and $P = \epsilon R^{\oplus n}$.

From now on, all projective modules will be assumed finitely generated (f.g.)

Projective modules “are” vector bundles:

- $R$ a ring $\to \operatorname{Spec} R$ a topological space (with the Zariski topology)
- basis of open sets $D(f)$ (think: complement of $f = 0$)
- $P$ a f.g. projective $R$-module, then there are $f_1, \ldots, f_n \in R$ such that “$P|_{D(f_i)}$ is free,” that is $P[\frac{1}{f_i}]$ is a free $R[\frac{1}{f_i}]$-module
- $P$ a projective module $\longrightarrow$ vector bundle on $\operatorname{Spec} R$ (locally trivial for the Zariski topology)

Vector bundles "are" projective modules:

- $M$ a (say) compact manifold;
- $C(M)$ = ring of continuous real-valued functions on $M$
- $\pi : E \to M$ a vector bundle, $P(\pi) = C(M)$-module of continuous sections of $\pi$

## Serre–Swan correspondence

$$\{ \text{ finite rank v.b. over } M\} \longleftrightarrow \{ \text{ f.g. projective } C(M) - \text{ modules } \}$$

## Serre's dictionary

If $R$ is a commutative ring, then locally free $R$-modules are projective:

$$\{ \text{ finite rank v.b. over } \operatorname{Spec} R\} \longleftrightarrow \{ \text{ f.g. projective } R - \text{ modules } \};$$

## Definition

Suppose $R$ is a Noetherian ring of Krull dimension $d$ and $P$ is a projective $R$-module of rank $r$.

- $\mathscr{V}_r(\operatorname{Spec} R)$ - $\cong$ classes of rank $r$ projective $R$-modules a.k.a. rank $r$ algebraic vector bundles on $\operatorname{Spec} R$
- corank $P = d - r$.

The assignment $P \mapsto P \oplus R$ determines a stabilization function

$$s_r : \mathscr{V}_{r-1}(\operatorname{Spec} R) \longrightarrow \mathscr{V}_r(\operatorname{Spec} R).$$

### Theorem (J.P. Serre '58)

*The function* $s_r$ *is surjective for* $r > d$ *(i.e., negative corank).*

### Remark

*The proof of Serre's theorem was based on an algebro-geometric analog of "general position" arguments: a generic section of a topological vector bundle of rank* $r > d$ *is nowhere vanishing.*

### Question (H. Bass '64)

*Can one characterize the image of* $s_r$*?*

# Corank 0: complications

Assume $k$ is a field, and $R$ is a smooth $k$-algebra of dimension $d$ and $P$ a projective $R$-module of rank $d$.

- obstruction: "the vanishing locus of a generic section", i.e., the $d$-th Chern class $c_d(P)$ lying in $CH^d(\operatorname{Spec} R)$ the Chow group of codimension $d$ algebraic cycles modulo rational equivalence
- obstruction is insufficient in general: (R. Swan '61) the tangent bundle $\tau_{S^2}$ to the real 2-sphere (hypersurface given by $x^2 + y^2 + z^2 = 1$) has $c_2(\tau_{S^2}) = 0$ but does not split off a free rank 1 summand.

When (if ever) is the obstruction sufficient?

# The Euler class in algebraic geometry

## Theorem

*If $k$ is algebraically closed, and $X$ is a smooth affine $d$-fold over $k$, then the image of $s_{d-1,X}$ consists of those $\mathcal{E}$ (of rank $d$) such that $0 = c_d(\mathcal{E}) \in CH^d(X)$.*

- $d = 2$ R. Swan–M.P. Murthy '76
- $d = 3$ N. Mohan Kumar–M.P. Murthy '82
- $d \geq 4$ M.P. Murthy '94
- Proof uses only "classical" tools.

What happens when $k$ is not algebraically closed?
Can one say anything in corank $> 0$?

# Vector bundles and motivic homotopy theory

- $Gr_r$ is an (infinite-dimensional) algebraic variety
- A map $\operatorname{Spec} R \to Gr_r$ corresponds to a vector bundle on $R$ and a collection of generating sections
- We may speak of "naive" homotopies between such maps, i.e., two maps $f, g : \operatorname{Spec} R \to Gr_r$ are naively homotopic if there exists a map $H : \operatorname{Spec} R[t] \to Gr_r$ with $H(0) = f$ and $H(1) = g$.

**Theorem**

*If k is a field, and R is a regular k-algebra, then*

$$[\operatorname{Spec} R, Gr_r]_{naive} \xrightarrow{\sim} \mathscr{V}_r(\operatorname{Spec} R);$$

- This theorem is much harder than the topological analog: for $R = k[t_1, \ldots, t_n]$ it follows from the Quillen–Suslin solution to Serre's problem. The general case uses Lindel's verification of the Bass–Quillen conjecture in the geometric case.
- Naive homotopy can be badly behaved (e.g., it is not an equivalence relation, in general).

