Bespoke Unitarity and Colored Yukawa Theory

with Rishabh Bhardwaj, Shounak De, Andrzej Pokraka, Marcos Skowronek,
Anastasia Volovich, and He-Chen Weng

2406.04410, 2406.04411

M. Spradlin

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On Unitarity of Bespoke Amplitudes

arXiv: 2406.04410

w/ Rishabh Bhardwaj, Anastasia Volovich, and He-Chen Weng

(see Gong Show)

Motivation

The proximal stimulus for our work was 2308.03833 (Cheung & Remmen) which provided a general construction of 4-point, tree-level (ie meromorphic) amplitudes

that describe an infinite tower of higher spins,

with tame UV behavior,

string

satisfying dual resonance,

Motivation

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that describe an infinite tower of higher spins,

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satisfying dual resonance,

with (almost) arbitrarily customizable spectra.

string-like

Motivation

Their work, in turn, lies along a path trodden by many recent papers that explore various "string-like" amplitudes consistent with standard bootstrap constraints

Cheung, Hillman, Remmen, Berman, Elvang, ...

Geiser, Lindwasser, Bhardwaj, De, Jepsen,

Figueroa, Tourkine, Volovich, MS, ...

soon

t

⋮ many

Caron-Huot, Komargodski, Sever, Zhiboedov, ...

⋮ many

Motivation

Working back, the ultimate distal stimulus for this work is one of the earliest and most famous examples of amplitudeology:

$$A(s,t,u) = \frac{\bar{\beta}}{\pi}\left[B(1-\alpha(t), 1-\alpha(s)) + B(1-\alpha(t), 1-\alpha(u)) + B(1-\alpha(s), 1-\alpha(u))\right] \qquad (3)$$

where we have introduced the Euler B-function $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

Veneziano 1968

Scanned from the original manuscript of the paper "Construction of a crossing symmetric, Regge behaved amplitude for linearly rising trajectories" by Gabriele Veneziano, 1968

Bespoke Amplitudes are built from two ingredients

(1) A seed amplitude, which can be

$$A_1(s,t) = -\frac{\Gamma(-1-s)\Gamma(-1-t)}{\Gamma(-2-s-t)} \qquad \text{(bosonic open string)}$$

$$A_0(s,t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)} \qquad \text{(open superstring)}$$

[A_a has poles at $s = -a, -a+1, -a+2, \ldots$;
lowest-lying state has $m^2 = -a$; mod some prefactors]

Bespoke Amplitudes are built from two ingredients

(2) A choice of spectral curve that generalizes the linear spectrum of string theory to

$$m_n^2 = P(n)/Q(n)$$

where P & Q are (for now) arbitrary polynomials (with $\deg P > \deg Q$ to avoid having an accumulation point and, worse, non-locality).

Bespoke Amplitudes

The bespoke amplitude is then defined as

seed amplitude

$$A_{\text{bespoke}}(s,t) = \sum_{\alpha,\beta} A_a\left(\nu_\alpha(s), \nu_\beta(t)\right)$$

where $\{\nu_\alpha(\mu)\}$ are the roots of the spectral curve

$$P(\nu) - \mu\, Q(\nu) = 0.$$

Bespoke Amplitudes

Cheung and Remmen showed that these amplitudes satisfy dual resonance:

(1) The residue $R_n(t)$ of the amplitude on the pole $s = m_n^2 = P(n)/Q(n)$ is always a polynomial in t, and

Bespoke Amplitudes

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(1) The residue $R_n(t)$ of the amplitude on the pole $s = m_n^2 = P(n)/Q(n)$ is always a polynomial in t, and

(2) the high energy behavior is sufficiently tame that (at least for sufficiently negative t) the dual resonant representation is convergent

$$\sum_{n=0}^{\infty} \frac{R_n(t)}{s - m_n^2}$$

What Rishabh & He-Chen Did: Are Bespoke Amplitudes Unitary?

(1) A necessary, but far from sufficient, condition for unitarity in D spacetime dimensions is that each residue $R_n(t)$ must be a (finite) linear combination of Gegenbauer polynomials with non-negative coefficients.

(2) By adapting a double-contour integral representation for these coefficients from the Veneziano case (Arkani-Hamed, Eberhardt, Huang, Mizera) and using a similar, intricate contour deformation argument, it was possible to evaluate their asymptotic large-n behavior.

