Local Systems in Arithmetic Geometry

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Abstract

A building block of homotopy theory is the fundamental group of varieties, in topology and in arithmetic geometry. We know very little on it. One way to approach it is via local systems, that is linear representations modulo conjugation. In doing so we lose a lot of information. Still it yields obstructions for a finitely presented group to be the fundamental group of a smooth complex quasi-projective variety. Among those there are the ones coming from geometry, that is the motivic ones. Deep conjectures predict over various fields when local systems should but motivic.

Geometric groups

For X a smooth connected complex quasi-projective variety, we denote by $\pi_1(X, x)$ its group of homotopy classes of continuous loops based at x. Up to isomorphism, the group does not depend on the choice of x. At path γ_{xy} from x to y induces an isomorphism $\gamma_{xy}^{-1} \circ \pi_1(X, y) \circ \gamma_{xy} = \pi_1(X, x)$ and another path is of the shape $\gamma_{xy} \circ \gamma_{xx}$. We write $\pi_1(X)$ for its isomorphism class.

$X(\mathbb{C})$ has the homotopy type of a finite CW complex, so is $\pi_1(X, x)$ finitely presented (**fp**). Any fp group is the fundamental group of a finite CW complex (the 1-cells are the generators, the 2 cells the relations), vice-versa the fundamental group of a finite CW complex is fp.

Definition

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General problem

How to recognize the geometric groups among all fp groups?

Magnitude

There is no general answer to this problem, which is of the same magnitude as the Hodge conjecture in complex geometry, the Tate conjecture in arithmetic geometry etc...we do not even have a conjectural answer (unlike for those ‘abelian’ cases). So we look for **obstructions** for fp groups to be geometric.

Lecture 1

Examples

Any finite group is geometric (Serre): the group acts freely on a complete intersection of very high dimension and Lefschetz theory implies that those complete intersections as simply connected, i.e. $\pi_1 = \{1\}$.

So in particular the abelianization of any fp group $\mathbb{Z}^b \oplus \Gamma$, with Γ finite (abelian) is geometric, as π_1 is compatible with products and $\mathbb{Z}^b = \pi_1(\mathbb{C}^{\times b})$.

Abelian Harmonic Analysis: The Malčev completion of $\pi_1(X, x)$ admits a mixed Hodge structure.

Local Systems

This is a representation $\rho\colon \pi_1(X,x) \to \mathrm{GL}_r(\mathbb{C})$ modulo conjugacy by $\mathrm{GL}_r(\mathbb{C})$ (so may be written $\rho\colon \pi_1(X) \to \mathrm{GL}_r(\mathbb{C})$). We write $\mathbb{L}_\rho$ for this conjugacy class. Can be thought of as a topological fibration $\mathbb{L}_\rho \to X$ in $\mathbb{C}^r$ vector spaces, with transition functions defined by ρ, or simply $(X_x \times \mathbb{C}^r)/\pi_1(X,x)$ where $X_x \to X$ is the universal covering based at x and $\pi_1(X,x)$ acts diagonally.

We can make this definition *abstract* by replacing $\pi_1(X,x)$ by any fp group.

Lecture 1

Warning

Study of local systems reflects only a small part of $\pi_1(X, x)$.

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As $\pi_1(X, x)$ is fp, so in particular fg (finitely generated), a representation ρ has values in $\mathrm{GL}_r(A)$ for $A \subset \mathbb{C}$ a $\mathbb{Z}$-algebra of finite type, so $\rho = \rho_A \otimes_A \mathbb{C}$. But any such A possesses a $\mathcal{O}_v$ point, so $A \hookrightarrow \mathcal{O}_v$, where $\mathcal{O}_v$ is an ℓ-adic ring, so with finite residue field. But $\mathrm{GL}_r(\mathcal{O}_v)$ is profinite, thus $\rho_{\mathcal{O}_v} := \rho_A \otimes_A \mathcal{O}_v$ factors through $\hat{\rho}_{\mathcal{O}_v} : \widehat{\pi_1(X, x)} \to \mathrm{GL}_r(\mathcal{O}_v)$, where the profinite completion $\widehat{\pi_1(X, x)}$ is by the Riemann existence theorem the étale fundamental group pf X (based at x), and $\hat{\rho}_{\mathcal{O}_v}$ is continuous for the profinite topology.

