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Positive Scattering Amplitudes

Based on work with Prashanth Raman (Max Planck Institute for Physics)

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Why do amplitudes have so few numerical features?

Example: Remainder function in N=4 super Yang-Mills (sYM)

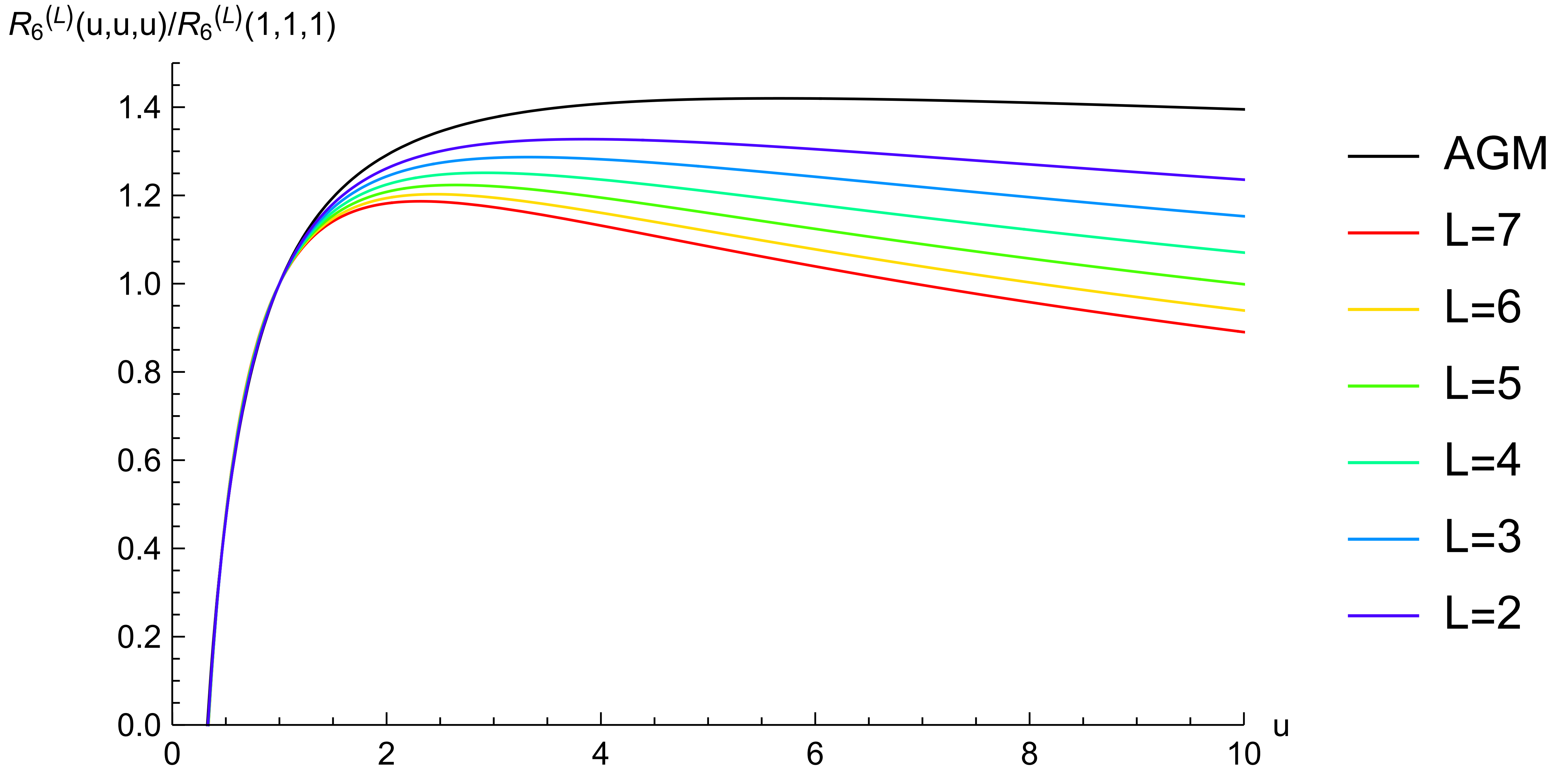
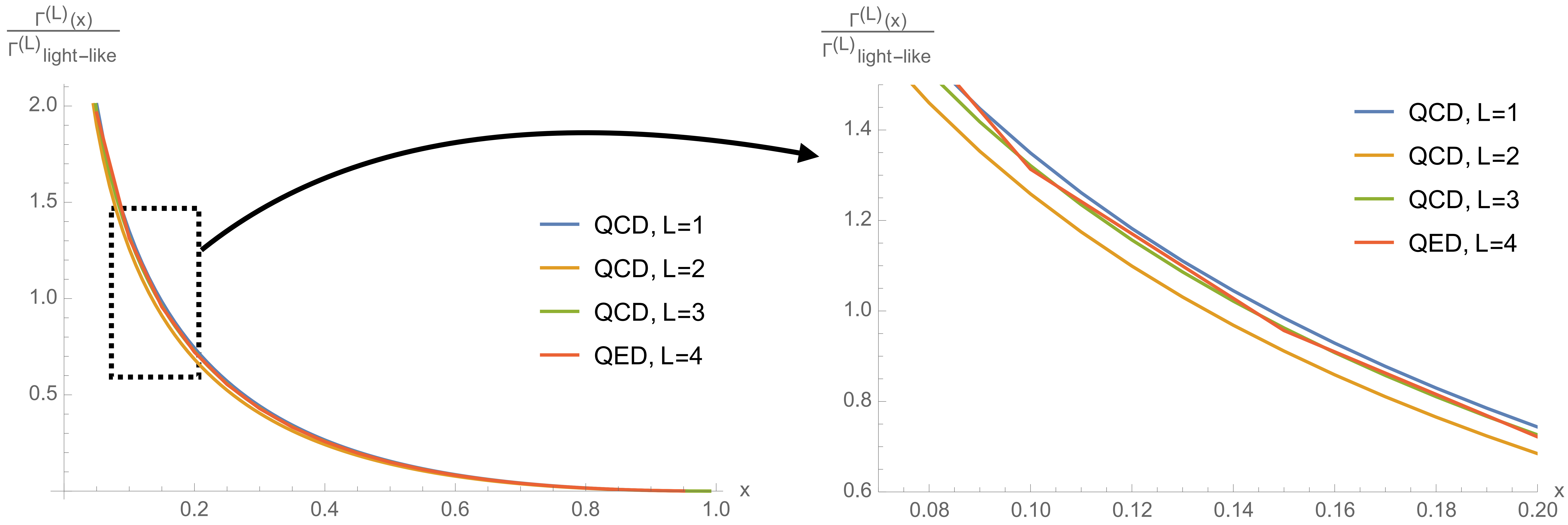