Results

Following CR let us parameterize the asymptotic spectrum as

$$m_n^2 \underset{n\to\infty}{\sim} (n+\delta)^p + k_1 (n+\delta)^{p-1} + k_2 (n+\delta)^{p-2} + \cdots$$

degree p, a positive integer

mass gap δ

"post-Regge parameters" k_i

Results

For asymptotically super-linear Regge trajectories $(p > 1)$

the coefficient of the spin-j Gegenbauer polynomial

$G_j^{(D)}$ in the n-th residue polynomial is

$$B_{n,j}^{a,(D)} \sim (\text{lots of stuff}) \cdot (-1)^j \qquad \text{as } n \to \infty$$

Results

For asymptotically super-linear Regge trajectories $(p>1)$

the coefficient of the spin-j Gegenbauer polynomial

$G_j^{(0)}$ in the n-th residue polynomial is

$$B_{n,j}^{a,(0)} \sim (\text{lots of stuff}) \cdot (-1)^j \qquad \text{as } n \to \infty$$

$\Rightarrow$ Infinitely many negative coefficients

$\Rightarrow$ non-unitary for all $p>1$

Results

For exactly linear, but with a shift δ

$$B_n \sim \left(1 - (-1)^{n+j}\, n^{3\delta}\, \frac{\Gamma(1-\delta+a)}{\Gamma(1+2\delta+a)}\right) \quad \text{as } n \to \infty$$

$\Rightarrow$ If $\delta > 0$ the second term will dominate, leading to infinitely many negative coefficients

$\Rightarrow$ unitarity demands $\delta \leq 0$

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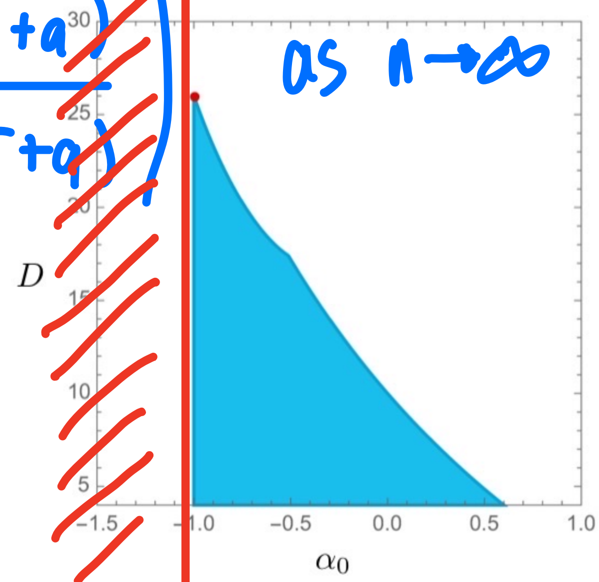


d

$\Rightarrow$ If $\delta > 0$ the second term will dominate, leading to infinitely many negative coefficients

$\Rightarrow$ unitarity demands $\delta \leq 0$

$\alpha_0 \geq -1$ in the context of Fig 1 of Arkani-Hamed, Cheung, Figueiredo, Remmen

Results

For the asymptotically linear spectrum, a combination of analytic results at large n, backed up by numerical checks to gain confidence in convergence of some asymptotic expansions, leads to the exclusion of a certain region in (δ, κ_1) parameter space.

$\Rightarrow$ see He-Chen's gong show talk.

Summary

Demanding the most primitive requirement of unitarity — non-negative partial wave coefficients — in the large-n fixed-j limit imposes constraints that complements those found at small-n, 1 allowing for example ruling out all asymptotically non-linear bespoke amplitudes.

Consistent with the no-go argument from Caron-Huot, Komargodski, Sever, Zhiboedov.

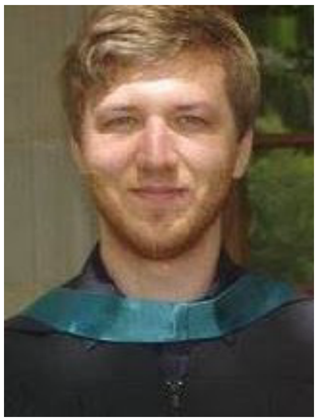
Summary

Demanding the most primitive requirement of unitarity — non-negative partial wave coefficients — in the large-n fixed-j limit imposes constraints that complements those found at small-n, 2 and those based on multi-particle factorization, e.g. by Arkani-Hamed, Cheung, Figueiredo, Remmen to rule out all asymptotically linear bespokes with $\deg(P) = 2$.