But (Toledo): the kernel $K := \mathrm{Ker}\big(\pi_1(X, x) \to \widehat{\pi_1(X, x)}\big)$ of the profinite completion may be **non-trivial**.

Problem

We know **nothing** on K.

Lecture 1

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ℓ-adic local systems

Similarly we define ℓ-adic local systems as continuous representations $\hat{\rho}\colon \widehat{\pi_1(X)} \to \mathrm{GL}_r(\mathcal{O}_v)$ where $\mathcal{O}_v$ is an ℓ-adic ring. Absolute irreducibility means that $\hat{\rho} \otimes_{\mathcal{O}_v} \bar{\mathbb{Q}}_\ell$ is irreducible.

Definition (de Jong-E '23)

A fp group π is *weakly integral* if whenever there is an irreducible representation $\rho\colon \pi \to \mathrm{GL}_r(\mathbb{C})$ with determinant of finite order dividing $\delta \in \mathbb{N}_{\geq 1}$, then for all prime numbers ℓ, there is an absolutely irreducible

$$\hat{\rho}_v\colon \hat{\pi} \to \mathrm{GL}_r(\mathcal{O}_v)$$

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Main Group Theoretic Theorem (de Jong-E '23)

$$\pi \text{ } \textit{geometric} \Longrightarrow \pi \text{ } \textit{weakly integral}.$$

Corollary

Being weakly integral is an obstruction for a fp π to be geometric.

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Breuillard's examples

Lecture 1

$\Gamma_w = \langle a, b | w = 1 \rangle$ with $w = a^2ba^{-2}b^{-2}, \delta = 1, r = 2$. Let

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} j & 0 \\ 0 & j^2 \end{pmatrix}$$

where $j = e^{2i\pi/3}$ is a primitive cube root of unity. Then irreducible, conjugate over $\mathbb{Q}(j)$ (quadratic over $\mathbb{Q}$) yield the only 2 irreducible $SL_2(\mathbb{C})$ local systems. However

$$AB = \frac{j}{\sqrt{2}} \begin{pmatrix} 1 & j \\ -1 & j \end{pmatrix}$$

so $\mathrm{Tr}(AB) = -\frac{1}{\sqrt{2}} \notin \bar{\mathbb{Z}}_2$ so Γ_w is not weakly integral as it is not integral by $\ell = 2$!

Not check myself: Now choose $w = a^2ba^{-2}ba^{-4}b^{-3}$ and A, B replace by C, D:

$$C = -\frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} \quad D = \begin{pmatrix} a & \frac{\phi}{\sqrt{3}} \\ 0 & a^{-1} \end{pmatrix}$$

where $\phi = (1+\sqrt{5})/2$ is the golden mean, and a is a solution to the quadratic equation $a^2 + a\phi + 1 = 0$ (so that $a + a^{-1} = -\phi = 2\cos(\frac{4\pi}{5})$). It does $\ell = 3$!

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Problem

Can we find for each separate ℓ a Breuillard type example? (It is interesting first to justify Breuillard's two statements, thus for $\ell = 2$ and for $\ell = 3$, above! See Ben's TA session.)

Non-Abelian Harmonic Analysis; Simpson's theory on X smooth connected **projective** over $\mathbb{C}$

Simpson constructed via GIT moduli spaces $M_B(X,r)$, $M_{dR}(X,r)$, $M_{Dol}(X,r)$ of rank r semi-simple local systems $\mathbb{L}$, semi-simple flat connections (E,∇), semi-stable Higgs bundles (V,θ) with vanishing Chern classes. $M_B(X,r)$ depends only on $\pi_1(X)$, while $M_{dR}(X,r)$, $M_{Dol}(X,r)$ depend on the analytic structure. He extended the complex analytic Riemann-Hilbert isomorphism between $M_B(X,r)$ and $M_{dR}(X,r)$ to a real analytic isomorphism between $M_{dR}(X,r)$ and $M_{Dol}(X,r)$. The Hitchin map $h: M_{Dol}(X,r) \to \mathbb{A}^N$, where

$$N = \dim_{\mathbb{C}} H^0(X, \oplus_{i=1}^r \mathrm{Sym}^i \Omega^1_X)$$

is proper, $M_B(X,r)$ is affine. See Ben's TA session.

It has tremendous consequences. e.g. if $N = 0$ then the 3 spaces are 0-dimensional. This is the case e.g. if X is a Shimura variety of exceptional type (Margulis superrigidity).