- The restriction to regular rings is necessary in order to have an analog of homotopy invariance.
- $k$ a fixed base-ring; $\mathrm{Sm}_k$, the category of smooth $k$-varieties
- $\mathrm{Sm}_k$ is not "big enough" to do homotopy theory (e.g., $Gr_r$ is not in this category, cannot form all quotients, etc.)
- $\mathrm{P}(\mathrm{Sm}_k)$ - space-valued presheaves on $\mathrm{Sm}_k$ an enlargement of $\mathrm{Sm}_k$ where we can perform all the constructions we will want later
- We now force two kinds of maps to be "equivalences":
  - Nisnevich local equivalences: roughly, if $X$ can be covered by $\{U_i\}$, then we may build a Cech object $C(u) \to X$, and we force $C(u) \to X$ to be an isomorphism
  - $\mathbb{A}^1$-equivalences: $X \times \mathbb{A}^1 \to X$
- $\mathrm{Spc}_k$ for the category of *motivic spaces* obtained by inverting both Nisnevich local and $\mathbb{A}^1$-weak equivalences; we write $[-,-]_{\mathbb{A}^1}$ for maps in the associated homotopy category (the Morel–Voevodsky motivic homotopy category)

If $k$ is regular, one may show $[X, \mathbb{P}^{\infty}]_{\mathbb{A}^1} = Pic(X)$ for any smooth $k$-scheme $X$. Unfortunately, this fails for $Gr_r$ with $r \geq 2$. Nevertheless,

**Theorem (Affine representability)**

*If $k$ is a field or $\mathbb{Z}$, then for any smooth affine $k$-scheme $X = \operatorname{Spec} R$,*

$$[\operatorname{Spec} R, Gr_r]_{naive} = [\operatorname{Spec} R, Gr_r]_{\mathbb{A}^1} \xrightarrow{\sim} \mathscr{V}_r(\operatorname{Spec} R).$$

- Morel '06 if $r \neq 2$ and $k$ a perfect field
- Schlichting '15 arbitrary $r$, $k$ perfect; simplifies part of Morel's argument
- A.–M. Hoyois–M. Wendt '15 (essentially self-contained: in essence, the theorem is equivalent to the Bass–Quillen conjecture for all smooth algebras over $k$)

**Theorem (F. Morel, L.F. Moser, M. Wendt, A.-Hoyois-Wendt)**

*If $k$ is either a field or a Dedekind ring with perfect residue fields, then there is an $\mathbb{A}^1$-fiber sequence of the form*

$$\mathbb{A}^r \setminus 0 \longrightarrow Gr_{r-1} \longrightarrow Gr_r$$

- We can build a version of obstruction theory in $\mathbb{A}^1$-homotopy theory.
- We need information about motivic homotopy type $\mathbb{A}^r \setminus 0$.

# Splitting algebraic vector bundles II: a motivic story

# Homotopy theory of $\mathbb{A}^n \setminus 0$

- There are two circles in $\mathbb{A}^1$-homotopy theory: $S^1$ and $\mathbb{G}_m$.
- There are equivalences in $\mathrm{Spc}_k$ of the form $\mathbb{P}^1 \sim S^1 \wedge \mathbb{G}_m$ and $\mathbb{A}^n \setminus 0 \sim S^{n-1} \wedge \mathbb{G}_m{}^{\wedge n}$; note that $\mathbb{A}^{n+1} \setminus 0 \sim \mathbb{P}^1 \wedge \mathbb{A}^n \setminus 0$. Set:

$$S^{p,q} := S^{p-q} \wedge \mathbb{G}_m{}^{\wedge q}.$$

- We must consider homotopy *sheaves* not homotopy groups (gives correct form of the Whitehead theorem, etc.), but those associated with $S^i$ are "most important."

**Theorem (F. Morel '12)**

*For any $n \geq 2$, the sphere $\mathbb{A}^n \setminus 0$ is $(n-2)$-$\mathbb{A}^1$-connected.*

- Intuition: a space is $\mathbb{A}^1$-connected, if any two points can be connected by a chain of affine lines (over any extension of the base field).
- At least for $\mathbb{A}^n \setminus 0$, higher connectivity can be examined analogously.

# On Milnor-Witt K-theory

**Theorem (F. Morel '12)**

*For any $n \geq 2$, there is an isomorphism $\pi^{\mathbb{A}^1}_{n-1}(\mathbb{A}^n \setminus 0) \cong \mathbf{K}^{MW}_n$.*

- Vague idea of proof: the Hurewicz theorem tells us first non-vanishing homotopy (when it is abelian) coincides with homology; the suspension isomorphism tells us that homology is a "free sheaf of abelian groups" on ${\mathbb{G}_m}^{\wedge n}$; Morel then describes this in terms of generators and relations.

# The Euler class

**Theorem (F. Morel '12)**

*Suppose $k$ is a field, $X$ is a smooth affine $k$-variety of dimension $d$, and $\xi : E \to X$ is a rank $d$ algebraic vector bundle on $X$. There exists a canonically defined "Euler class"*

$$e(\xi) \in H^d(X, \mathbf{K}_n^{MW}(\det \xi)) = \widetilde{CH}^d(X, \det \xi)$$

*whose vanishing is sufficient to guarantee $E$ splits off a free rank 1 summand.*

**Proof.**

Obstruction theory! □

## Conjecture (M.P. Murthy '97)

*Suppose $k$ is an algebraically closed field and $d \geq 1$ is an integer. If $X$ is a smooth affine $k$-variety of dimension $d+1$, and $E$ is a rank $d$ vector bundle on $X$, then $E \cong E' \oplus \mathbf{1}_X$ if and only if $0 = c_d(E) \in CH^d(X)$.*

## Theorem (A.–T. Bachmann–M.J. Hopkins '24)

*If $k$ is an algebraically closed field having characteristic $0$, $X$ is a smooth affine $d$-fold over $k$, then Murthy's conjecture holds.*

(building on work of A.–J. Fasel)