Surfaceology for Colored Yukawa Theory

arXiv: 2406.04411

w/ Shounak De, Andrzej Pokraka, Marcos Skowronek, and Anastasia Volovich



(see Gong Show)

Introduction to Surfaceology

Our paper builds on the three papers

All Loop Scattering as a Counting Problem

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Abstract: This is the first in a series of papers presenting a new understanding of scattering amplitudes based on fundamentally combinatorial ideas in the kinematic space of the scattering data. We study the simplest theory of colored scalar particles with cubic interactions, at all loop orders and to all orders in the topological 't Hooft expansion. We find a novel formula for loop-integrated amplitudes, with no trace of the conventional sum over Feynman diagrams, but instead determined by a beautifully simple counting problem attached to any order of the topological expansion. These results represent a significant step forward in the decade-long quest to formulate the fundamental physics of the real world in a radically new language, where the rules of spacetime and quantum mechanics, as reflected in the principles of locality and unitarity, are seen to emerge from deeper mathematical structures.

All Loop Scattering for All Multiplicity

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Abstract: This is part of a series of papers describing the new curve integral formalism for scattering amplitudes of the colored scalar trϕ^3 theory. We show that the curve integral manifests a very surprising fact about these amplitudes: the dependence on the number of particles, n, and the loop order, L, is effectively decoupled. We derive the curve integrals at tree-level for all n. We then show that, for higher loop-order, it suffices to study the curve integrals for L-loop tadpole-like amplitudes, which have just one particle per color trace-factor. By combining these tadpole-like formulas with the the tree-level result, we find formulas for the all n amplitudes at L loops. We illustrate this result by giving explicit curve integrals for all the amplitudes in the theory, including the non-planar amplitudes, through to two loops, for all n.

Tropical Amplitudes For Colored Lagrangians

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Abstract: Recently a new formulation for scattering amplitudes in Tr(Φ^3) theory has been given based on simple combinatorial ideas in the space of kinematic data. This allows all-loop integrated amplitudes to be expressed as "curve integrals" defined using tropical building blocks — the "headlight functions". This paper shows how the formulation extends to the amplitudes of more general Lagrangians. We will present a number of different ways of introducing tropical "numerator functions" that allow us to describe general Lagrangian interactions. The simplest family of these "tropical numerators" computes the amplitudes of interesting Lagrangians with infinitely many interactions. We also describe methods for tropically formulating the amplitudes for general Lagrangians. One uses a variant of "Wick contraction" to glue together numerator factors for general interaction vertices. Another uses a natural characterization of polygons on surfaces to give a novel combinatorial description of all possible diagrams associated with arbitrary valence interactions.

that total 178 pages and provide a new geometric/combinatorial formalism for amplitudes in any colored theory.

See Figueiredo's talk for a review and much subsequent work.

Introduction to Surfaceology

In quantum realms where particles entwine,
Two tales of scattering through curves we trace,
With colored scalars, paths and loops align,
In simple forms, complexities embrace.

From counting curves, first paper's truths unfold,
All loop orders, through pure math revealed,
No Feynman diagrams in sight, yet bold,
A new perspective, elegant, unsealed.

The second paper deepens this grand view,
Decoupling n and L with stunning grace,
For all multiplicities, insights true,
In tadpole forms and matrices we place.

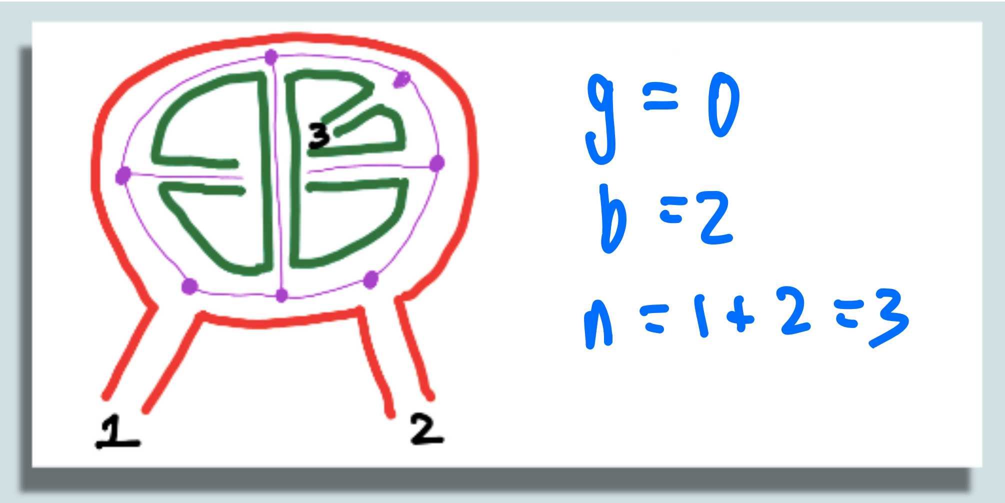
Two works as one, they sing of curves' pure might,
In physics' dance, they shed the brightest light.