These isomorphisms extend to the stable locus on all sides, which are then fine moduli. For B those points are the *irreducible* local systems, so dR they are *simple* flat connections, for Dol there are stable Higgs bundles with vanishing Chern classes.

It yields an **obstruction for a fp group π to be geometric with X projective**: its (affine) character variety $M_B(\pi, r)$ should be real analytically endowed with a proper map h to an affine space.

The correspondence holds if we fix a order δ of the determinant as above. The theory does not quite extend in this shape to the non-proper case.

Betti moduli with boundary and determinant conditions

Fix $X \subset \bar{X}$ a good compatification, with $\bar{X} \setminus X = \cup D_i$ a strict normal crossings divisor. We fix δ so $\det(\mathbb{L})^\delta = \mathbb{I}$. We also fix the eigenvalues $\lambda_{ij} \in \mu_\infty \subset \mathbb{C}^\times$ of the monodromies (assumed to be quasi-unipotent) at infinity. The resulting Betti moduli is an affine finite type variety defined over $\mathbb{Z}$, on which GL_r acts by conjugacy. Then $M_B^{\mathrm{all}}(X, r; \delta, \lambda_{ij})$ is the GIT quotient. It is an affine finite type variety defined over $\mathbb{Z}$. Its stable points over $\mathbb{C}$ correspond to irreducible local systems with the (δ, λ_{ij}) conditions. Its points over $\mathbb{C}$ correspond to semi-simple local systems with the (δ, λ_{ij}) conditions. We then consider the *open* $M_B(X, r; \delta, \lambda_{ij})$ *in* $M_B^{\mathrm{all}}(X, r; \delta, \lambda_{ij})$ *which is the complement in* $M_B^{\mathrm{all}}(X, r; \delta, \lambda_{ij})$ *of the Zariski closure of the non-absolutely irreducible points in characteristic* 0.

Lecture 2

We denote by
$\epsilon\colon M_B(X, r; \delta, \lambda_{ij}) \to \mathrm{Spec}(\mathbb{Z}),\ \epsilon^{\mathrm{all}}\colon M_B^{\mathrm{all}}(X, r; \delta, \lambda_{ij}) \to \mathrm{Spec}(\mathbb{Z})$
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Main Geometric Theorem (de Jong-E '23)

ϵ *dominant* $\Longrightarrow$ for all prime numbers ℓ, ϵ^{all} has a $\bar{\mathbb{Z}}_\ell$-point which meets $M_B(X,r;\delta,\lambda_{ij}) \otimes_{\mathbb{Z}} \mathbb{C}$ non-trivially.

In fact the theorem is stronger and equivalent to it under a geometric assumption in $M_B(X,r;\delta,\lambda_{ij})$.

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Diophantine Corollary (Rumely's Theorem)

If $M_B(X,r;\delta,\lambda_{ij})(\mathbb{C})$ is non-empty and consists of irreducible local systems, and if $M_B^{\mathrm{all}}(X,r;\delta,\lambda_{ij})$ and $M_B^{\mathrm{all}}(X,r;\delta,\lambda_{ij}) \otimes_{\mathcal{O}} \mathbb{C}$ are irreducible, then $M_B^{\mathrm{all}}(X,r;\delta,\lambda_{ij})$ possesses an integral (i.e. $\bar{\mathbb{Z}}$) point which meets $M_B(X,r;\delta,\lambda_{ij}) \otimes_{\mathbb{Z}} \mathbb{C}$ non-trivially.

Lecture 2

Problem

The conditions in the Diophantine Corollary are difficult to meet. Can we find examples where the conditions are met? Can we do a little better? *We thank Peter Sarnak for his interest and questions on the diophantine consequences of our Main Geometric Theorem.*

Initial dream with Johan: Prove or disprove that $M_B(X, r; \delta) \to \mathrm{Spec}(\mathbb{Z})$ is pure in the sense of Gruson-Peskine (would be sufficient for Diophantine Applications). Impossible in the present state of knowledge (the problem being the use of companions)