Figure 4: Normalized perturbative coefficients of the remainder function, $R_6^{(L)}(u,u,u)/R_6^{(L)}(1,1,1)$, for $L = 2$ to 7, plotted along with the strong-coupling result of AGM. The curves all have a remarkably similar shape for $u \lesssim 1$.

Figure from [Caron-Huot *et al*, 2005.06735]

Why do amplitudes have so few numerical features?

Angle-dependent ($x = e^{i\phi}$) QCD and QED cusp anomalous dimensions



Based on analytic QCD and QED expressions from [Grozin *et al*, 1409.0023],[Brüser *et al*, 2007.04851].

Completely monotonous functions for scattering amplitudes

Let me introduce functions with an *infinite number of positivity properties*:

$$(-\partial_x)^n f(x) \geq 0\,, \quad \text{for} \quad n \geq 0\,, x \in I \subset \mathbb{R}$$

All their signed derivatives are positive.

In particular, such functions are:

- positive: $f(x) \geq 0$
- monotonically decreasing: $\partial_x f(x) \leq 0$
- convex: $\partial_x^2 f(x) \geq 0$

We will see that such functions, called *completely monotonous*, appear in several scattering amplitudes and Feynman diagrams.

Examples of completely monotonous (CM) functions

Simple examples:

$$\frac{1}{x}, \frac{1}{1+x}, \frac{\log x}{x-1} \quad \text{on } x \in (0,\infty). \quad -\mathrm{Li}_2\left(1-\frac{1}{x}\right) \text{ on } x \in (0,1)$$

Closure under products and convex linear combinations:

$$f_1(x) f_2(x)\,, \quad c_1 f_1(x) + c_2 f_2(x)\,, \quad c_1, c_2 \geq 0$$

are CM if $f_1(x)\,, f_2(x)$ are CM.

Also closure under differentiation and integration (with a condition on the choice of integration constant).

Haussdorff-Bernstein-Widder theorem (HBW)

A function $f(x)$ is completely monotonic on $x \in (0,\infty)$ if and only if it is the Laplace transform of a non-negative function $\mu(t)$.

$$f(x) = \int_0^\infty e^{-tx}\mu(t)dt\,.$$

[Widder, *Absolutely and Completely Monotonic Functions*, Chapter 4, Princeton University Press, 1941]

Example: $f(x) = \dfrac{\log x}{x-1}$ has $\mu(t) = \displaystyle\int_0^\infty \frac{e^{-ty}}{y+1}dy \geq 0.$

Extension to multiple variables

Complete monotonicity in n variables x_i:

$$(-1)^{\sum_{i=1}^{n} m_i} \partial_{x_1}^{m_1} \ldots \partial_{x_n}^{m_n} f(x_1, \ldots, x_n) \geq 0 \,.$$

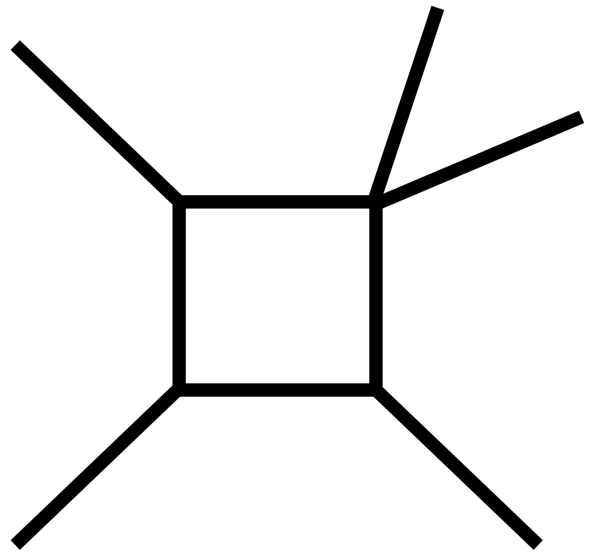
The HBW theorem exists for this multi-variable case as well.

Two-variable example:

$$f(x_1, x_2) = \frac{1}{1 - x_1 - x_2} \left[\mathrm{Li}_2(1 - x_1) + \mathrm{Li}_2(1 - x_2) + \log x_1 \log x_2 - \zeta_2 \right]$$

is CM on $x_1, x_2 > 0$.

This function appears in one-loop integrals:



Abundance of CM functions in quantum field theory

(1) Positive Geometry motivates that we think of scattering amplitudes as volumes. Evidence for positive integrated amplitudes in planar N=4 sYM was found in Amplituhedron kinematics.

[Arkani-Hamed, Hodges, Trnka, 1412.8478] [Dixon, von Hippel, McLeod, Trnka, 1611.08325]

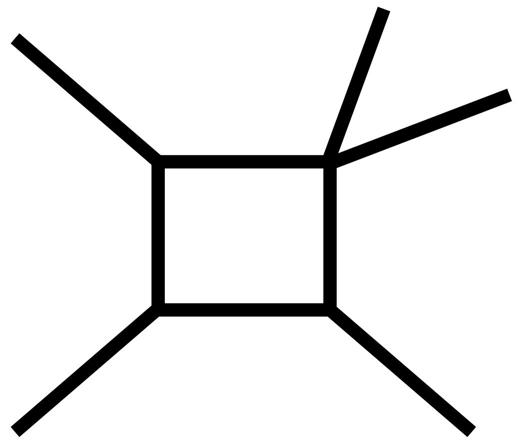
In this talk, we discuss evidence of a CM property for this and several other quantities in quantum field theory (QFT).

(2) The CM property appears in many QFT building blocks: Any scalar Feynman diagram in Euclidean kinematics has this property.

(3) There is a close connection to integral representations, such as dispersion relations.

CM functions from integral representations

Back to two-variable example from a Feynman integral.



$$f(x_1, x_2) = \frac{1}{1 - x_1 - x_2} \left[\mathrm{Li}_2(1 - x_1) + \mathrm{Li}_2(1 - x_2) + \log x_1 \log x_2 - \zeta_2 \right]$$

This has the dispersive (Mandelstam) representation:

$$f(x_1, x_2) = \int_0^\infty \int_0^\infty \frac{dy_1 dy_2}{(x_1 + y_1)(x_2 + y_2)(1 + y_1 + y_2)}$$

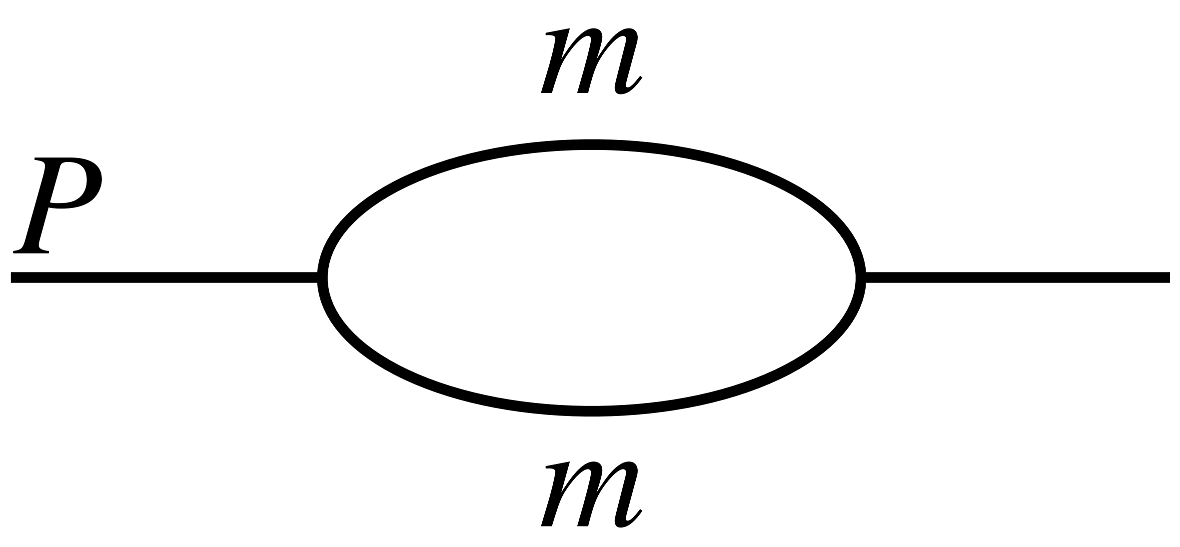
The CM property follows if one assumes that taking derivatives and integrations may be interchanged.

Feynman integrals with Euclidean region

The CM property holds for any scalar Feynman diagram that has a Euclidean region. This follows from the Feynman parametrization formula.

Example (massive bubble diagram):

$$f(x_1, x_2) = \int_0^\infty \frac{d\alpha_1 d\alpha_2}{\mathrm{GL}(1)} \frac{1}{x_1\alpha_1\alpha_2 + x_2(\alpha_1 + \alpha_2)^2},$$



is CM in $x_1 = -P^2, x_2 = m^2$, for $x_1, x_2 > 0$.

This widely applicable property may be useful for constraining Feynman integrals via semidefinite programming, as in [Zeng, 2303.15624] .

Other relevant occurrences of CM functions

- Non-planar integrals with a Euclidean region
- Cosmological Correlators, e.g. [Arkani-Hamed et al, 2312.05303]

$$\Psi_{\mathrm{FRW}} \propto \int_0^\infty \int_0^\infty \frac{dx_1 dx_2 (x_1 x_2)^\epsilon}{(X_1 + X_2 + x_1 + x_2)(X_1 + x_1 + Y)(X_2 + x_2 + Y)}$$

- Stringy integrals, e.g. [Arkani-Hamed, He, Lam, 1912.08707]
- Gelfand-Aomoto hypergeometric functions

[Kozhasov, Michałek, Sturmfels, 1908.04191]

- …

Recap:

We discussed completely monotonic functions (CM), which satisfy an infinite number of positivity conditions.

Although this is a strong constraint, we saw that many building blocks in quantum field theory (QFT) have this property.

QFT observables are typically linear combinations of such building blocks, with plus and minus signs. Are any of them CM?

Positivity of six-particle MHV amplitudes in N=4 sYM

The Amplituhedron determines the loop integrands, which are rational functions. These are positive within the Amplituhedron region.
[Hodges, 0905.1473],[Arkani-Hamed and Trnka, 1312.2007]

What happens when one integrates the integrand over Minkowski space?
[Arkani-Hamed, Hodges, Trnka, 1412.8478] found evidence that the finite part of *integrated* amplitudes is also positive, when evaluated within the tree Amplituhedron kinematic region.

There is a subtlety in which quantity to study, due to infrared divergences. For MHV amplitudes, [Dixon, von Hippel, McLeod, Trnka, 1611.08325] suggested to consider the 'BDS-like-subtracted remainder function' $\mathcal{E}(u, v, w)$.

Positivity of six-particle MHV amplitudes in N=4 sYM

The six-particle MHV Amplituhedron region is:

$$\Gamma: \quad u > 0, v > 0, w > 0, u + v + w < 1, (u + v + w - 1)^2 < 4uvw$$

[Dixon *et al*, 1611.08325] found that $(-1)^L \mathcal{E}^{(L)}(u, v, w) > 0$ in Γ, up to $L = 4$.

They showed this analytically in certain limits / kinematic slices, and by numerical evaluation for kinematic points in Γ. They also found evidence for monotonicity in a double scaling limit.

Two-loop positivity

For $u = v = w$, the Amplituhedron region becomes $0 < u < 1/4$.

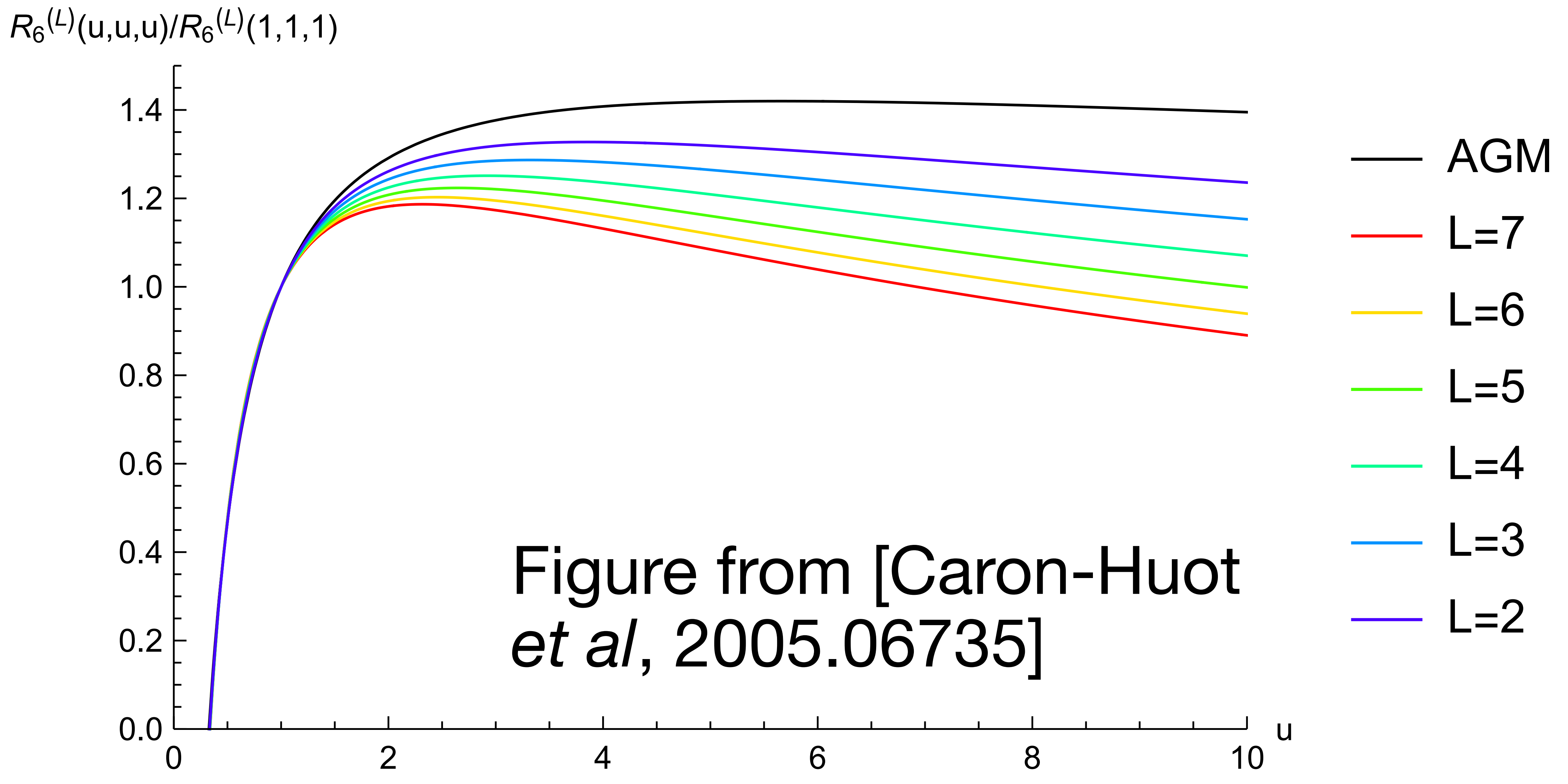
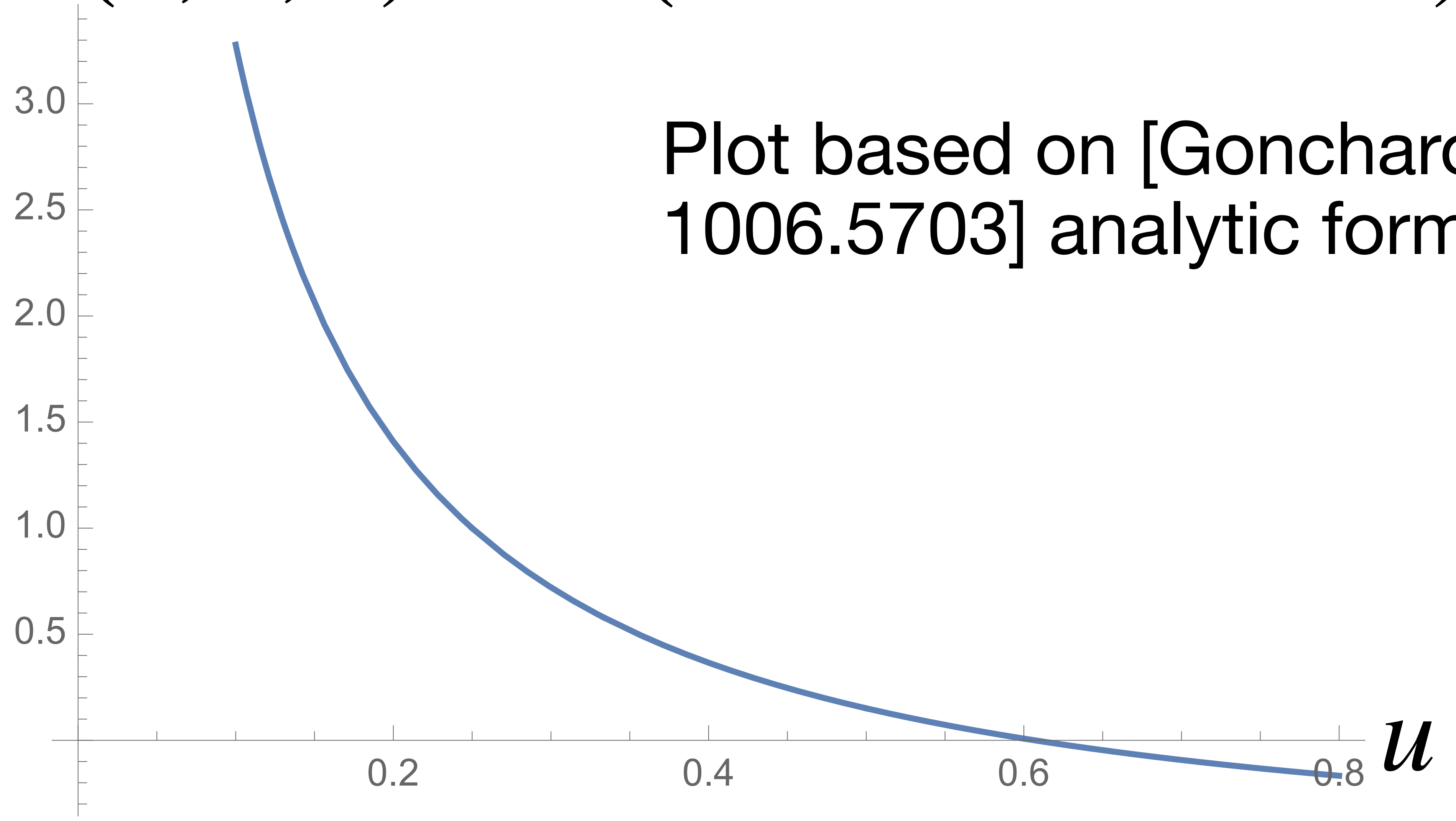


Figure from [Caron-Huot *et al*, 2005.06735]

$\mathcal{E}^{(2)}(u,u,u)/\mathcal{E}^{(2)}(u=v=w=1/4)$



Plot based on [Goncharov *et al*, 1006.5703] analytic formula

Function looks monotonically decreasing and convex in Γ. Can this be proven? Goal: show that $(-1)^L \mathcal{E}^{(L)}(u,v,w)$ is CM in Γ.

Proof of CM property at one loop

$-\mathcal{E}^{(1)}(u,v,w) = f(u) + f(v) + f(w)$ with $f(x) = -\mathrm{Li}_2(1-1/x)$

Note that Γ implies that $0 < u, v, w < 1$.

(1) $\dfrac{\log x}{(x-1)}$ and $\dfrac{1}{x}$ are CM for $0 < x$. Therefore their product $\dfrac{\log x}{(x-1)x}$ is also CM.

(2) We have that $-\partial_x f(x) = \dfrac{\log x}{(x-1)x}$. This is CM.

(3) One checks that $f(1) = 0$.

This completes the proof that $f(x)$ is CM on $0 < x < 1$, and hence so is $-\mathcal{E}^{(1)}(u,v,w)$ in the Amplituhedron region Γ.

Sketch of CM property proof at two loops

We prove that $\mathcal{E}^{(2)}(u,v,w)$ is CM as follows:

(1) We use the following representation [Dixon, Drummond, Henn, 1111.1704]:

$$\mathcal{E}^{(2)}(u,v,w) = \Omega^{(2)}(u,v,w) + \tilde{r}(u) + \text{cyclic}$$

(2) Proof of CM property of $\tilde{r}(u)$ via a suitable dispersive representation for its first derivative.

(3) Proof that $\Omega^{(2)}(u,v,w)$ is CM via an integral representation:

$$\Omega^{(2)}(u,v,w) = \int_0^w H_6(u,v,t)dt + \Psi^{(2)}(u,v),$$

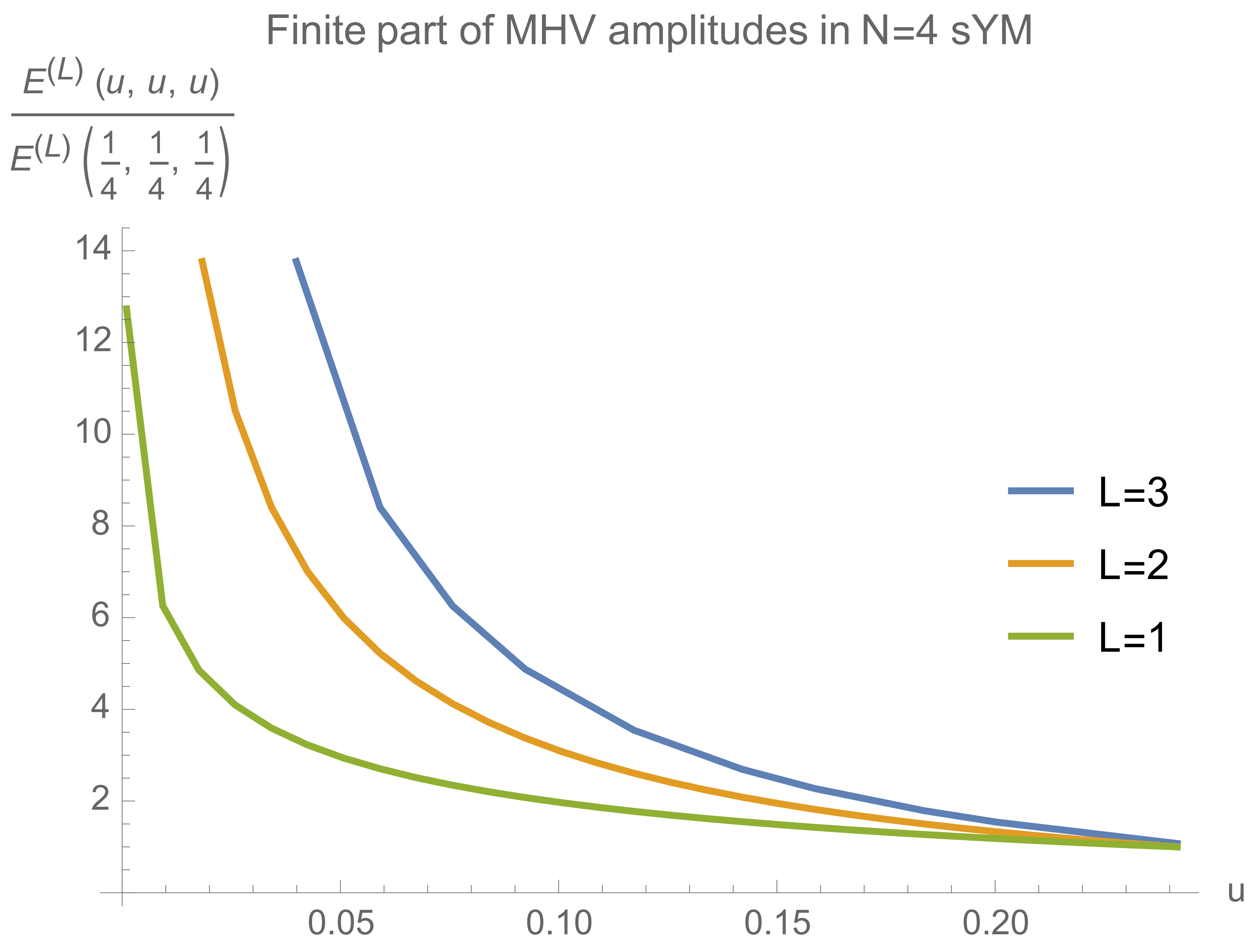
where H_6 and $\Psi^{(2)}$ are known CM functions.

Numerical CM evidence at higher loops

We verified positivity of $(-1)^L \mathcal{E}^{(L)}(u, v, w)$ and of its first two signed derivatives numerically at various points (u, v, w) chosen randomly within the Amplituhedron region Γ, for $L = 2,3$.

($L = 4$ check is in progress.)

Plot for symmetric configuration:



Evidence for CM property of other quantities

We have numerical evidence, and in some cases proofs, of the CM property for several further quantities in planar N=4 super-Yang-Mills:

- Four-point Wilson loop with Lagrangian insertion
- Angle-dependent cusp anomalous dimension
- Deformed Amplituhedron and Coulomb branch amplitudes
- Four-point correlation functions

Furthermore, the three-loop angle-dependent cusp anomalous dimension in QCD, and the four-loop QED one are CM.

Summary

We showed evidence that *completely monotonic (CM)* functions appear in many quantum field theory quantities.

In some cases there is an elementary explanation (positive integral representations), while in other cases the CM property was motivated by the Amplituhedron, or more generally by *Positive Geometry,* but remains to be proven.

Much more remains to be explored!

Research directions related to Complete Monotonicity (CM)

1) Explore data: in what cases does CM hold?

- Develop methods to systematically prove the CM property.
- Explore CM property for helicity configurations beyond MHV.

2) Understand from first principles why this property holds.

- Can the CM property be derived from Positive Geometry?
- What is the relationship to dispersion relations and partial wave analysis?

3) What is it good for?

- Use CM information for numerical approximation or bootstrap.
- Study implications for analytically continued kinematic configurations.

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Thank you!

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