Introduction to Surfaceology

Curves count particles,
Hidden truths in loops revealed,
Physics redefined.

Introduction to Surfaceology

Feynman diagrams in a colored theory are naturally organized by fatgraph topology (genus g, number of boundaries b, and number of external particle insertions on each boundary $n_1 + n_2 + \cdots + n_b = n$). Example:



$g = 0$
$b = 2$
$n = 1 + 2 = 3$

$L = 3$ loops

$E = n - 3 + 3L + 2g = 9$ propagators

Introduction to Surfaceology

Fundamentally, every field theory is based on cubic interactions; higher valent vertices can be built on a cubic skeleton by introducing suitable numerators.

A goal of the surfaceology program is to solve the underlying problem of parameterizing the space of all cubic graphs (ie "Feynman diagrams"), for any given fatgraph topology, once and for all.

Introduction to Surfaceology

Specifically, the idea is to write the sum over all

Feynman diagrams of a given fatgraph topology

(any g, b, $\{n_i\}$), as a single curve integral.

Starting just with $\mathrm{Tr}[\phi^3]$ theory, which only has the scalar propagator skeleton and nothing else.

Introduction to Surfaceology

The idea is to Schwinger parameterize each propagator

$$\frac{1}{p^2+m^2} = \int_0^\infty dt \quad \exp(-(p^2+m^2)\alpha)$$

so each Feynman diagram has the form

$$\int_0^\infty d^E t \quad \exp(-\text{something})$$

$\uparrow$ integral over $\mathbb{R}_+^E$
the positive orthant

$E = n-3+3L+2g$

Introduction to Surfaceology

The idea is to Schwinger parameterize each propagator

$$\frac{1}{p^2+m^2} = \int_0^\infty dt \;\; \exp(-(p^2+m^2)\alpha)$$

so each feynman diagram has the form

$$\text{each Feynman diagram} = \int_0^\infty d^E t \;\; \exp(-\text{something})$$

and then do a giant change of variables to glue together all of those orthants so the integral combines to

$$\sum \text{all feynman diagrams} = \int_{-\infty}^{\infty} d^E t \;\; \exp(-\text{something reasonable})$$

Introduction to Surfaceology

It's not surprising that this is possible; it is very surprising that this is actually achievable in practice.

In fact it is shockingly simple, and all of the necessary ingredients can be read off by drawing curves on your fatgraph, i.e.

Introduction to Surfaceology

The curve integral is

sum over all curves

$$\mathcal{A} = \int d^{DL}\ell \qquad \int \frac{d^E t}{MCG} \qquad \exp\Big(-\sum_c \alpha_c(t)\, P_c^2\Big)$$

piece wise linear functions that "turn on" certain curves in appropriate cones in $\mathbb{R}^E$

a momentum assigned to each curve

Introduction to Surfaceology

The curve integral is

$$\mathcal{A} = \int d^{DL}\ell \int \frac{d^E t}{MCG} \exp\left(-\sum_c \alpha_c(t)\, P_c^2\right)$$

L-loop integration in D spacetime dimensions

Schwinger parameter space for $E = n-3+3L+2g$ propagators

divide by overcounting loops (MCG = mapping class group)

Introduction to Surfaceology

The curve integral is

$$\mathcal{A} = \int d^{DL}\ell \quad \int \frac{d^E t}{MCG} \quad \exp\left(-\sum_C \alpha_C(t)\, P_C^2\right)$$

Every ingredient can be read off from the fatgraph.

There is a rule for how to insert appropriate tropical numerator factors to get more interesting scalar interactions (Arkani-Hamed, Figueiredo, Frost, Salvatori) and there is a curve formalism for gluons
(Arkani-Hamed, Cao, Dong, Figueiredo, He)

What Shounak, Andrzej and Marcos Did: Colored Yukawa Theory?

Develop a curve integral formalism for a simple fermionic theory: colored Yukawa theory.

They worked out numerous examplitudes at tree level, one loop, and two loops — both at the integrand and integrated level. In order to prep you for Skowronek's gong show I'll just highlight some key features.

Key Features of Fermionic Curve Integrals

We now have to distinguish color traces — built into the fatgraph structure, from Dirac traces — which require extra "decoration" on top of the fatgraph.

Key Features of Fermionic Curve Integrals

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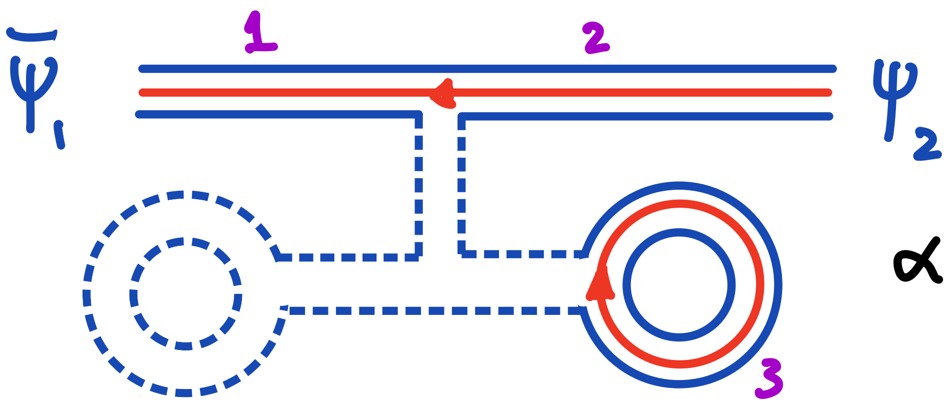
Dirac traces can be external or internal, for example



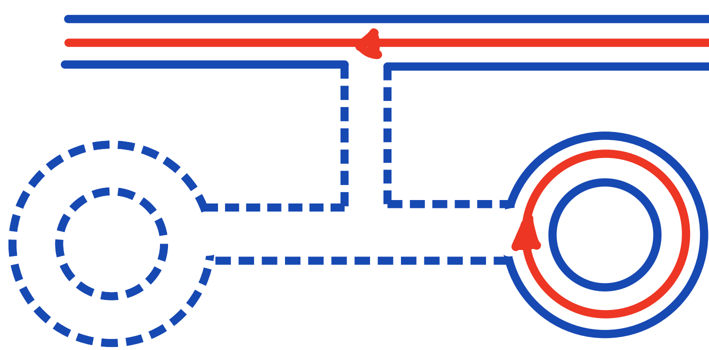
$$\propto \overbrace{\bar{v}(p_1)\,\not{p}_1\,\not{p}_2\,u(p_2)}^{\text{external}}\ \overbrace{\mathrm{Tr}(\not{p}_3)}^{\text{internal}}$$

(schematically)

Key Features of Fermionic Curve Integrals

There is a simple combinatorial rule to determine which P_c's appear inside each Dirac trace (internal or external) and in which order inside the trace.

Since we want these to be labeled by curves, not by edges in a Feynman diagram, it turns out to be natural to assign each curve to be fermionic or bosonic, and, naturally, also each puncture.



two kinds of punctures

Key Features of Fermionic Curve Integrals

In Marcos's talk look out for

- examples of fermion/boson curve/puncture assignments
- a sum over 2^L of the latter
- formulas for "tropical numerator" factors for external and internal traces, the latter with a -1.
- the numerator contains a bunch of theta functions that kill contributions disallowed for certain combinations of color & Dirac traces.
- post-loop integration factors involving $m \times m$ determinants where $m = \#$ of fermionic curves.

Many Open Questions

- quarks (i.e. fundamental fermions instead of adjoint)?
- is there a combinatorial way to combine the 2^L terms naturally? (GS vs RNS)
- "stringy" Yukawa theory, i.e. turn on α'/"δ-shifts"
- super Riemann surfaces/"super u-equations"

Conclusions

In the world of quantum strings,
Bespoke amplitudes do many things.
With equations precise,
Their beauty will suffice,
But their unitarity sometimes stings.

In the world of Yukawa's delight,
Colored fermions come into sight.
Through curves they express,
Scattering's finesse,
Their formulas pack quite a bite.