de Jong's conjecture

It predicts that an arithmetic $\mathrm{GL}_r(\overline{k_v[[t]]})$ representation, where k_v is a finite char. ℓ field and X is defined over $\mathbb{F}_q$ of char. prime to ℓ, is geometrically finite. From the conjecture, Johan deduces the following: let $\bar\rho\colon \pi_1^t(X_{\bar{\mathbb{F}}_p}) \to \mathrm{GL}_r(k_v)$ be an absolutely irreducible representation of the tame fundamental group (i.e. tame on all curves, i.e. the Galois cover defined by $\bar\rho$ is such that in restriction to all curves, the wild ramification index at punctures is trivial and the residual field extension is separable), $\mathrm{Def}(\bar\rho)$ be Mazur's deformation's space which represents the functor assigning to a formal local $W(\mathbb{F}_v)$-algebra $\hat R$ with residue field k_v the set of equivalence classes of representations $\pi_1(X_{\bar{\mathbb{F}}_p}) \to \mathrm{GL}_r(\hat R)$ lifting $\bar\rho$. It is a formal scheme formally of finite type over $W(\mathbb{F}_v)$. We can also decorate the functor with a determinant condition and conditions at infinity etc. See Ben's TA session.

$\bar{\mathbb{Q}}_\ell$-points of $\mathrm{Def}(\bar{\rho})$ are precisely ℓ-adic local systems with residual local system equals to $\bar{\rho}$. The group $\pi_1^t(X_{\bar{\mathbb{F}}_p})$ is topologically finitely generated and $\mathrm{GL}_r(k_v)$ is finite, thus there are only finitely many such $\bar{\rho}$, thus a power m say of Frobenius Φ fixes $\bar{\rho}$ thus $\mathrm{Def}(\bar{\rho})$. Replacing $\mathbb{F}_q$ by $\mathbb{F}_{q^m}$, Φ acts $\mathrm{Def}(\bar{\rho})$ and we can study its fix points. Those are precisely the ℓ-adic local systems which lift $\bar{\rho}$ and are *arithmetic*, that is descend to $X/\mathbb{F}_q$ once we have assumed that $X(\mathbb{F}_q) \neq \varnothing$ to represent $\pi_1(X)$ as an extension of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_q) = \hat{\mathbb{Z}}$ with $\pi_1(X_{\bar{\mathbb{F}}_p})$.

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de Jong's Theorem

If $\mathrm{Def}(\bar\rho)_{\mathrm{red}}$ if formally smooth, then $\mathrm{Def}(\bar\rho)^{\Phi}_{\mathrm{red}} \to \mathrm{Spf}(W(\mathbb{F}_v))$ is finite flat. Moreover $\cup_{n\in\mathbb{N}_{\geq 1}}\mathrm{Def}(\bar\rho)^{\Phi^n}_{\mathrm{red}} \subset \mathrm{Def}(\bar\rho)$ is Zariski dense.

Similarly with decoration.

de Jong's conjecture implies: Main Geometric Theorem $\Longrightarrow$ Main Group Theoretic Theorem

First let us remark that there is a big difference between the two theorems. In the geometric one, we do have the loops γ_i at infinity while in the group theoretic one, π is abstract, so there are no privileged γ_i. Should it be geometric, it means in particular that in it, there are hidden some γ_i with Γ_i quasi-unipotent etc.

We use the notation $M_B(X, r; \delta)$ dropping the λ_{ij} conditions at infinity. This defines a forgetful morphism $M_B(X, r; \delta, \lambda_{ij}) \hookrightarrow M_B(X, r; \delta)$ which is a closed embedding. Note again the left hand side sees the geometry as we need the boundary divisor to define it, the right hand side depends solely on $\pi_1(X)$. As a application of de Jong's conjecture proved by Gaitsgory in general (for $p \geq 3$) one obtains:

Lecture 2

Proposition

$$\cup_{\lambda_{ij}\in\mu_\infty} M_B(X, r; \delta, \lambda_{ij}) \subset M_B(X, r; \delta)$$

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Thus if ϵ on $M_B(X, r; \delta)$ is dominant, so is it on some $M_B(X, r; \delta, \lambda_{ij})$ and we can apply the geometric theorem.

Lecture 3

The goal of this Lecture is to
sketch the proof of the Main Geometric Theorem.

Companions

We refer to our **Lecture Notes 2337**, Sections 7.1,7.2,7.3 for the *motivation on the complex side of the existence of companions*, for the meaning in rank one and for some remarks on geometricity.

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Motivation on the Complex Side

Given a field automorphism τ of $\mathbb{C}$, we can postcompose the underlying monodromy representation of a complex local system $\mathbb{L}_{\mathbb{C}}$ by τ to define a *conjugate* complex local system $\mathbb{L}_{\mathbb{C}}^{\tau}$.

Topology on the Field of Coefficients

Given a field automorphism σ of $\bar{\mathbb{Q}}_\ell$, which then can only be continuous if this is the identity on $\mathbb{Q}_\ell$, or more generally given a field isomorphism σ between $\bar{\mathbb{Q}}_\ell$ and $\bar{\mathbb{Q}}_{\ell'}$ for some prime number ℓ', the postcomposition of a *continuous* monodromy representation is no longer continuous (unless $\ell = \ell'$ and σ is the identity on $\mathbb{Q}_\ell$), so *we cannot define a conjugate* $\mathbb{L}_\ell^\sigma$ of an ℓ-adic local system by this simple postcomposition procedure.

Returning to the complex side, as a consequence of $\pi_1(X(\mathbb{C}))$ being finitely generated (we do not even need the finite presentation here), there are finitely many elements $(\gamma_1, \ldots, \gamma_s) \in \pi_1(X(\mathbb{C}))$ such that the characteristic polynomial map

$$M_B(X,r)^{\square}_{\mathbb{C}} \xrightarrow{\psi^{\square}} N_{\mathbb{C}} = \prod_{i=1}^{s} (\mathbb{A}^{r-1} \times \mathbb{G}_m)_{\mathbb{C}}$$

$$\rho \mapsto (\det(T - \rho(\gamma_1)), \ldots, \det(T - \rho(\gamma_s)))$$

which factors through $M_B(X,r)_{\mathbb{C}} \xrightarrow{\psi} N_{\mathbb{C}}$ has the property that ψ is a *closed embedding*. The reason is that a semi-simple representation is determined uniquely up to conjugation by the characteristic polynomial function on all $\gamma \in \pi_1(X(\mathbb{C}), x(\mathbb{C}))$.

By finite generation of $\pi_1(X(\mathbb{C}))$, finitely many suitably chosen ones among them are enough to recognize a semi-simple representation up to conjugation. So denoting $\tau \circ \det(T - \rho(\gamma))$ by $\det(T - \rho(\gamma))^\tau$ to unify the notation, we can summarize the discussion as follows:

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An automorphism τ of $\mathbb{C}$ yields a commutative diagram

$$\begin{array}{ccccc} M_B^{irr}(X,r)(\mathbb{C}) & \xrightarrow{\text{incl.}} & M_B(X,r)(\mathbb{C}) & \xrightarrow{\psi} & N(\mathbb{C}) \\ \downarrow{\scriptstyle \mathbb{L}_{\mathbb{C}} \mapsto \mathbb{L}_{\mathbb{C}}^\tau} & & \downarrow{\scriptstyle \mathbb{L}_{\mathbb{C}} \mapsto \mathbb{L}_{\mathbb{C}}^\tau} & & \downarrow{\scriptstyle (-)^\tau} \\ M_B^{irr}(X,r)(\mathbb{C}) & \xrightarrow{\text{incl.}} & M_B(X,r)(\mathbb{C}) & \xrightarrow[\psi]{} & N(\mathbb{C}) \end{array}$$

The upper script ^{irr} means the irreducible locus.

Analogy with a Finite Field

Let us now assume that X is smooth quasi-projective over a finite field $\mathbb{F}_q$. We denote by p the characteristic of $\mathbb{F}_q$. We fix a prime ℓ different from p. We denote by $M_\ell^{irr}(X_{\bar{\mathbb{F}}_p}, r)$ the *set* of all rank r ℓ-adic (rather Weil but we do not discuss this) local systems $\mathbb{L}_\ell$ defined over $X_{\bar{\mathbb{F}}_p}$ which

- are arithmetic, that is descend to some $X_{\mathbb{F}_{q'}}$ for some finite extension
$$\mathbb{F}_q \subset \mathbb{F}_{q'} (\subset \bar{\mathbb{F}}_p);$$
- • on $X_{\mathbb{F}_{q'}}$ are irreducible over $\bar{\mathbb{Q}}_\ell$.

Lecture 3

We give ourselves an abstract field isomorphism $\sigma : \bar{\mathbb{Q}}_\ell \xrightarrow{\cong} \bar{\mathbb{Q}}_{\ell'}$. The only "continuous"' information it contains is that it sends a number field $K \subset \bar{\mathbb{Q}}_\ell$ to another one $K^\sigma \subset \bar{\mathbb{Q}}_{\ell'}$. So the right vertical arrow $(-)^\sigma$ makes sense only on a $\gamma \in \pi_1(X_{\mathbb{F}_{q'}}, x_{\bar{\mathbb{F}}_p})$, where $\mathbb{F}_q \subset \mathbb{F}_{q'} (\subset \bar{\mathbb{F}}_p)$ is a finite extension, which has the property that *the characteristic polynomial of γ has values in a number field.*

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Furthermore, we wish the set of such γ to be able to recognize completely $M_\ell^{irr}(X, r)$ as ψ does over $\mathbb{C}$. We know the following fact: The set of *conjugacy classes of the Frobenii at all closed points $|X|$ of X* has those two properties (L. Lafforgue and Čebotarev.)

So we set $N^\infty = \prod_{|X|}(\mathbb{A}^{r-1} \times \mathbb{G}_m)$ and ψ^∞ for the characteristic polynomial map on those Frobenii of all closed points. Then *Deligne's companion conjecture may be visualized on the diagram*

$$\begin{array}{ccc} M_\ell^{irr}(X,r) & \xrightarrow{\psi^\infty} & N^\infty(\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_\ell) \\ {\scriptstyle ?\exists\ \mathbb{L}_\ell \mapsto \mathbb{L}_\ell^\sigma} \downarrow & & \downarrow {\scriptstyle (-)^\sigma} \\ M_{\ell'}^{irr}(X,r) & \xrightarrow[\psi^\infty]{} & N^\infty(\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_{\ell'}) \end{array}$$

The wished $\mathbb{L}_\ell^\sigma$ is called the *companion* of $\mathbb{L}_\ell$ for σ. This is Deligne's companion conjecture proved by Drinfeld-L. Lafforgue (arithmetic Langlands) on curves and Drinfeld in higher dimension.

Algebraic Geometry Facts

M of finite type $\Longrightarrow \epsilon$ generically smooth on M_{red}. Let $z \in M$ be a closed point, of residue field $\mathbb{F}_{\ell^m}$. It corresponds to an absolutely irreducible

$$\bar{\rho}\colon \pi_1(X) \to \mathrm{GL}_r(\mathbb{F}_{\ell^m}).$$

Lecture 3

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Grothendieck's specialization

For a good model of X, Grothendieck's specialization homomorphism $\Longrightarrow$ factorization

$$\bar{\rho}: \pi_1(X) \xrightarrow{\mathrm{sp}} \pi_1^t(X_{\bar{\mathbb{F}}_p}) \xrightarrow{\bar{\rho}_{\bar{\mathbb{F}}_p}} \mathrm{GL}_r(\mathbb{F}_{\ell^m}).$$

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Mazur's Deformation Space $\mathrm{Def}(X_{\bar{\mathbb{F}}_p}, \bar{\rho}_{\bar{\mathbb{F}}_p}; \delta, \lambda_{ij})$

See Ben's TA session

$$\widehat{M}_z \cong \mathrm{Def}(X_{\bar{\mathbb{F}}_p}, \bar{\rho}_{\bar{\mathbb{F}}_p}; \delta, \lambda_{ij})$$

as $W(\mathbb{F}_v)$-local formal schemes, with residue field $\mathbb{F}_v = \mathbb{F}_{\ell^m}$.

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argument stemming from the proof of Simpson's integrality conjecture with Michael Groechenig: for any algebraic isomorphism $\sigma : \bar{\mathbb{Q}}_\ell \cong \bar{\mathbb{Q}}_{\ell'}$ construct the companion (which respects after Deligne the boundary conditions and the determinant) and consider sp^{-1} of it. **It yields the theorem for all $\ell \neq p$.**

Lecture 3

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there is a ℓ-adic local system $\mathbb{L}$ on $X_{\mathbb{F}_q}$ with residual local system $\bar{\rho}_{\bar{\mathbb{F}}_p}$.

use companions:

argument stemming from the proof of Simpson's integrality conjecture with Michael Groechenig: for any algebraic isomorphism $\sigma : \bar{\mathbb{Q}}_\ell \cong \bar{\mathbb{Q}}_{\ell'}$ construct the companion (which respects after Deligne the boundary conditions and the determinant) and consider sp^{-1} of it. **It yields the theorem for all $\ell \neq p$.**

catching $\ell = p$:

redo for a larger p. □

To: the organizers, Benjamin Church, all of you: