MAX-PLANCK-INSTITUT FÜR PHYSIK

The dark side of precision calculations: subtractions

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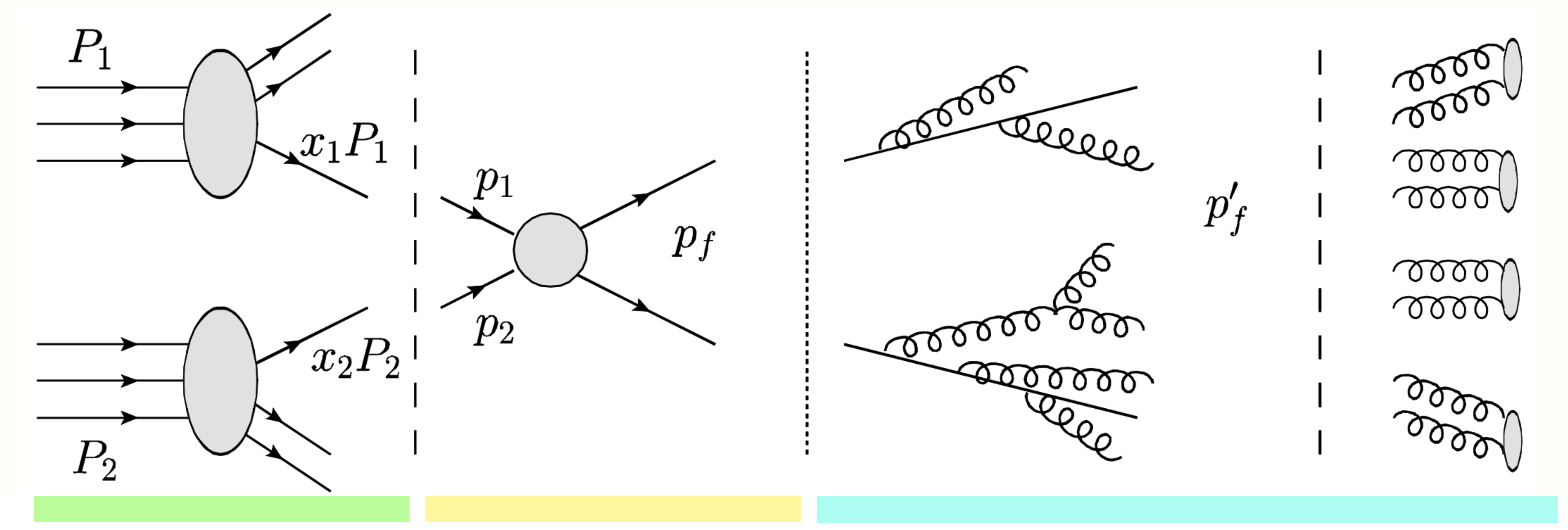
Based on: JHEP02(2024)016

Take-home message

When the complexity of the problem increases, look at simple, recurring structures!

Rudiments of particle physics at colliders

The success of a percent level phenomenology program relies on our ability to interpret and predict the outcome of LHC measurements. *[Snowmass'2021 whitepaper]*



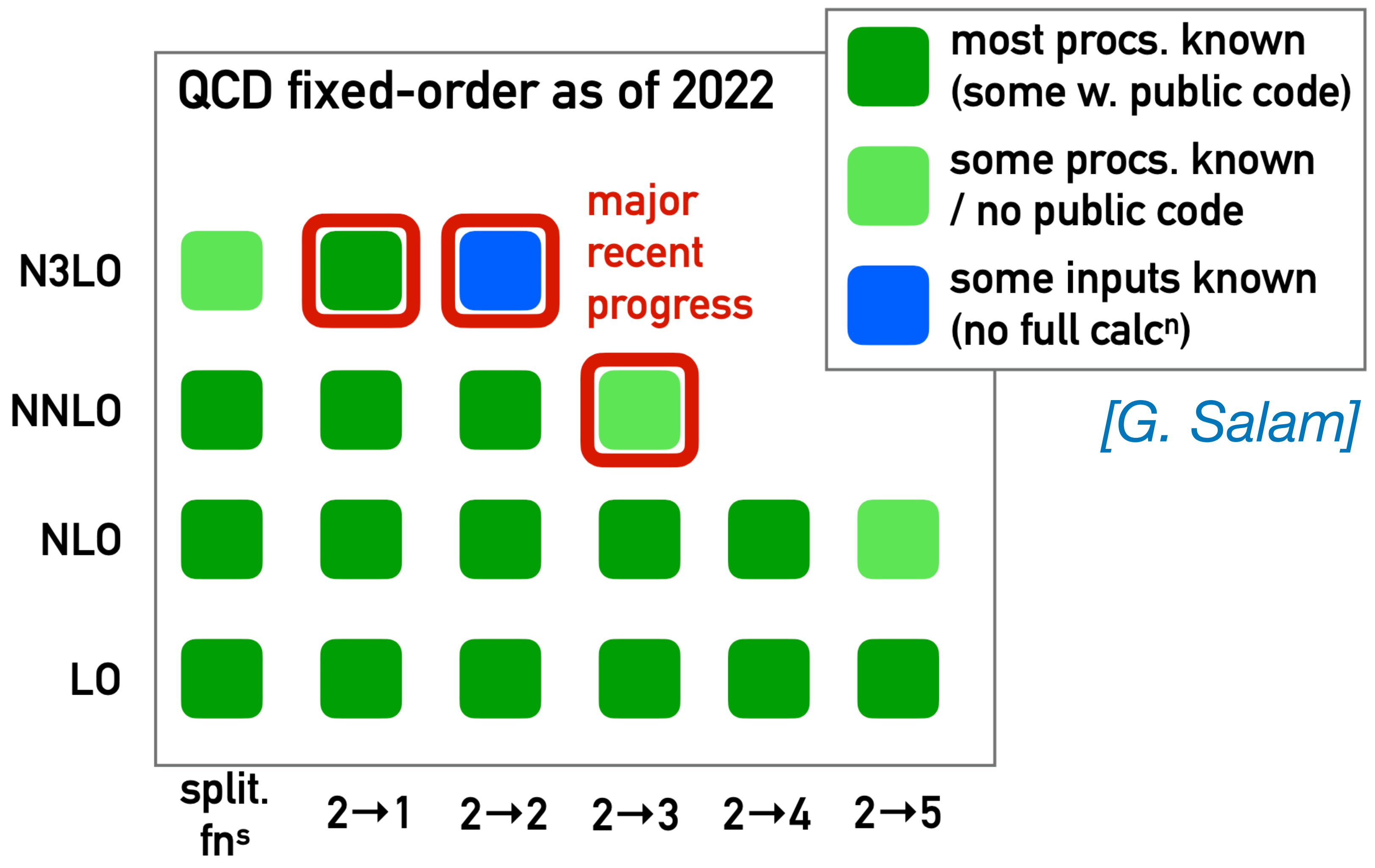
[Phys. Proc. 51(2014)25-30]

→ Collinear factorisation theorem *[Collins, Soper, Sterman '04]:* ***separate energy scale → different treatment***

$$d\sigma = \sum_{ij} \int dx_1\, dx_2\, f_{i/p}(x_1) f_{j/p}(x_2)\, d\hat{\sigma}_{ij}(x_1 x_2 s) \left(1 + \mathcal{O}\left(\frac{\Lambda^n_{\text{QCD}}}{Q^n}\right)\right), \quad n \geq 1$$

No large hierarchies of scales + no strong sensitivity to infrared physics

→ fixed order calculations provide a robust and reliable framework to obtain precision predictions at the LHC



[G. Salam]

Ingredients for higher-order corrections and main difficulties

$$\frac{d\sigma}{dX} = \frac{d\sigma_{\mathrm{LO}}}{dX} + \alpha_s \frac{d\sigma_{\mathrm{NLO}}}{dX} + \boxed{\alpha_s^2 \frac{d\sigma_{\mathrm{N^2LO}}}{dX}} + \alpha_s^3 \frac{d\sigma_{\mathrm{N^3LO}}}{dX} + \dots$$

X = IRC-safe, $\delta_{X_i} = \delta(X - X_i)$

Strong coupling:
$\alpha_s \sim 0.1$

$\mathcal{O}(\alpha_s) \sim 10\,\%$ $\quad$ $\mathcal{O}(\alpha_s^2) \sim 1\,\%$ $\quad$ $\mathcal{O}(\alpha_s^3) \sim 0.1\,\%$

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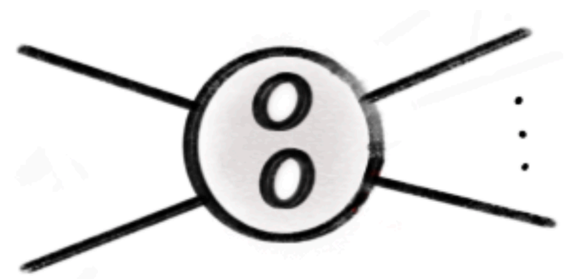
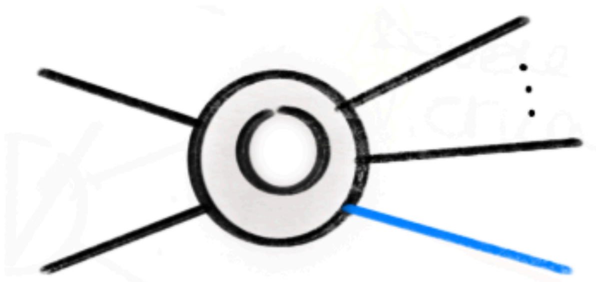
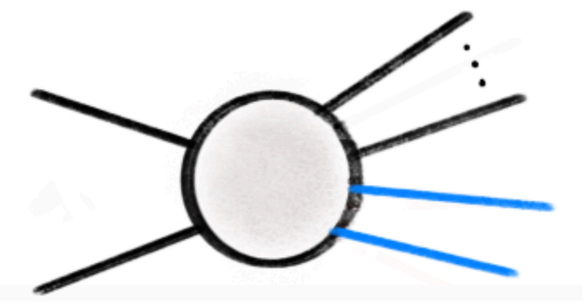
$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n \, VV \, \delta_{X_n} + \int d\Phi_{n+1} \, RV \, \delta_{X_{n+1}} + \int d\Phi_{n+2} \, RR \, \delta_{X_{n+2}}$$







Each ingredient presents significant **technical challenges**. Overcoming these issues requires **profound insight from QFT**

Virtual amplitudes:

- **Multi-loop integrals** involving **multiple scales, arising from different masses and many legs**

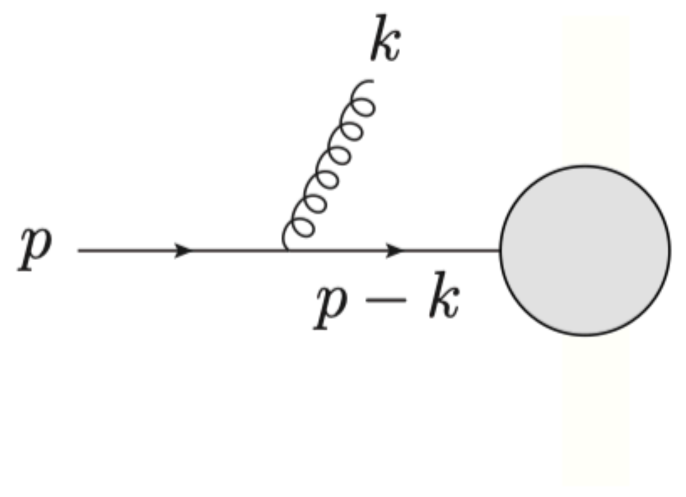
Real radiation singularities

- Extraction of **soft and collinear** singularities

IR singularities

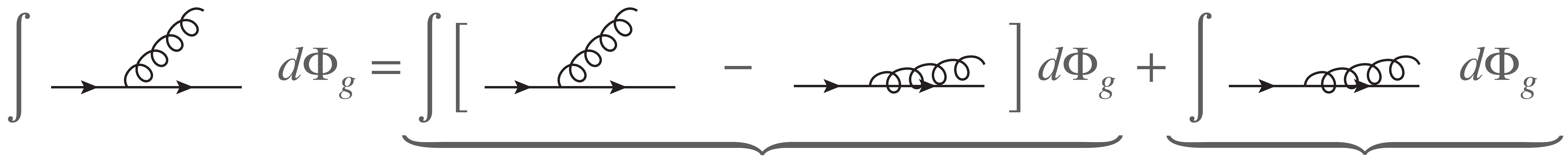
Real corrections:

- Singularities **arising from unresolved radiation** after integration over full phase space of radiated parton
- Goal: **extract IR singularities without integrating** over the resolved phase space → obtain **fully differential prediction**



$$\sim \frac{1}{(p-k)^2} = \frac{1}{2E_pE_k(1-\cos\theta)} \underset{\substack{E_k \to 0 \\ \text{or} \\ \theta \to 0}}{\longrightarrow} \infty . \qquad \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2E_k} |M(\{p\},k)|^2 \underset{\substack{E_k \to 0 \\ \theta \to 0}}{\sim} \int \frac{\mathrm{d}E_k}{E_k^{1+2\epsilon}} \frac{\mathrm{d}\theta}{\theta^{1+2\epsilon}} \times |M(\{p\})|^2 \sim \frac{1}{4\epsilon^2} .$$

Unresolved limits are universal and known (even at N3LO) → a **general procedure is in principle feasible**



Finite in d=4
integrable numerically

Counterterm

exposes the same $1/\epsilon$ poles as
the virtual correction

Integrated counterterm

Subtraction: conceptually non-trivial, but if local and analytic then extremely versatile and numerically stable

Subtractions: status

NLO:
solved conceptually in the 90s and now implemented in automatic frameworks

NNLO:
still **looking for the optimal scheme** $\rightarrow$ **the problem is highly non-trivial and a simple generalisation of NLO not doable due to overlapping singularities**

Example: di-jet two-loop amplitudes ~ 20 years ago *[Anastasiou et al. '01]*

di-jet production at NNLO ~ 5 years ago *[Currie et al. '17]*

Antenna *[Gehrmann-De Ridder et al. '05]*, ColorfullNNLO *[Del Duca et al. '16]*, STRIPPER *[Czakon '10],* Nested soft-collinear *[Caola et al. '17],*

Local analytic sector *[Magnea, CSS et al. '18],* Geometric IR subtraction *[Herzog '18],* Unsubtraction *[Sborlini et al. '16],* FDR *[Pittau '12],*

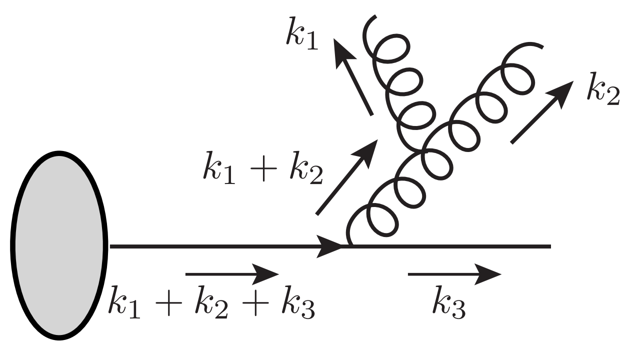
Universal Factorisation *[Sterman et al. '20]*, …

Most of them feature a **relevant degree of complexity**, and are **not ready to tackle multi-patron scattering**.

Simplifications and recurring patterns seem to be elusive!

Why is NNLO so difficult?

1. Clear understanding of which singular configurations do actually contribute
2. Get to the point where the problem is well defined
3. Solve the phase space integrals of the relevant limits

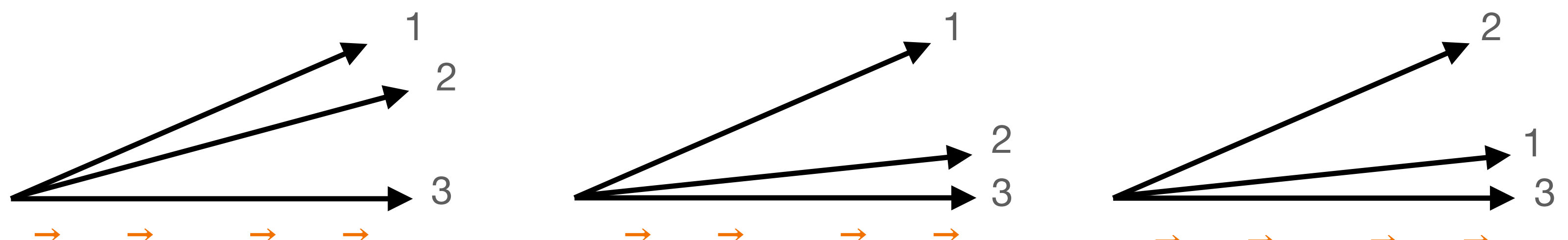


$$\sim \frac{1}{E_1\, E_2\,(1-\vec{n}_1\cdot\vec{n}_2)}\;\frac{1}{E_1\,E_2(1-\vec{n}_1\cdot\vec{n}_2)+E_1\,E_3(1-\vec{n}_1\cdot\vec{n}_3)+E_2\,E_3(1-\vec{n}_2\cdot\vec{n}_3)}$$

$E_1 \to 0 \quad E_2 \to 0 \quad E_1, E_2 \to 0$

$\vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3 \qquad \vec{n}_1 \parallel \vec{n}_2$

Strongly-ordered configurations have also to be included: $E_1 \ll E_2\,, \quad E_2 \ll E_1$



$\vec{n}_1\cdot\vec{n}_2 < \vec{n}_1\cdot\vec{n}_3 \qquad \vec{n}_2\cdot\vec{n}_3 < \vec{n}_1\cdot\vec{n}_3 \qquad \vec{n}_1\cdot\vec{n}_3 < \vec{n}_2\cdot\vec{n}_3$

Non-trivial structures to integrate → double-soft and triple-collinear kernels

$$I_{12}^{(gg)(56)} = \frac{(1-\epsilon)(s_{51}s_{62}+s_{52}s_{61})-2s_{56}s_{12}}{s_{56}^2(s_{51}+s_{61})(s_{52}+s_{62})} + s_{12}\,\frac{s_{51}s_{62}+s_{52}s_{61}-s_{56}s_{12}}{s_{56}s_{51}s_{62}s_{52}s_{61}}\left[1-\frac{1}{2}\,\frac{s_{51}s_{62}+s_{52}s_{61}}{(s_{51}+s_{61})(s_{52}+s_{62})}\right]$$

$s_{ab} = 2p_a\cdot p_b$

$$I_{S_{56}}^{(gg)} = \int [\mathrm{d}k_5]\,[\mathrm{d}k_6]\,\theta(E_{\max}-E_5)\,\theta(E_5-E_6)\,I_{12}^{(gg)(56)}\big(k_1,k_2,k_5,k_6\big)$$

$$[\,\mathrm{d}f_i\,] = \frac{\mathrm{d}^d k_i}{(2\pi)^d}\,(2\pi)\,\delta_+(k_i^2)$$

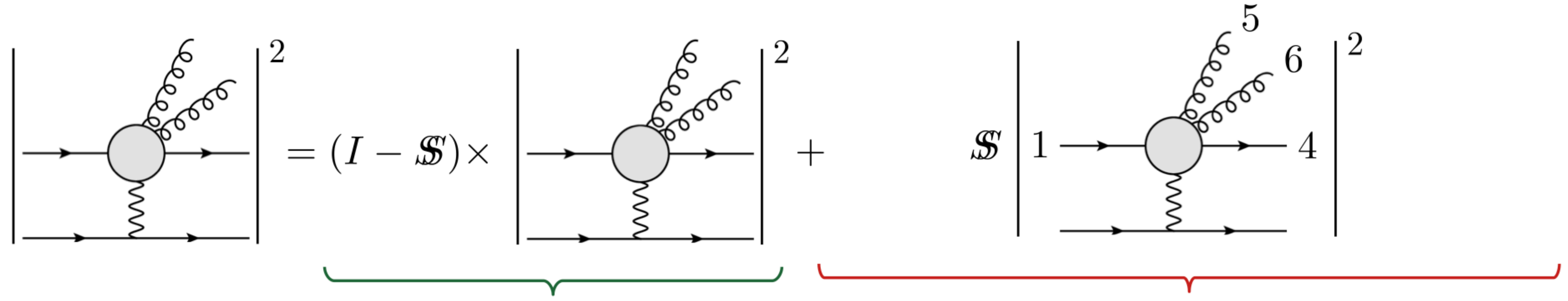
Nested soft-collinear subtraction at NNLO: generalities *[Caola, Melnikov, Röntsch '17]*

[figures curtsy of K. Asteriadis]

Example: DIS *[Asteriadis, Caola, Melnikov, Röntsch '19]*

- Extract double soft singularities first ($E_5 \sim E_6 \to 0$)

  $$I = (I - \mathbb{S}) + \mathbb{S}$$

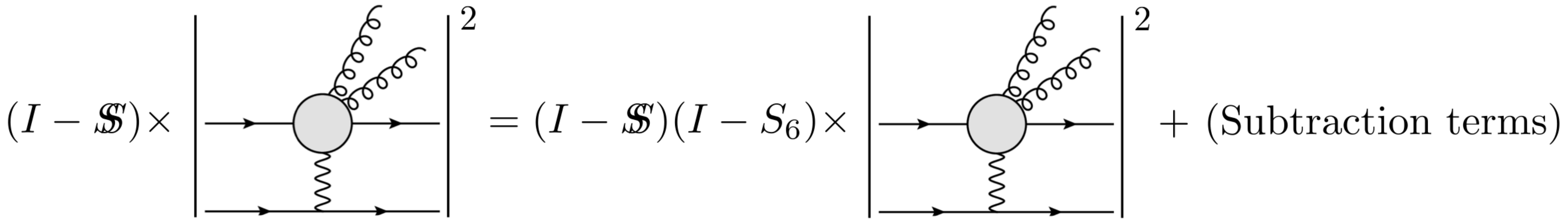


Double-soft singularity regularized but still contains single soft and collinear singularities.

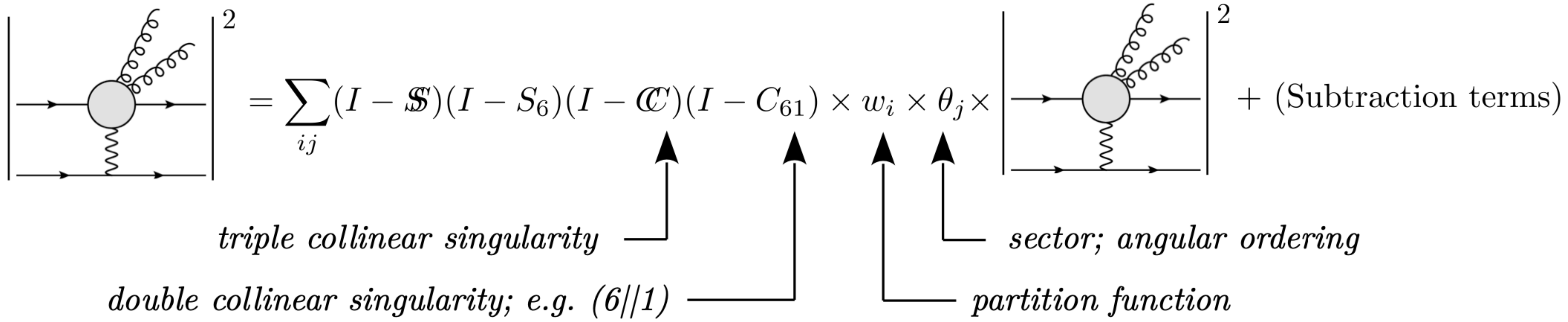
Subtraction term; soft gluons decouple; integrate analytically over phase space of gluons 5 and 6

- Gluons ordered in energy → only one single soft singularity

  $$I = (I - S_6) + S_6$$



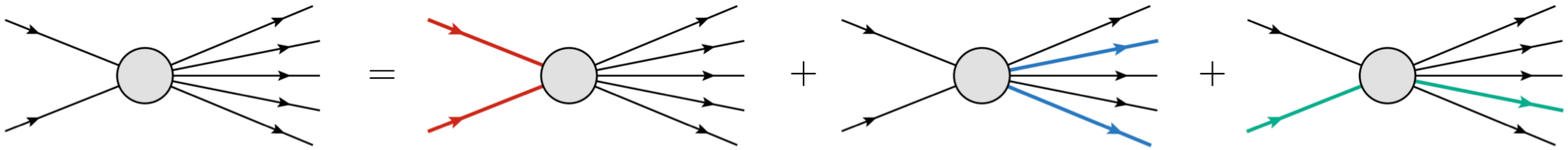
- Collinear singularities: partition function + sectoring [separate overlapping singularities]



- Integrate subtraction terms analytically using Reverse Unitarity *[Anastasiou, Melnikov '02]*: map phase space integrals onto loop integrals *[Caola, Delto, Frellesvig, Melnikov '18, '19]*

State of the art:

Separation of complex $pp \to N$ processes into simpler building blocks



QCD corrections to Drell-Yan
Both **initial state momenta**
[Caola, Melnikov, Röntsch ‘19]

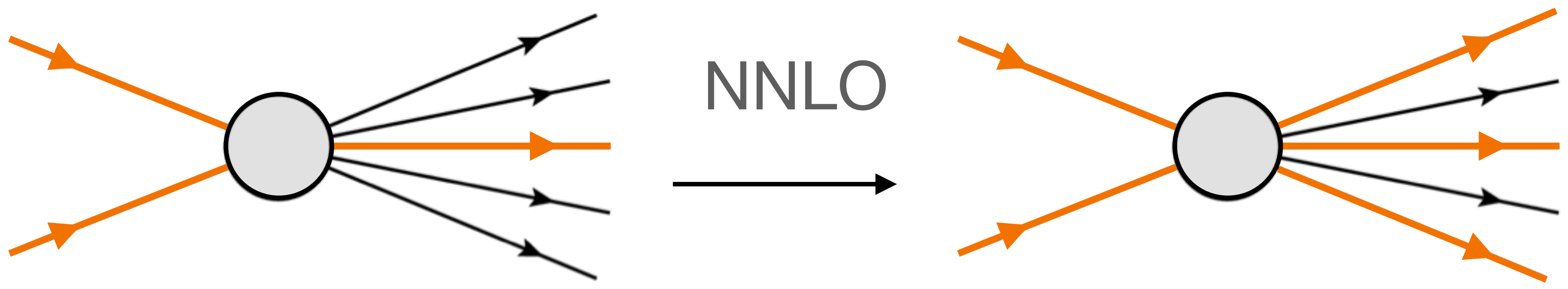
Higgs decay
Both **final state momenta**
[Caola, Melnikov, Röntsch ‘19]

Deep Inelastic Scattering
One initial and **one final** state momentum
[Asteriadis, Calola, Melnikov Röntsch ‘19]

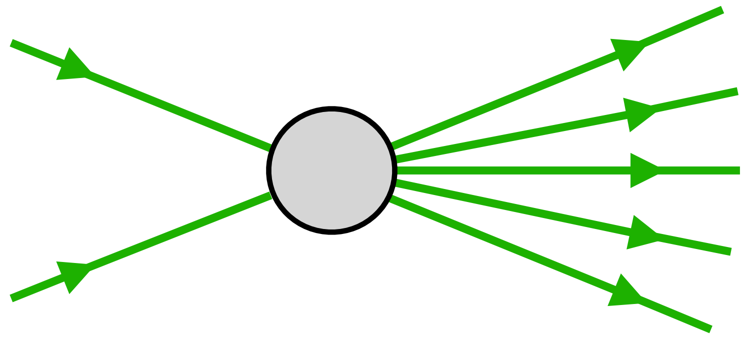
Focus on simple processes→ full control of the procedure, check against analytic results sometime possible.

Application to Z+j production

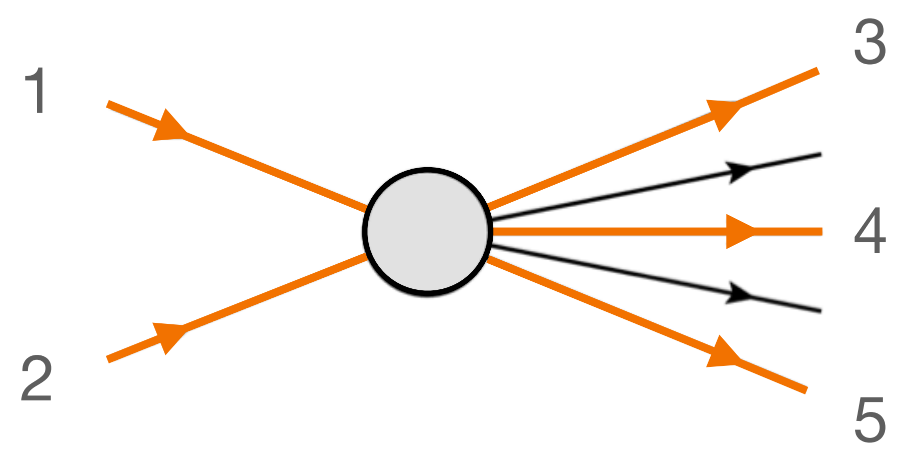
New!



→ **Prototype for**



Application to Z+j production



$$\frac{1}{3!}\langle F_{\rm LM}(1_q,2_{\bar q};3_g,4_g,5_g)\rangle = \langle S_{45}\,\Delta^{(45)}F_{\rm LM}^{4>5}\rangle + \langle (I-S_4)S_5\,\Delta^{(45)}F_{\rm LM}^{4>5}\rangle$$

$$+\Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in{\rm TC}}\Big[\Theta^{(a)}C_{45,i}(I-C_{5i})+\Theta^{(b)}C_{45,i}(I-C_{45})$$

$$+\Theta^{(c)}C_{45,i}(I-C_{4i})+\Theta^{(d)}C_{45,i}(I-C_{45})\Big]\omega_{4i5i}\Big\}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle$$

$$-\Big\langle (I-S_{45})(I-S_5)\sum_{(ij)\in{\rm DC}} C_{4i}C_{5j}\,\omega_{4i5j}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle$$

$$+\Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in{\rm TC}}\big[\Theta^{(a)}C_{5i}+\Theta^{(b)}C_{45}+\Theta^{(c)}C_{4i}+\Theta^{(d)}C_{45}\big]\,\omega_{4i5i}$$

$$+\sum_{(ij)\in{\rm DC}}\big[C_{4i}+C_{5j}\big]\,\omega_{4i5j}\Big\}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle$$

$$+\Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in{\rm TC}}\Big[\Theta^{(a)}(I-C_{45,i})(I-C_{5i})+\Theta^{(b)}(I-C_{45,i})(I-C_{45})$$

$$+\Theta^{(c)}(I-C_{45,i})(I-C_{4i})+\Theta^{(d)}(I-C_{45,i})(I-C_{45})\Big]\omega_{4i5i}$$

$$+\sum_{(ij)\in{\rm DC}}(I-C_{4i})(I-C_{5j})\,\omega_{4i5j}\Big\}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle$$

In principle generalisable to n-partons

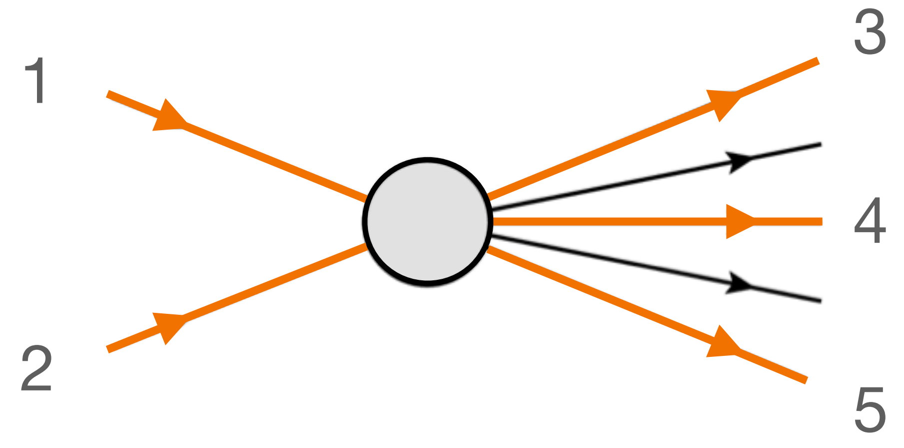
Subtraction terms

Implemented numerically → no issues in increasing the number of partons

Fully regulated contribution

$$(ij)\in{\rm DC}\longrightarrow (ij)\in\{(12),(13),(21),(23),(31),(32)\}$$
$$i\in{\rm TC}\longrightarrow i\in\{1,2,3\}\,.$$

Application to Z+j production



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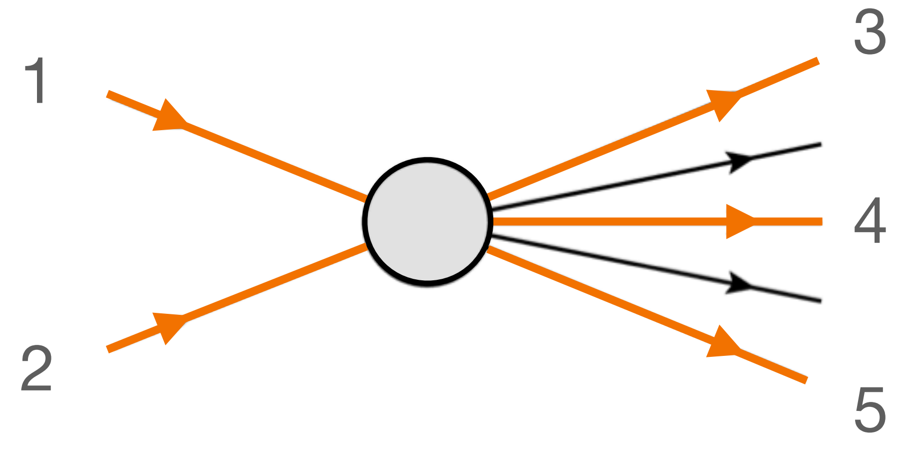
$$+ \Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in{\rm TC}}\Big[\Theta^{(a)}C_{45,i}(I-C_{5i}) + \Theta^{(b)}C_{45,i}(I-C_{45})$$
$$+\Theta^{(c)}C_{45,i}(I-C_{4i}) + \Theta^{(d)}C_{45,i}(I-C_{45})\Big]\omega_{4i5i}\Big\}\,\Delta^{(45)} F_{\rm LM}^{4>5}\Big\rangle$$
$$-\Big\langle (I-S_{45})(I-S_5)\sum_{(ij)\in{\rm DC}} C_{4i}C_{5j}\,\omega_{4i5j}\,\Delta^{(45)} F_{\rm LM}^{4>5}\Big\rangle$$
$$+ \Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in{\rm TC}}\big[\Theta^{(a)}C_{5i} + \Theta^{(b)}C_{45} + \Theta^{(c)}C_{4i} + \Theta^{(d)}C_{45}\big]\,\omega_{4i5i}$$
$$+\sum_{(ij)\in{\rm DC}}\big[C_{4i}+C_{5j}\big]\,\omega_{4i5j}\Big\}\,\Delta^{(45)} F_{\rm LM}^{4>5}\Big\rangle$$
$$+ \Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in{\rm TC}}\Big[\Theta^{(a)}(I-C_{45,i})(I-C_{5i}) + \Theta^{(b)}(I-C_{45,i})(I-C_{45})$$
$$+\Theta^{(c)}(I-C_{45,i})(I-C_{4i}) + \Theta^{(d)}(I-C_{45,i})(I-C_{45})\Big]\omega_{4i5i}$$
$$+\sum_{(ij)\in{\rm DC}}(I-C_{4i})(I-C_{5j})\,\omega_{4i5j}\Big\}\,\Delta^{(45)} F_{\rm LM}^{4>5}\Big\rangle$$

Drawbacks identified with Z+j

- The **bookkeeping** becomes immediately **cumbersome** → large number of subtraction terms.
- Calculating all **subtraction terms separately** may **hide a number of simplifications** that can occur **before explicit evaluation**.

$$(ij)\in{\rm DC}\longrightarrow (ij)\in\{(12),(13),(21),(23),(31),(32)\}$$
$$i\in{\rm TC}\longrightarrow i\in\{1,2,3\}\,.$$

Summary of the talk




```
Series[RRpoles, {ϵ, 0, -1}]

Out[•]= 1/(4 ϵ^4) (3 asontwopi^2 CA^2 FLM[p1_q, p2_q, p3_g] + 10 asontwopi^2 CA CF FLM[p1_q, p2_q, p3_g] + 2 asontwopi^2 CF^2 FLM[p1_q, p2_q, p3_g] +
    5 asontwopi^2 CF^2 delta[1 - z] × FLM[p1_q, z p2_q, p3_g, z] + 5 asontwopi^2 CF^2 delta[1 - z] × FLM[z p1_q, p2_q, p3_g, z] -
    4 asontwopi^2 CF^2 delta[1 - z1] × delta[1 - z2] × FLM[z1 p1_q, z2 p2_q, p3_g, z1, z2]) +
  (77 asontwopi^2 CA^2 FLM[p1_q,p2_q,p3_g] + 110 asontwopi^2 CA CF FLM[p1_q,p2_q,p3_g] + 72 asontwopi^2 CA^2 L3 FLM[p1_q,p2_q,p3_g] + 36 asontwopi^2 CA CF FLM[p1_q,z p2_q,p3_g,z] + ⋯100⋯)/(24 ϵ^3) +
  (1381/72 asontwopi^2 CA^2 FLM[p1_q,p2_q,p3_g] + 67/4 asontwopi^2 CA CF FLM[p1_q,p2_q,p3_g] + ⋯509⋯ + 1/2 asontwopi^2 CA CF delta[1-z] × FLM[z p1_q,p2_q,p3_g,z] PolyLog[2,1-eta[2,3]])/ϵ^2 +
  (3151/27 asontwopi^2 CA^2 FLM[p1_q,p2_q,p3_g] + 2813/36 asontwopi^2 CA CF FLM[p1_q,p2_q,p3_g] + 265/18 asontwopi^2 CA^2 L3 FLM[p1_q,p2_q,p3_g] - 11/2 ⋯3⋯ FLM[p1_q,p2_q,p3_g] + ⋯2016⋯)/ϵ + O[ϵ]^0

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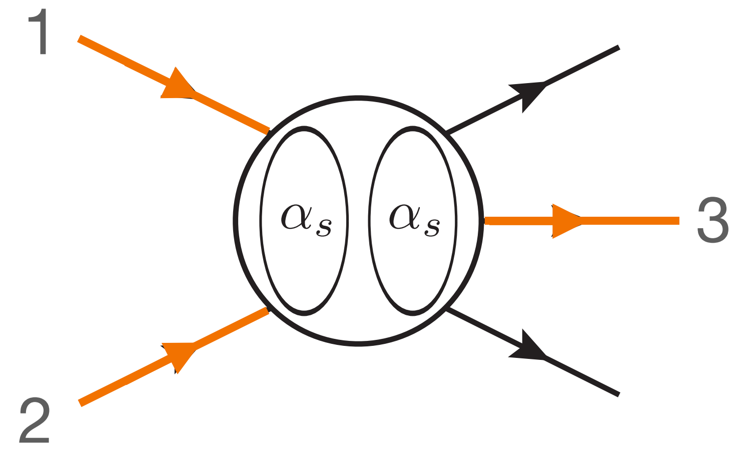

[Catani '98]

$$\langle\mathcal{M}|\mathcal{M}\rangle_{\alpha_s^2} = \left\langle \mathcal{M}_0 \left| \frac{1}{2} I_1^2(\epsilon) + \frac{1}{2}\left(I_1^\dagger(\epsilon)\right)^2 + I_1^\dagger(\epsilon) I_1(\epsilon) + \left(\mathcal{H}_2 + \mathcal{H}_2^\dagger\right) \right| \mathcal{M}_0 \right\rangle$$
$$+ \left\langle \mathcal{M}_0 \left| -\frac{\beta_0}{\epsilon}\left(I_1(\epsilon) + I_1^\dagger(\epsilon)\right) + c_\epsilon \left(\frac{\beta_0}{\epsilon} + K\right)\left(I_1(2\epsilon) + I_1^\dagger(2\epsilon)\right) \right| \mathcal{M}_0 \right\rangle$$
$$+ 2\mathrm{Re}\left[\langle \mathcal{M}_0 \,|\, I_1(\epsilon) + I_1^\dagger(\epsilon) \,|\, \mathcal{M}_1^{\mathrm{fin}} \rangle\right] + 2\mathrm{Re}\left[\langle \mathcal{M}_0 | \mathcal{M}_2^{\mathrm{fin}} \rangle\right] + \langle \mathcal{M}_1^{\mathrm{fin}} | \mathcal{M}_1^{\mathrm{fin}} \rangle \,.$$

```
Series[VVpoles, {ϵ, 0, -1}]

asontwopi^2 (CA + 2 CF)^2 FLM[p1_q, p2_q, p3_g]/(2 ϵ^4) +
 asontwopi^2 (CA + 2 CF) FLM[p1_q, p2_q, p3_g] (12 CF + 7 β0 + 4 (CA - 2 CF) Log[s12/mu2] - 4 CA (Log[s13/mu2] + Log[s23/mu2]))/(4 ϵ^3) + 1/ϵ^2 asontwopi^2
  ((-CA - 2 CF) FLVfin[p1_q, p2_q, p3_g] + FLM[p1_q, p2_q, p3_g] (1/72 (-2 CA CF (67 + 9 π^2) + CA^2 (-67 + 33 π^2) + 20 CA Nf TR + 4 CF (CF (81 - 42 π^2) + 10 Nf TR)) +
      9 CF β0/2 + β0^2 + 1/4 (8 CF (-CA + 2 CF) Log[s12/mu2]^2 + 4 CA (CA + CF) Log[s13/mu2]^2 + Log[s13/mu2] (-3 CA (CA + 6 CF) - 4 (2 CA + CF) β0 + 4 CA^2 Log[s23/mu2]) + Log[s23/mu2]
        (-3 CA (CA + 6 CF) - 4 (2 CA + CF) β0 + 4 CA (CA + CF) Log[s23/mu2]) - 2 (CA - 2 CF) Log[s12/mu2] (-3 (CA + 4 CF + β0) + 2 CA (Log[s13/mu2] + Log[s23/mu2]))))) +
 1/ϵ asontwopi^2 (FLVfin[p1_q, p2_q, p3_g] (-3 CF - β0 - (CA - 2 CF) Log[s12/mu2] + CA (Log[s13/mu2] + Log[s23/mu2])) +
   1/864 FLM[p1_q, p2_q, p3_g] (3 CA^2 (60 + 227 π^2) + 2 CA (CF (-1922 + 333 π^2) - 2 Nf (232 + 3 π^2) TR + 3 (-268 + 51 π^2) β0) +
     4 (-27 CF^2 (3 + 52 π^2) + 2 CF Nf (184 + 9 π^2) TR - 315 CF π^2 β0 + 40 Nf TR (2 Nf TR + 3 β0)) -
     12 (24 (CA - 4 CF) (CA - 2 CF) Log[s12/mu2]^3 + 24 CA (2 CA + CF) Log[s13/mu2]^3 + 9 Log[s13/mu2]^2 (-9 CA (CA + 2 CF) - 2 (5 CA + 2 CF) β0 + 4 CA^2 Log[s23/mu2]) -
       18 (CA - 2 CF) Log[s12/mu2]^2 (3 CA - 2 (12 CF + β0) + 2 CA (Log[s13/mu2] + Log[s23/mu2])) + 2 (CA - 2 CF) Log[s12/mu2]
        (67 CA + 3 (CA + 28 CF) π^2 - 2 (81 CF + 10 Nf TR + 27 β0) + 9 (9 CA + 2 β0) Log[s13/mu2] - 18 CA Log[s13/mu2]^2 + 9 Log[s23/mu2] (9 CA + 2 β0 - 2 CA Log[s23/mu2])) +
       2 Log[s13/mu2] (CA (-67 CA + 81 CF + 3 (5 CA - 16 CF) π^2 + 20 Nf TR) + 27 (CA + 2 CF) β0 + 18 β0^2 + 18 CA Log[s23/mu2] (-3 CA - 2 β0 + CA Log[s23/mu2])) +
       Log[s23/mu2] (2 CA (-67 CA + 81 CF + 3 (5 CA - 16 CF) π^2 + 20 Nf TR) + 54 (CA + 2 CF) β0 + 36 β0^2 +
         3 Log[s23/mu2] (-27 CA (CA + 2 CF) - 6 (5 CA + 2 CF) β0 + 8 CA (2 CA + CF) Log[s23/mu2])) + 6 (CA^2 - 62 CA CF + 88 CF^2) Zeta[3]))) + O[ϵ]^0
```

"Asymmetry": VV very simple pole structure, RR structure obscured by energy ordering, partitioning…

Summary of the talk

Can we identify structures **early on** in the calculations so that cancellation of divergences can be seen "by eye", even for a **generic process**?

Main idea: look at the pole structure of the virtual corrections to infer similar structures for the subtraction terms

→ by product: **get rid of color correlations and reduce the rest to a sum over external-leg contributions.**

Case of study: $q\bar{q} \to X + Ng$

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A fresh look at the nested soft-collinear subtraction scheme: NNLO QCD corrections to N-gluon final states in $q\bar{q}$ annihilation

Federica Devoto,[a] Kirill Melnikov,[b] Raoul Röntsch,[c] Chiara Signorile-Signorile[b,d,e] and Davide Maria Tagliabue[c]

Work in progress: $gq \to X + (N-1)g + q$

NLO and NNLO QCD contributions to the channel $gq \to X + (N-1)g + q$

Federica Devoto,[a] Kirill Melnikov,[b] Raoul Röntsch,[c] Chiara Signorile-Signorile,[d] Davide Maria Tagliabue[c]

Warm up @NLO: $q\bar{q} \to X + Ng$

$$2s\, \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO}} = \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{V}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{R}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{pdf}}$$

Virtual corrections:
color-correlations, elastic terms

$$I_{\mathrm{V}}(\epsilon) = \bar{I}_1(\epsilon) + \bar{I}_1^{\dagger}(\epsilon)$$

$$\bar{I}_1(\epsilon) = \frac{1}{2}\sum_{(ij)}^{N_p} \frac{\mathcal{V}_i^{\mathrm{sing}}(\epsilon)}{\boldsymbol{T}_i^2} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \left(\frac{\mu^2}{2p_i \cdot p_j}\right)^{\epsilon} e^{i\pi\lambda_{ij}\epsilon}$$

$$\mathcal{V}_i^{\mathrm{sing}}(\epsilon) = \frac{\boldsymbol{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}.$$

Real corrections:

soft: color-correlations, elastic terms

$$I_{\mathrm{S}}(\epsilon) = -\frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)}^{N_p} \eta_{ij}^{-\epsilon} K_{ij} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)$$

hard-collinear: no color-correlations, elastic terms+boosts

$$I_{\mathrm{C}}(\epsilon) = \sum_{i=1}^{N_p} \frac{\Gamma_{i,f_i}}{\epsilon}$$

$$\mathcal{P}_{aa}^{\mathrm{gen}} \otimes F_{\mathrm{LM}}$$

Warm up @NLO: $q\bar{q} \to X + Ng$

$$2s\, d\hat{\sigma}_{ab}^{\rm NLO} = d\hat{\sigma}_{ab}^{\rm V} + d\hat{\sigma}_{ab}^{\rm R} + d\hat{\sigma}_{ab}^{\rm pdf}$$

Virtual corrections:
color-correlations, elastic terms

$$I_{\rm V}(\epsilon) = \bar{I}_1(\epsilon) + \bar{I}_1^\dagger(\epsilon)$$

$$\bar{I}_1(\epsilon) = \frac{1}{2}\sum_{(ij)}^{N_p} \frac{\mathcal{V}_i^{\rm sing}(\epsilon)}{\boldsymbol{T}_i^2} (\boldsymbol{T}_i\cdot\boldsymbol{T}_j) \left(\frac{\mu^2}{2p_i\cdot p_j}\right)^\epsilon e^{i\pi\lambda_{ij}\epsilon}$$

$$\mathcal{V}_i^{\rm sing}(\epsilon) = \frac{\boldsymbol{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}.$$

Real corrections:

soft: color-correlations, elastic terms

$$I_{\rm S}(\epsilon) = -\frac{(2E_{\rm max}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)}^{N_p} \eta_{ij}^{-\epsilon} K_{ij} (\boldsymbol{T}_i\cdot\boldsymbol{T}_j)$$

hard-collinear: no color-correlations, elastic terms+boosts

$$I_{\rm C}(\epsilon) = \sum_{i=1}^{N_p} \frac{\Gamma_{i,f_i}}{\epsilon}$$

$$\mathcal{P}_{aa}^{\rm gen} \otimes F_{\rm LM}$$

$$I_{\rm V}(\epsilon) + I_{\rm S}(\epsilon)$$

- Highest pole trivially cancels
- Color correlations cancel

} Remnant elastic single pole

"generalised anomalous dimensions"

$$\Gamma_{i,f_i} = \gamma_i + 2\boldsymbol{T}_i^2 L_i + \mathcal{O}(\epsilon)\,.$$

Warm up @NLO: $q\bar{q} \to X + Ng$

$$2s\, \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO}} = \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{V}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{R}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{pdf}}$$

Virtual corrections:
color-correlations, elastic terms

$$I_{\mathrm{V}}(\epsilon) = \overline{I}_1(\epsilon) + \overline{I}_1^{\dagger}(\epsilon)$$

$$\overline{I}_1(\epsilon) = \frac{1}{2}\sum_{(ij)}^{N_p} \frac{\mathcal{V}_i^{\mathrm{sing}}(\epsilon)}{\boldsymbol{T}_i^2} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \left(\frac{\mu^2}{2p_i \cdot p_j}\right)^{\epsilon} e^{i\pi\lambda_{ij}\epsilon}$$

$$\mathcal{V}_i^{\mathrm{sing}}(\epsilon) = \frac{\boldsymbol{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}.$$

Real corrections:

soft: color-correlations, elastic terms

$$I_{\mathrm{S}}(\epsilon) = -\frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)}^{N_p} \eta_{ij}^{-\epsilon} K_{ij} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)$$

hard-collinear: no color-correlations, elastic terms+boosts

$$I_{\mathrm{C}}(\epsilon) = \sum_{i=1}^{N_p} \frac{\Gamma_{i,f_i}}{\epsilon}$$

$$\mathcal{P}_{aa}^{\mathrm{gen}} \otimes F_{\mathrm{LM}}$$

$I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon)$

- Highest pole trivially cancels
- Color correlations cancel

} Remnant elastic single pole

"generalised anomalous dimensions"

$$\Gamma_{i,f_i} = \gamma_i + 2\boldsymbol{T}_i^2 L_i + \mathcal{O}(\epsilon)\,.$$

$$\Longrightarrow \quad I_{\mathrm{T}}(\epsilon) = I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon) + I_{\mathrm{C}}(\epsilon) \quad \text{FINITE!}$$

$$2s\, \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO}} = \frac{\alpha_s(\mu)}{2\pi}\left\langle I_{\mathrm{T}}^{(0)} \cdot F_{\mathrm{LM}}\right\rangle + \frac{\alpha_s(\mu)}{2\pi}\Big[\left\langle \mathcal{P}_{aa}^{\mathrm{NLO}} \otimes F_{\mathrm{LM}}\right\rangle + \left\langle F_{\mathrm{LM}} \otimes \mathcal{P}_{bb}^{\mathrm{NLO}}\right\rangle\Big] + \left\langle F_{\mathrm{LV}}^{\mathrm{fin}}\right\rangle + \left\langle \mathcal{O}_{\mathrm{NLO}}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}(\mathfrak{m})\right\rangle$$

Lesson from NLO

Simple interplay between $\underbrace{\left[V + S_i R + (I - S_i) C_{ij} R\right]_{\text{elastic}}}_{I_{\rm T}(\epsilon) = I_{\rm V}(\epsilon) + I_{\rm S}(\epsilon) + I_{\rm C}(\epsilon)}$ **and** $\underbrace{\left[(1 - S_i) C_{ij} R\right]_{\text{boost}} + \text{PDFs}}_{\langle \mathcal{P}_{aa}^{\rm NLO} \otimes F_{\rm LM}\rangle + \langle F_{\rm LM} \otimes \mathcal{P}_{bb}^{\rm NLO}\rangle}$

New approach at NNLO:

Starting from **IR poles of double-virtual** *[Catani '98]* we want to find **subtraction terms** that can "**complete**" it:

- identify structures similar to those encountered at NLO → ideally the result will be $\sim$ NLO^2 as much as possible

$$\langle F_{\rm VV}\rangle = [\alpha_s]^2 \left\langle \left[\frac{1}{2} I_{\rm V}^2(\epsilon) - \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \left(\frac{\beta_0}{\epsilon} I_{\rm V}(\epsilon) - \left(\frac{\beta_0}{\epsilon} + K\right) I_{\rm V}(2\epsilon)\right)\right] \cdot F_{\rm LM}\right\rangle$$
$$+ [\alpha_s]^2 \left\langle \left[-\frac{1}{2}\left[\overline{I}_1(\epsilon), \overline{I}_1^\dagger(\epsilon)\right] + \mathcal{H}_{2,\rm tc} + \mathcal{H}_{2,\rm tc}^\dagger + \mathcal{H}_{2,\rm cd} + \mathcal{H}_{2,\rm cd}^\dagger\right] \cdot F_{\rm LM}\right\rangle$$
$$+ [\alpha_s] \langle I_{\rm V}(\epsilon) \cdot F_{\rm LV}^{\rm fin}\rangle + \langle F_{\rm LV^2}^{\rm fin}\rangle + \langle F_{\rm VV}^{\rm fin}\rangle \,.$$

single structure

$\overline{I}_1$

(already encountered at NLO)

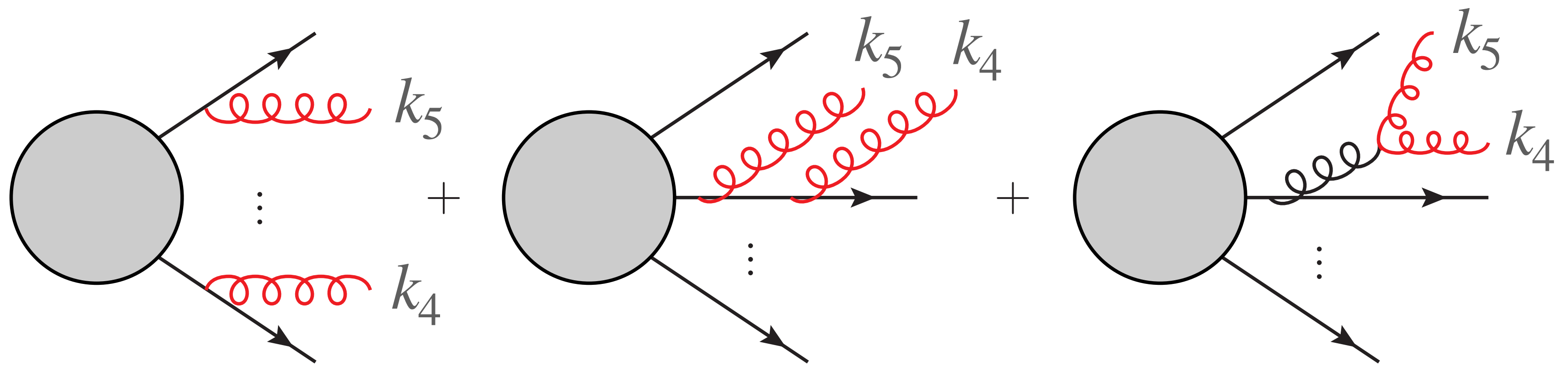
- different **powers/arguments/prefactors**
- different type of **color-correlations**

$$\begin{cases} T_i \cdot T_j \\ T_i \cdot T_j \cdot T_k \\ (T_i \cdot T_j) \cdot (T_k \cdot T_l) \end{cases}$$

→ **specific pattern of cancellation.**

Follow the (colored) crumbs

Color correlations can only arise from soft real emissions and loop corrections



Double soft

[Catani, Grazzini '99]

Non-factorised term

$T_i \cdot T_j$

$$\langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\rm LM}(\mathfrak{m},\mathfrak{n}) \rangle_{T^2}$$

Factorised term

$(T_i \cdot T_j) \cdot (T_k \cdot T_l)$

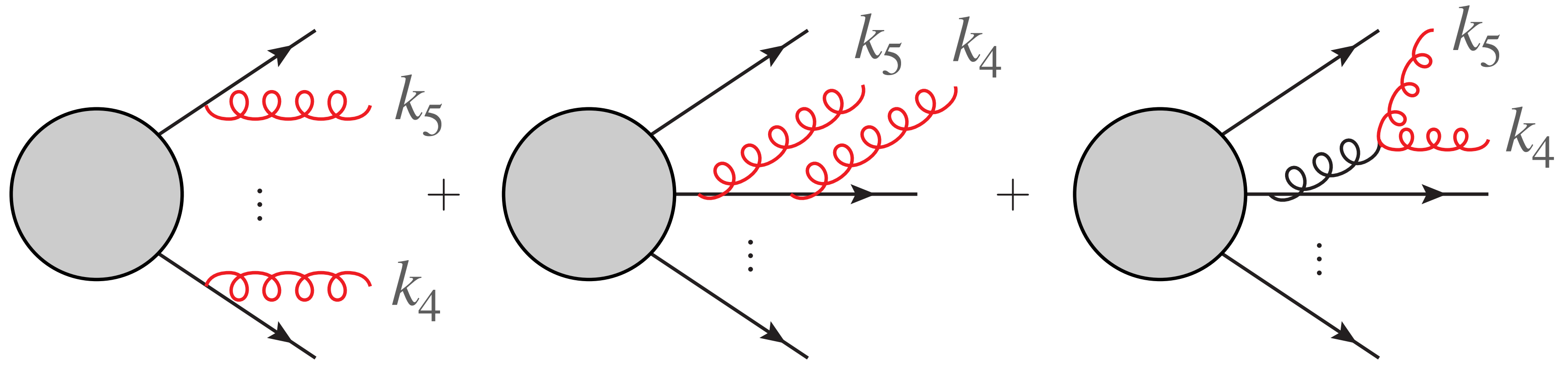
$$\langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\rm LM}(\mathfrak{m},\mathfrak{n}) \rangle_{T^4} = [\alpha_s]^2 \frac{1}{2} \langle I_{\rm S}^2(\epsilon) \cdot F_{\rm LM} \rangle$$

$I_S^2(\epsilon) + I_V^2(\epsilon)$ takes care of "quartic" color-correlated poles

Iterations of NLO!

Follow the (colored) crumbs

Color correlations can only arise from soft real emissions and loop corrections



Double soft

[Catani, Grazzini '99]

Non-factorised term

$T_i \cdot T_j$

$$\langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\rm LM}(\mathfrak{m},\mathfrak{n}) \rangle_{T^2}$$

$\Downarrow$

$$\begin{aligned}
&(2E_{\max})^{-4\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]^2 \{ \frac{1}{2\epsilon^4}+\frac{1}{\epsilon^3}\left[\frac{11}{12}-\ln(s^2)\right] \\
&+\frac{1}{\epsilon^2}\left[2\mathrm{Li}_2(c^2)+\ln^2(s^2)-\frac{11}{6}\ln(s^2)+\frac{11}{3}\ln 2-\frac{\pi^2}{4}-\frac{16}{9}\right] \\
&+\frac{1}{\epsilon}[6\mathrm{Li}_3(s^2)+2\mathrm{Li}_3(c^2)+\left(2\ln(s^2)+\frac{11}{3}\right)\mathrm{Li}_2(c^2)-\frac{2}{3}\ln^3(s^2) \\
&\quad+\left(3\ln(c^2)+\frac{11}{6}\right)\ln^2(s^2)-\left(\frac{22}{3}\ln 2+\frac{\pi^2}{2}-\frac{32}{9}\right)\ln(s^2) \\
&\quad-\frac{45}{4}\zeta_3-\frac{11}{3}\ln^2 2-\frac{11}{36}\pi^2-\frac{137}{18}\ln 2+\frac{217}{54}] \\
&+4\mathrm{G}_{-1,0,0,1}(s^2)-7\mathrm{G}_{0,1,0,1}(s^2)+\frac{22}{3}\mathrm{Ci}_3(2\delta)+\frac{1}{3\tan(\delta)}\mathrm{Si}_2(2\delta) \\
&+2\mathrm{Li}_4(c^2)-14\mathrm{Li}_4(s^2)+4\mathrm{Li}_4\left(\frac{1}{1+s^2}\right)-2\mathrm{Li}_4\left(\frac{1-s^2}{1+s^2}\right) \\
&+2\mathrm{Li}_4\left(\frac{s^2-1}{1+s^2}\right)+\mathrm{Li}_4(1-s^4)+[10\ln(s^2)-4\ln\left(1+s^2\right) \\
&+\frac{11}{3}]\mathrm{Li}_3(c^2)+\left[14\ln(c^2)+2\ln(s^2)+4\ln\left(1+s^2\right)+\frac{22}{3}\right]\mathrm{Li}_3(s^2) \\
&+4\ln(c^2)\mathrm{Li}_3(-s^2)+\frac{9}{2}\mathrm{Li}_2^2(c^2)-4\mathrm{Li}_2(c^2)\mathrm{Li}_2(-s^2)+[7\ln(c^2)\ln(s^2) \\
&-\ln^2(s^2)-\frac{5}{2}\pi^2+\frac{22}{3}\ln 2-\frac{131}{18}]\mathrm{Li}_2(c^2)+\left[\frac{2}{3}\pi^2-4\ln(c^2)\ln(s^2)\right]\times \\
&\mathrm{Li}_2(-s^2)+\frac{\ln^4(s^2)}{3}+\frac{\ln^4\left(1+s^2\right)}{6}-\ln^3(s^2)\left[\frac{4}{3}\ln(c^2)+\frac{11}{9}\right] \\
&+\ln^2(s^2)\left[7\ln^2(c^2)+\frac{11}{3}\ln(c^2)+\frac{\pi^2}{3}+\frac{22}{3}\ln 2-\frac{32}{9}\right]-\frac{\pi^2}{6}\ln^2\left(1+s^2\right) \\
&+\zeta_3\left[\frac{17}{2}\ln(s^2)-11\ln(c^2)+\frac{7}{2}\ln\left(1+s^2\right)-\frac{21}{2}\ln 2-\frac{99}{4}\right]+\ln(s^2)\times \\
&\left[-\frac{7\pi^2}{2}\ln(c^2)+\frac{22}{3}\ln^2 2-\frac{11}{18}\pi^2+\frac{137}{9}\ln 2-\frac{208}{27}\right]-12\mathrm{Li}_4\left(\frac{1}{2}\right) \\
&+\frac{143}{720}\pi^4-\frac{\ln^4 2}{2}+\frac{\pi^2}{2}\ln^2 2-\frac{11}{6}\pi^2\ln 2+\frac{125}{216}\pi^2+\frac{22}{9}\ln^3 2 \\
&+\frac{137}{18}\ln^2 2+\frac{434}{27}\ln 2-\frac{649}{81}+\mathcal{O}(\epsilon) \},
\end{aligned}$$

[Caola, Delto, Frellesvig, Melnikov '18]

$$\delta=\frac{\delta_{12}}{2},\ s=\sin\frac{\delta_{12}}{2},\ c=\cos\frac{\delta_{12}}{2} \qquad \mathrm{Ci}_n(z)=\frac{\mathrm{Li}_n(e^{iz})+\mathrm{Li}_n(e^{-iz})}{2},\ \mathrm{Si}_n(z)=\frac{\mathrm{Li}_n(e^{iz})-\mathrm{Li}_n(e^{-iz})}{2i}$$

Factorised term

$(T_i \cdot T_j)\cdot(T_k \cdot T_l)$

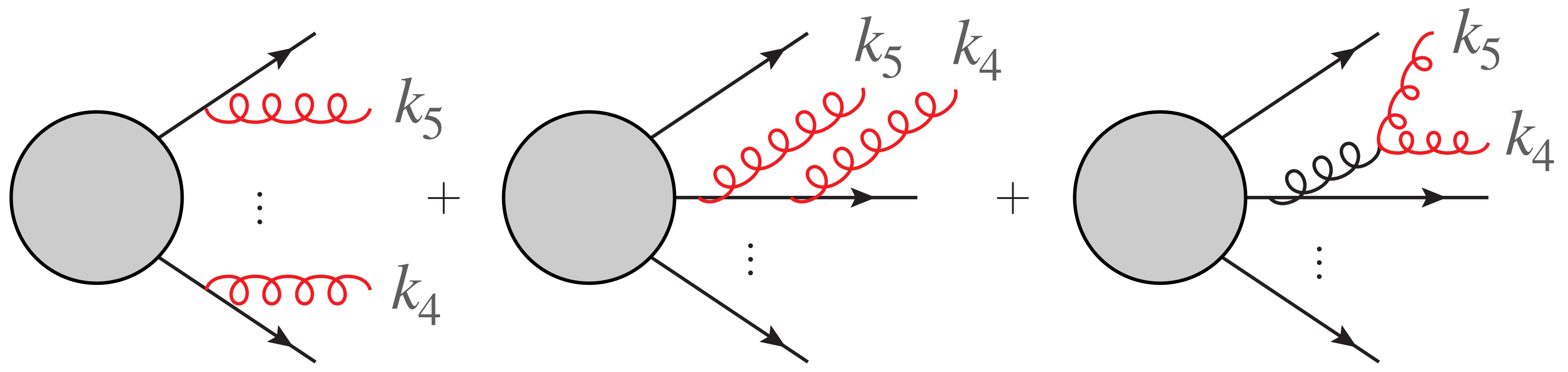
$$\langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\rm LM}(\mathfrak{m},\mathfrak{n}) \rangle_{T^4} = [\alpha_s]^2 \frac{1}{2} \langle I_{\rm S}^2(\epsilon) \cdot F_{\rm LM} \rangle$$

$I_S^2(\epsilon)+I_V^2(\epsilon)$ takes care of "quartic" color-correlated poles

Iterations of NLO!

Follow the (colored) crumbs

Color correlations can only arise from soft real emissions and loop corrections



Double soft

[Catani, Grazzini '99]

Non-factorised term

$T_i \cdot T_j$

$$\langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \rangle_{T^2}$$

$$= [\alpha_s]^2 \left[\frac{C_A}{\epsilon^2} c_1(\epsilon) + \frac{\beta_0}{\epsilon} c_2(\epsilon) + \beta_0 c_3(\epsilon) \right] \langle \widetilde{I}_{\mathrm{S}}(2\epsilon) \cdot F_{\mathrm{LM}} \rangle + \langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \rangle_{T^2}^{\mathrm{fin}}$$

Factorised term

$(T_i \cdot T_j) \cdot (T_k \cdot T_l)$

$$\langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \rangle_{T^4} = [\alpha_s]^2 \frac{1}{2} \langle I_{\mathrm{S}}^2(\epsilon) \cdot F_{\mathrm{LM}} \rangle$$

New structure, but pole content reducible to "variants" of NLO

$$\widetilde{I}_{\mathrm{S}}(2\epsilon) = I_{\mathrm{S}}(2\epsilon) + \mathcal{O}(\epsilon)$$

$I_S^2(\epsilon) + I_V^2(\epsilon)$ takes care of "quartic" color-correlated poles

Iterations of NLO!

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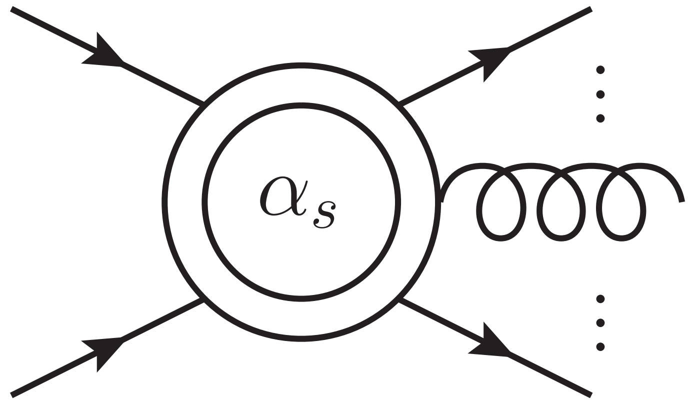

Soft real-virtual

[Catani, Grazzini '00]

$$
\begin{aligned}
&S_{\mathfrak{m}}\, F_{\rm RV}(\mathfrak{m}) \\
&= -\, g_{s,b}^2 \sum_{(ij)}^{N_p} \Big\{ 2\, S_{ij}(p_{\mathfrak{m}})\, (\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LV} - \frac{\alpha_s(\mu)}{2\pi}\frac{\beta_0}{\epsilon}\, 2\, S_{ij}(p_{\mathfrak{m}})\, (\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM} \\
&\quad - 2\,\frac{[\alpha_s]}{\epsilon^2}\, C_A\, A_K(\epsilon)\, \Big(S_{ij}(p_{\mathfrak{m}})\Big)^{1+\epsilon}\, (\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM} \\
&\quad - [\alpha_s]\, \frac{4\pi\,\Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\epsilon\,\Gamma(1-2\epsilon)} \sum_{\substack{k=1\\ k\neq i,j}}^{N_p} \kappa_{ij}\, S_{ki}(p_{\mathfrak{m}})\, \Big(S_{ij}(p_{\mathfrak{m}})\Big)^{\epsilon} f_{abc}\, T_k^a\, T_i^b\, T_j^c\, F_{\rm LM} \Big\}
\end{aligned}
$$

$$
A_K = \frac{\Gamma^3(1+\epsilon)\,\Gamma^5(1-\epsilon)}{\epsilon^2\,\Gamma(1+2\epsilon)\,\Gamma^2(1-2\epsilon)}
$$

Triple-color correlations:

- Vanish for $N_p \geq 4$
- Non-trivial phase space integral
- Finite after integration for FSR

The integrated subtraction term can be almost fully written in terms of NLO-like operators

$$
\begin{aligned}
\big\langle S_{\mathfrak{m}}\, F_{\rm RV}(\mathfrak{m})\big\rangle &= [\alpha_s]^2 \Big\langle \frac{1}{2}\Big[I_{\rm S}(\epsilon)\cdot I_{\rm V}(\epsilon) + I_{\rm V}(\epsilon)\cdot I_{\rm S}(\epsilon)\Big]\cdot F_{\rm LM}\Big\rangle \\
&+ [\alpha_s]\,\Big\langle I_{\rm S}(\epsilon)\cdot F_{\rm LV}^{\rm fin}\Big\rangle - [\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}}\frac{\beta_0}{\epsilon}\,\Big\langle I_{\rm S}(\epsilon) F_{\rm LM}\Big\rangle \\
&- \frac{[\alpha_s]^2}{\epsilon^2}\, C_A\, A_K(\epsilon)\,\Big\langle \widetilde{I}_{\rm S}(2\epsilon)\cdot F_{\rm LM}\Big\rangle \\
&+ [\alpha_s]^2 \Big\langle \Big(\frac{1}{2}\Big[I_{\rm S}(\epsilon)\,,\, \overline{I}_1(\epsilon) - \overline{I}_1^{\dagger}(\epsilon)\Big] + I_{\rm tri}^{\rm RV}(\epsilon)\Big)\cdot F_{\rm LM}\Big\rangle
\end{aligned}
$$

Follow the (colored) crumbs

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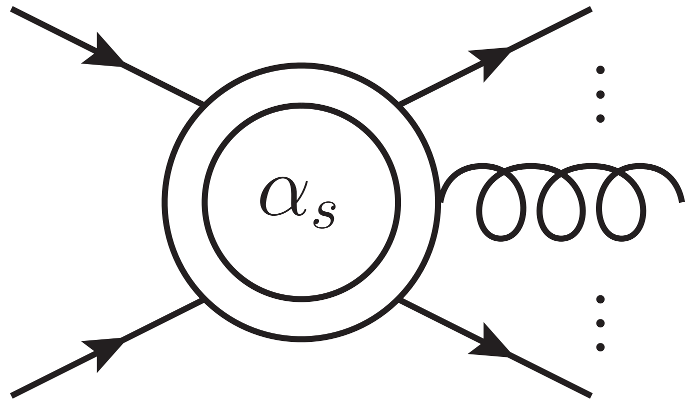

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[Catani, Grazzini '00]

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&\quad - 2\, \frac{[\alpha_s]}{\epsilon^2}\, C_A\, A_K(\epsilon) \Big(S_{ij}(p_{\mathfrak{m}}) \Big)^{1+\epsilon} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\mathrm{LM}} \\
&\quad - [\alpha_s]\, \frac{4\pi\, \Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\epsilon\, \Gamma(1-2\epsilon)} \sum_{\substack{k=1 \\ k\neq i,j}}^{N_p} \kappa_{ij}\, S_{ki}(p_{\mathfrak{m}}) \Big(S_{ij}(p_{\mathfrak{m}}) \Big)^{\epsilon} f_{abc}\, T_k^a\, T_i^b\, T_j^c\, F_{\mathrm{LM}} \Big\}
\end{aligned}
$$

$$A_K = \frac{\Gamma^3(1+\epsilon)\,\Gamma^5(1-\epsilon)}{\epsilon^2\,\Gamma(1+2\epsilon)\,\Gamma^2(1-2\epsilon)}$$

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$$
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&+ [\alpha_s] \big\langle I_{\mathrm{S}}(\epsilon) \cdot F_{\mathrm{LV}}^{\mathrm{fin}} \big\rangle - [\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon \gamma_E}} \frac{\beta_0}{\epsilon} \big\langle I_{\mathrm{S}}(\epsilon)\, F_{\mathrm{LM}} \big\rangle \\
&- \frac{[\alpha_s]^2}{\epsilon^2}\, C_A\, A_K(\epsilon) \big\langle \widetilde{I}_{\mathrm{S}}(2\epsilon) \cdot F_{\mathrm{LM}} \big\rangle \\
&+ [\alpha_s]^2 \Big\langle \Big(\frac{1}{2} \Big[I_{\mathrm{S}}(\epsilon)\,,\, \overline{I}_1(\epsilon) - \overline{I}_1^{\dagger}(\epsilon) \Big] + I_{\mathrm{tri}}^{\mathrm{RV}}(\epsilon) \Big) \cdot F_{\mathrm{LM}} \Big\rangle
\end{aligned}
$$

Structures and **color coefficients** already encountered in **double-virtual** and **double-soft**.

A pattern begins to arise…

Partial recap

$$I_{\mathrm{T}}(\epsilon) = I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon) + I_{\mathrm{C}}(\epsilon)$$ FINITE

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R n_f \,.$$

$\langle F_{\mathrm{LVV}} \rangle$	$\frac{1}{2} I_{\mathrm{V}}^2(\epsilon)$	$\frac{\beta_0}{\epsilon} I_{\mathrm{V}}(\epsilon)$	$K\, I_{\mathrm{V}}(2\epsilon)$	$\frac{\beta_0}{\epsilon} I_{\mathrm{V}}(2\epsilon)$
$\langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m}, \mathfrak{n}) \rangle$	$\frac{1}{2} I_{1,R}^2(\epsilon)$		$\frac{C_A}{\epsilon^2} c_1(\epsilon)\, \widetilde{I}_{\mathrm{S}}(2\epsilon)$	$\frac{\beta_0}{\epsilon} \widetilde{I}_{1,R}(2\epsilon)$
$\langle S_{\mathfrak{m}}\, F_{\mathrm{RV}}(\mathfrak{m}) \rangle$	$\frac{1}{2}\left[I_{\mathrm{S}}(\epsilon) \cdot I_{\mathrm{V}}(\epsilon) + I_{\mathrm{V}}(\epsilon) \cdot I_{\mathrm{S}}(\epsilon)\right]$	$\frac{\beta_0}{\epsilon} I_{\mathrm{S}}(\epsilon)$	$-\frac{C_A}{\epsilon^2} A_K(\epsilon)\, \widetilde{I}_{\mathrm{S}}(2\epsilon)$	

Almost reconstruct $I_{\mathrm{T}}^2(\epsilon)$ → look at collinear

Almost reconstruct $I_{\mathrm{T}}(\epsilon)$ but with extra $1/\epsilon$ → look at collinear

Clear interplay → $C_A, 2\epsilon$

non-transparent cancellation

Almost reconstruct $I_{\mathrm{T}}(2\epsilon)$ but with extra $1/\epsilon$ → look at collinear

Cancellation of double color-correlated poles

$$I_{\mathrm{T}}(\epsilon) = I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon) + I_{\mathrm{C}}(\epsilon)$$ FINITE

Some relevant collinear limits have to be added.

Here we focus on contributions that contain at least one virtual or one soft operator and feature elastic, LO-like kinematics:

$= I_{\mathrm{T}}^2 - I_{\mathrm{C}}^2$ No singular color-correlations

$$\begin{aligned}\Sigma_N^{(\mathrm{V+S}),\mathrm{el}} = {} & [\alpha_s]^2 \frac{1}{2} \langle \left[I_{\mathrm{V}}^2 + I_{\mathrm{V}} I_{\mathrm{S}} + I_{\mathrm{S}} I_{\mathrm{V}} + I_{\mathrm{S}}^2 + 2 I_{\mathrm{C}} I_{\mathrm{V}} + 2 I_{\mathrm{C}} I_{\mathrm{S}} \right] \cdot F_{\mathrm{LM}} \rangle \\ & + [\alpha_s]^2 \frac{\beta_0}{\epsilon} \frac{\Gamma(1-\epsilon)}{e^{\epsilon \gamma_{\mathrm{E}}}} \left\langle \left[-[I_{\mathrm{S}}(\epsilon) + I_{\mathrm{V}}(\epsilon)] + I_{\mathrm{V}}(2\epsilon) + \tilde{c}(\epsilon)\, \widetilde{I}_{\mathrm{S}}(2\epsilon) \right] \cdot F_{\mathrm{LM}} \right\rangle \\ & + [\alpha_s]^2 \left\langle \left[K \frac{\Gamma(1-\epsilon)}{e^{\epsilon \gamma_{\mathrm{E}}}} I_{\mathrm{V}}(2\epsilon) + C_A \left(\frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} - 2^{2+2\epsilon} \delta_g^{C_A}(\epsilon) \right) \widetilde{I}_{\mathrm{S}}(2\epsilon) \right] \cdot F_{\mathrm{LM}} \right\rangle \\ & + [\alpha_s] \langle \left[I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon) \right] \cdot F_{\mathrm{LV}}^{\mathrm{fin}} \rangle ,\end{aligned}$$

$= I_{\mathrm{T}} - I_{\mathrm{C}}$

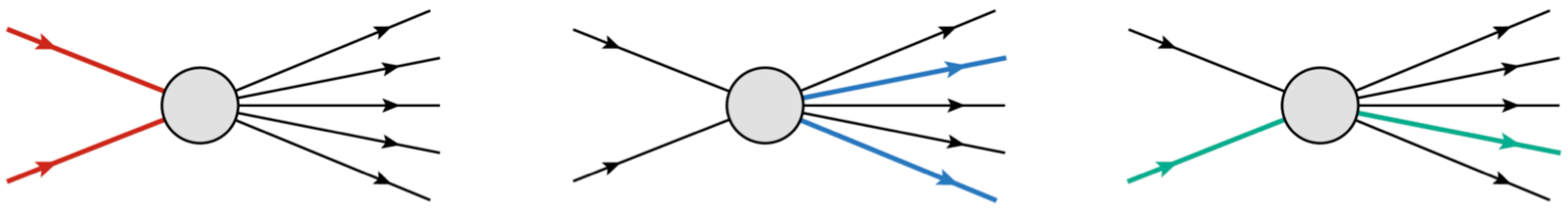
No singular color-correlations

$$\sim -I_{V+S}(\epsilon) + I_{V+S}(2\epsilon) + (\tilde{c}(\epsilon) - 1)\tilde{I}_S(2\epsilon) + \tilde{I}_S(2\epsilon) - I_S(2\epsilon)$$

$\mathcal{O}(\epsilon)$ $\mathcal{O}(\epsilon^2)$ $\mathcal{O}(\epsilon)$

Graphical conclusions

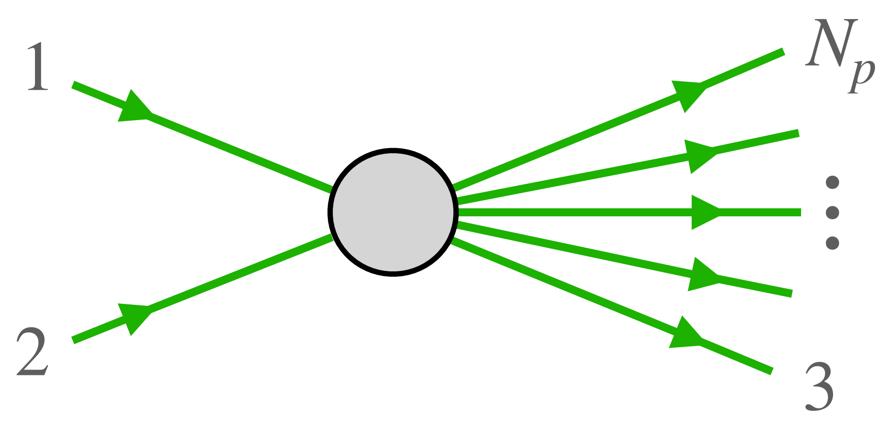
Previous studies:



Can we generalise? yes!

[Devoto, Melnikov, Röntsch, CSS, Tagliabue '24] **New!**

❖ Introduce **NLO-like universal operators** that describe **virtual, soft and collinear singularities,** and combine into **finite quantities**



Virtual component · **Soft component** · **Collinear component**

$$I_{\rm T}(\epsilon) = I_{\rm V}(\epsilon) + I_{\rm S}(\epsilon) + I_{\rm C}(\epsilon)$$

Free of poles
Fully general in the number of partons

❖ Reduce **NNLO** corrections to **iterations of these operators** → demonstrate **cancellations prior to explicit evaluation**

$$\mathrm{d}\hat{\sigma}^{\rm NNLO} = \mathrm{d}\hat{\sigma}^{\rm VV} + \mathrm{d}\hat{\sigma}^{\rm RV} + \mathrm{d}\hat{\sigma}^{\rm RR} + \mathrm{d}\hat{\sigma}^{\rm pdf} = \frac{[\alpha_s]^2}{2}\left\langle M_0 \right| \left[I_{\rm V} + I_{\rm S} + I_{\rm C}\right]^2 \left| M_0 \right\rangle + \ldots \equiv \langle M_0 | I_{\rm T}^2 | M_0 \rangle + \ldots$$

Double-Virtual · Real-Virtual · Double-Real · PDFs Renor.

[slide courtesy of DMT]

Standard conclusions

1. Subtraction schemes are necessary ingredients to obtain precise theoretical predictions.

2. **Nested-soft collinear subtraction** provides an efficient method to deal with n-parton processes:

   I. combine different subtraction terms **to get rid of color-correlations** (and boosted contributions),

   II. reduce the subtraction terms to **few, recurring structures**.

3. Pole cancellation proven analytically for the case-study $q\bar{q} \to X + Ng$.

→ **Finite remainders in agreement with the standard approach for** $q\bar{q} \to X + g\,@NNLO$

Work in progress

Generalisation to arbitrary final- and initial-state partons.

Thank you!

Backup

Cancellation of single-color-correlated contributions

$$-\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\left\langle\left[[\alpha_s]I_{1,R}(\epsilon)+\frac{\alpha_s}{2\pi}2\Re\left(\mathcal{I}_1(\epsilon)\right)+I_C(\epsilon)\right]F_{\rm LM}\right\rangle$$

$$\frac{\beta_0}{\epsilon}[\alpha_s]I_{1,T}(\epsilon)$$

$$+\left(\frac{\alpha_s}{2\pi}\right)^2\frac{\beta_0}{\epsilon}c_\epsilon\left\langle 2\Re(\mathcal{I}_1(2\epsilon))\,F_{\rm LM}\right\rangle+[\alpha_s]^2\frac{\beta_0}{\epsilon}c_2(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle+[\alpha_s]^2\beta_0\,c_3(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle$$

$$+\left\langle\left[-[\alpha_s]^2C_A\,A_K\,\widetilde{I}_{1,R}(2\epsilon)+[\alpha_s]^2\frac{C_A}{\epsilon^2}c_1(\epsilon)\,\widetilde{I}_{1,R}(2\epsilon)+\left(\frac{\alpha_s}{2\pi}\right)^2c_\epsilon\,K\,2\Re\left(\mathcal{I}_1(2\epsilon)\right)\right]F_{\rm LM}\right\rangle$$

$$\frac{\alpha_s}{2\pi}[\alpha_s]\frac{\beta_0}{\epsilon}\left\langle I_{1,T}(2\epsilon)\,F_{\rm LM}\right\rangle-\frac{\alpha_s}{2\pi}[\alpha_s]\frac{\beta_0}{\epsilon}\left\langle I_C(2\epsilon)\,F_{\rm LM}\right\rangle+\Sigma^{(1)}_{T_i\cdot T_j,\,{\rm fin}}$$

No singular, color-correlated contributions

Cancellation of single-color-correlated contributions

$$- \frac{\alpha_s}{2\pi} \frac{\beta_0}{\epsilon} \left\langle \left[[\alpha_s] I_{1,R}(\epsilon) + \frac{\alpha_s}{2\pi} 2\Re\left(\mathcal{I}_1(\epsilon)\right) + I_C(\epsilon) \right] F_{\rm LM} \right\rangle$$

$$+ \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{\beta_0}{\epsilon} c_\epsilon \left\langle 2\Re(\mathcal{I}_1(2\epsilon))\, F_{\rm LM} \right\rangle + [\alpha_s]^2 \frac{\beta_0}{\epsilon} c_2(\epsilon) \left\langle \widetilde{I}_{1,R}(2\epsilon)\, F_{\rm LM} \right\rangle + [\alpha_s]^2 \beta_0\, c_3(\epsilon) \left\langle \widetilde{I}_{1,R}(2\epsilon)\, F_{\rm LM} \right\rangle$$

$$+ \left\langle \left[- [\alpha_s]^2 C_A A_K \widetilde{I}_{1,R}(2\epsilon) + [\alpha_s]^2 \frac{C_A}{\epsilon^2} c_1(\epsilon) \widetilde{I}_{1,R}(2\epsilon) + \left(\frac{\alpha_s}{2\pi}\right)^2 c_\epsilon K\, 2\Re\left(\mathcal{I}_1(2\epsilon)\right) \right] F_{\rm LM} \right\rangle$$

$$\frac{\alpha_s}{2\pi} [\alpha_s] \left\langle c_\epsilon K\, I_{1,T}(2\epsilon)\, F_{\rm LM} \right\rangle - \frac{\alpha_s}{2\pi} [\alpha_s] \left\langle c_\epsilon K\, I_{1,R}(2\epsilon)\, F_{\rm LM} \right\rangle - \frac{\alpha_s}{2\pi} [\alpha_s] \left\langle c_\epsilon K\, I_C(2\epsilon)\, F_{\rm LM} \right\rangle$$

finite

Singular and color-correlated

color-uncorrelated

Cancellation of single-color-correlated contributions

$$-\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\left\langle\left[[\alpha_s] I_{1,R}(\epsilon)+\frac{\alpha_s}{2\pi}2\Re\left(\mathcal{I}_1(\epsilon)\right)+I_C(\epsilon)\right]F_{\rm LM}\right\rangle$$

$$+\left(\frac{\alpha_s}{2\pi}\right)^2\frac{\beta_0}{\epsilon}c_\epsilon\left\langle 2\Re(\mathcal{I}_1(2\epsilon))\,F_{\rm LM}\right\rangle+[\alpha_s]^2\frac{\beta_0}{\epsilon}c_2(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle+[\alpha_s]^2\beta_0\,\underbrace{c_3(\epsilon)}_{\text{finite}}\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle$$

$$+\left\langle\left[-[\alpha_s]^2 C_A A_K\,\widetilde{I}_{1,R}(2\epsilon)+[\alpha_s]^2\frac{C_A}{\epsilon^2}c_1(\epsilon)\,\widetilde{I}_{1,R}(2\epsilon)+\left(\frac{\alpha_s}{2\pi}\right)^2 c_\epsilon\,K\,2\Re\left(\mathcal{I}_1(2\epsilon)\right)\right]F_{\rm LM}\right\rangle$$

$\widetilde{I}_{1,R}(2\epsilon)\longrightarrow I_{1,R}(2\epsilon)$

$-C_A A_K+\frac{C_A}{\epsilon^2}c_1$ finite

$\widetilde{I}_{1,R}(2\epsilon)\longrightarrow I_{1,R}(2\epsilon)$

$$\frac{\alpha_s}{2\pi}[\alpha_s]\left\langle c_\epsilon\,K\,I_{1,T}(2\epsilon)\,F_{\rm LM}\right\rangle-\frac{\alpha_s}{2\pi}[\alpha_s]\left\langle c_\epsilon\,K\,I_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle-\frac{\alpha_s}{2\pi}[\alpha_s]\left\langle c_\epsilon\,K\,I_C(2\epsilon)\,F_{\rm LM}\right\rangle$$

finite — Singular and color-correlated — color-uncorrelated

Cancellation of single-color-correlated contributions

$$-\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\Big\langle\Big[[\alpha_s]I_{1,R}(\epsilon)+\frac{\alpha_s}{2\pi}2\Re\left(\mathcal{I}_1(\epsilon)\right)+I_C(\epsilon)\Big]F_{\rm LM}\Big\rangle$$

$$+\left(\frac{\alpha_s}{2\pi}\right)^2\frac{\beta_0}{\epsilon}c_\epsilon\left\langle 2\Re(\mathcal{I}_1(2\epsilon))\,F_{\rm LM}\right\rangle+[\alpha_s]^2\frac{\beta_0}{\epsilon}c_2(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle+[\alpha_s]^2\,\beta_0\,c_3(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle$$

$$+\Big\langle\Big[-[\alpha_s]^2\,C_A\,A_K\,\widetilde{I}_{1,R}(2\epsilon)+[\alpha_s]^2\frac{C_A}{\epsilon^2}c_1(\epsilon)\,\widetilde{I}_{1,R}(2\epsilon)+\left(\frac{\alpha_s}{2\pi}\right)^2c_\epsilon\,K\,2\Re\left(\mathcal{I}_1(2\epsilon)\right)\Big]F_{\rm LM}\Big\rangle$$

$1/\epsilon$ color-uncorrelated

$$[\alpha_s]\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\Big\langle\Big(I_{1,T}(2\epsilon)-I_{1,T}(\epsilon)\Big)F_{\rm LM}\Big\rangle$$

$$\propto\frac{C_A(C_A+2C_F)}{\epsilon^2}\left(-\frac{131}{72}+\frac{\pi^2}{6}+\frac{11}{6}\log 2\right)+\frac{1}{\epsilon}[\text{color}-\text{correlations}]$$

$$+\Big\langle\Big[[\alpha_s]^2\Big(-C_A\,A_K+\frac{C_A}{\epsilon^2}c_1(\epsilon)+\beta_0\,c_3(\epsilon)\Big)-\frac{\alpha_s}{2\pi}[\alpha_s]\,c_\epsilon\,K\Big]I_{1,R}(2\epsilon)\,F_{\rm LM}\Big\rangle$$

$$-\frac{\alpha_s}{2\pi}[\alpha_s]\Big\langle\Big(c_\epsilon\,K+\frac{\beta_0}{\epsilon}\Big)\,I_C(2\epsilon)\,F_{\rm LM}\Big\rangle$$

$1/\epsilon^2$ color-uncorrelated

Peculiar dependence in the color-correlations, that fits perfectly a contribution from triple-collinear sectors $\Theta^{(b)}$

$$\Big\langle\sum_{i\in\mathrm{TC}}(I-S_{45})\,C_{45}\Theta^{(b)}(F_{\rm LM}-2S_5F_{\rm LM}^{4>5})\,\omega_{4i5i}\,\Delta^{(45)}\Big\rangle$$

$$\rightarrow -4[\alpha_s]^2C_A2^{-2\epsilon}\delta_g(\epsilon)\Big\langle I_{1,R}(2\epsilon)F_{\rm LM}\Big\rangle+\Sigma^{(2)}_{T_i\cdot T_j,\,\rm fin}\;\propto-\frac{C_A(C_A+2C_F)}{\epsilon^2}\left(-\frac{131}{72}+\frac{\pi^2}{6}+\frac{11}{6}\log 2\right)+\mathcal{O}(\epsilon^{-1})$$

Useful relations:

$$I_{1,R}(\epsilon) = -\frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2}\sum_{i\neq j}^{n}\eta_{ij}^{-\epsilon}K_{ij}\mathbf{T}_i\cdot\mathbf{T}_j\,,$$

$$K_{ij} = \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,\eta_{ij}^{1+\epsilon}\,{}_2F_1(1,1,1-\epsilon,1-\eta_{ij})$$
$$= \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,{}_2F_1(-\epsilon,-\epsilon,1-\epsilon,1-\eta_{ij})$$

$$\widetilde{I}_{1,R}(2\epsilon) = -\frac{(2E_{\max}/\mu)^{-4\epsilon}}{(2\epsilon)^2}\sum_{i\neq j}^{n}\eta_{ij}^{-2\epsilon}\widetilde{K}_{ij}\mathbf{T}_i\cdot\mathbf{T}_j$$

$$\widetilde{K}_{ij} = \frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-4\epsilon)}\,\eta_{ij}^{1+3\epsilon}{}_2F_1(1+\epsilon,1+\epsilon,1-\epsilon,1-\eta_{ij})$$
$$= \frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-4\epsilon)}\,{}_2F_1(-2\epsilon,-2\epsilon;1-\epsilon,1-\eta_{ij})\ .$$

$$\widetilde{K}_{ij}(\epsilon) = K_{ij}(2\epsilon)\left[\frac{{}_2F_1(-2\epsilon,-2\epsilon;1-\epsilon,1-\eta_{ij})}{{}_2F_1(-2\epsilon,-2\epsilon,1-2\epsilon,1-\eta_{ij})}\right] = K_{ij}(2\epsilon)\left[1+\mathcal{O}(\epsilon^3)\right]$$

$$\tilde{I}_{1,R}(2\epsilon) = I_{1,R}(2\epsilon) + \mathcal{O}(\epsilon)$$

Useful definitions:

$$\hat{\Gamma}_q = \frac{1}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{2E_1}{\mu}\right)^{-2\epsilon}\left[\gamma_q + \frac{C_F}{\epsilon}(1-e^{-2\epsilon L_1})\right]F_{\rm LM}(1\ldots N) \sim \frac{1}{\epsilon}(\gamma_q + 2C_F\,L_1) + \mathcal{O}(\epsilon^0)$$

$$\hat{\Gamma}_g = \frac{1}{\epsilon}C_A\left(\frac{2E_n}{\mu}\right)^{-2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\gamma^{22}_{z,g\to gg} + \frac{1}{\epsilon}(1-e^{-2\epsilon L_n})\right] \qquad \gamma^{22}_{z,g\to gg} \sim \frac{11}{6} + \frac{1}{9}\left(67-6\pi^2\right)\epsilon + \ldots$$

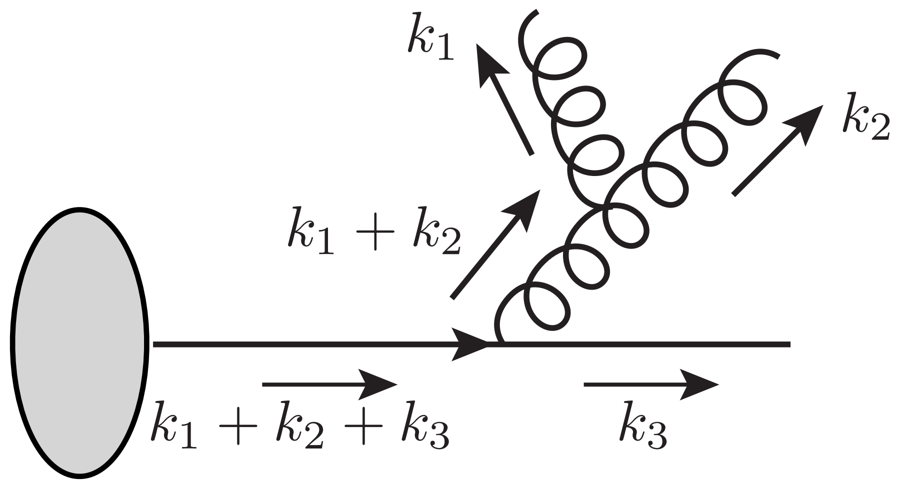
$$\hat{\Gamma}_g(2\epsilon) = \frac{1}{2\epsilon}C_A\left(\frac{2E_n}{\mu}\right)^{-4\epsilon}\frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-4\epsilon)}\left[\gamma^{44}_{z,g\to gg} + \frac{1}{2\epsilon}(1-e^{-4\epsilon L_n})\right]$$

$$P^{\rm gen}_{qq}(z) = -\frac{1}{\epsilon}\hat{P}^{\rm AP,0}_{qq}(z) + P'_{\rm fin,qq}(z)$$

$$G^{(1)}(z)\,F^{(1)}_{\rm LM} = \frac{1}{2}[\alpha_s]^2\left[-P^{\rm gen}_{qq}\otimes\Gamma^{(1)}_q(z)F^{(1)}_{\rm LM}(1_q,2_{\bar{q}};3_g|z) + \Gamma^{(1)}_q P^{\rm gen}_{qq}\otimes F^{(1)}_{\rm LM}(1_q,2_{\bar{q}};3_g|z)\right]$$

$$G^{(3)}(L_3) = \frac{1}{2}\frac{[\alpha_s]^2}{\epsilon^2}C_A^2\left(\frac{2E_3}{\mu}\right)^{-4\epsilon}\left(\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right)^2\left(\gamma^{22}_{z,g\to gg} + \frac{1}{\epsilon}\right)\left(\gamma^{42}_{z,g\to gg} - \gamma^{22}_{z,g\to gg}\right)$$

1. Clear understanding of which singular configurations do actually contribute



$$\sim \frac{1}{(k_1+k_2)^2}\frac{1}{(k_1+k_2+k_3)^2} = \frac{1}{2k_1\cdot k_2}\frac{1}{2k_1\cdot k_2 + 2k_1\cdot k_3 + 2k_2\cdot k_3} \Longleftrightarrow k_1 \to 0 \text{ and } k_2 \parallel k_3$$

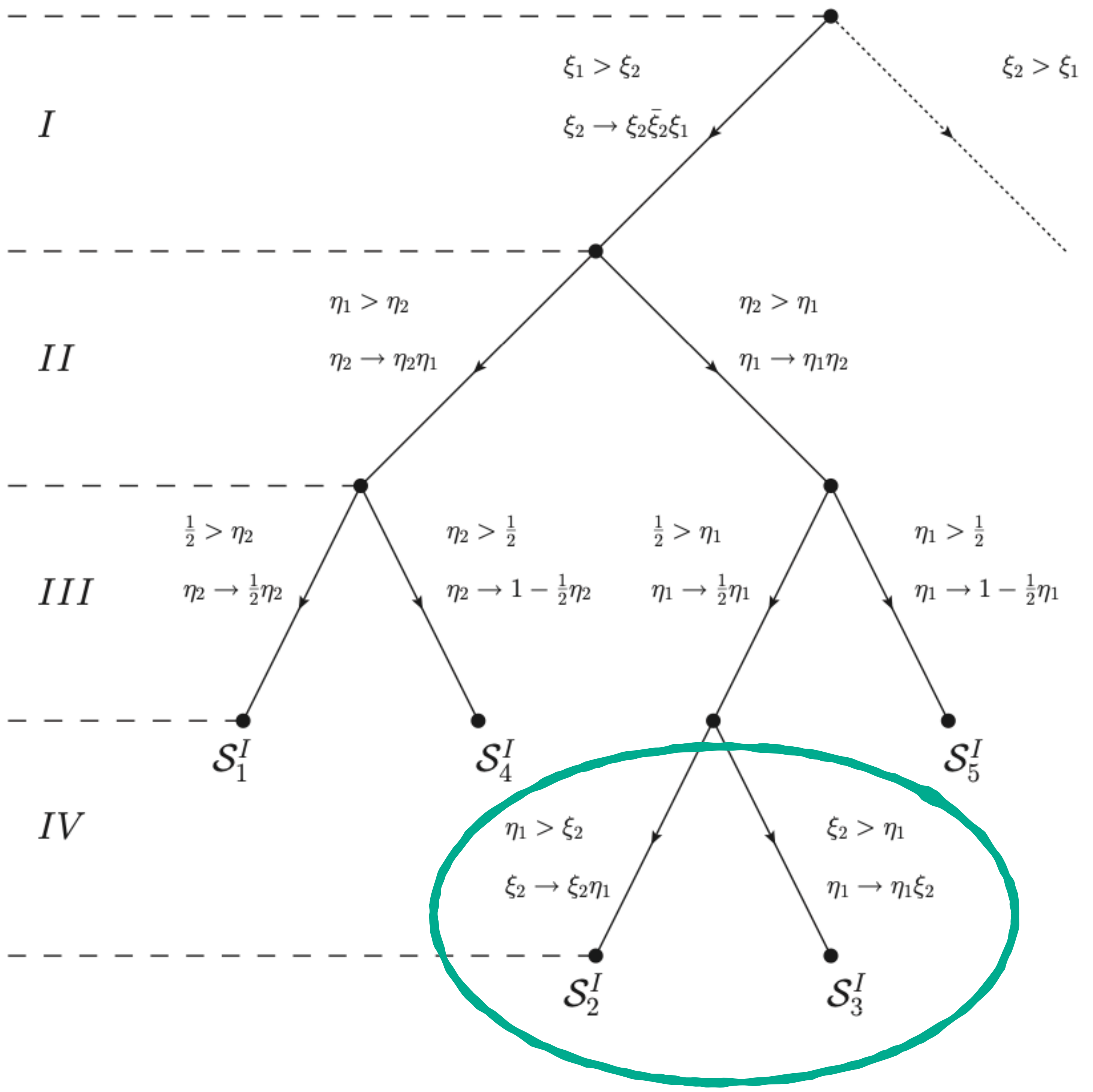
Entangled soft-collinear limits of diagrams can not be treated in a process-independent way.

Do non-commutative limits actually contribute?

STRIPPER was implemented taking into account all the possible choices of soft and collinear limits order -> redundant configurations were included

Gauge invariant amplitudes are free of entangled singularities thanks to color coherence: soft parton does not resolve angles of the collinear partons

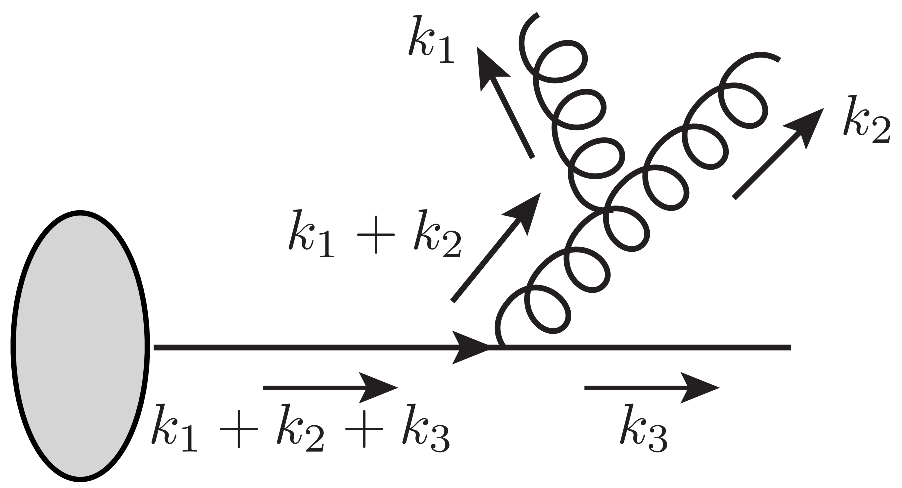
Soft-collinear limits can be described by taking the known soft and collinear limits sequentially



[Czakon 1005.0274]

2. Get to the point where the problem is well defined

a) Identify the overlapping singularities
b) Regulate them



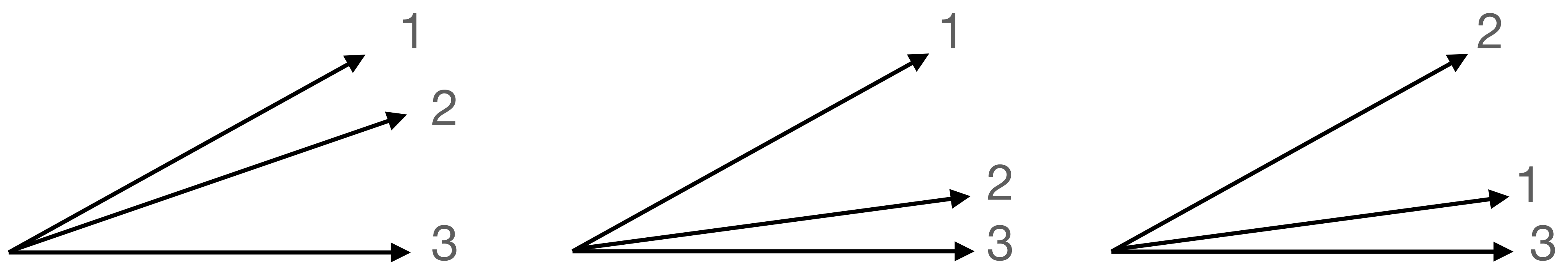
$$\sim \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2)} \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2) + E_1 E_3 (1 - \vec{n}_1 \cdot \vec{n}_3) + E_2 E_3 (1 - \vec{n}_2 \cdot \vec{n}_3)}$$

Soft origin: $E_1 \to 0 \quad E_2 \to 0 \quad E_1, E_2 \to 0$

Collinear origin: $\vec{n}_1 \parallel \vec{n}_2 \quad \vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3$

$$E_1 \ll E_2 \,, \quad E_2 \ll E_1$$

Includes **strongly ordered** configurations



$\vec{n}_1 \cdot \vec{n}_2 < \vec{n}_1 \cdot \vec{n}_3$ $\quad \vec{n}_2 \cdot \vec{n}_3 < \vec{n}_1 \cdot \vec{n}_3$ $\quad \vec{n}_1 \cdot \vec{n}_3 < \vec{n}_2 \cdot \vec{n}_3$

Soft and collinear modes do not intertwine: soft subtraction can be done globally. Collinear singularities have still to be regulated.
Strongly ordered configurations have to be properly taken into account.

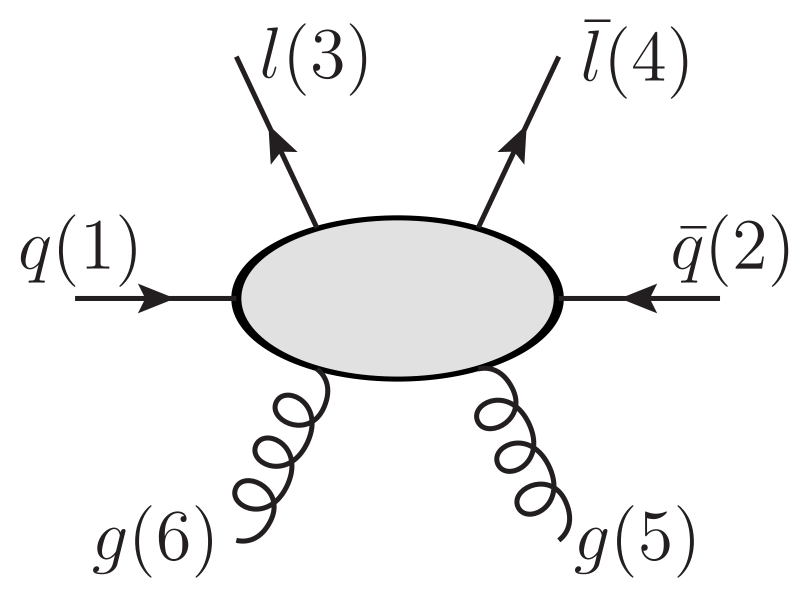
Phase space partitions

Efficient way to simplify the problem: introduce **partition functions** (following FKS philosophy):

- **Unitary partition**
- Select a **minimum number of singularities** in each sector
- Do **not affect** the **analytic integration** of the counterterms

Definition of partition functions benefits from remarkable degree of **freedom**: different approaches can be implemented

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ \, g \, g$ [Caola, Melnikov, Röntsch 1702.01352]



$$1 = \omega^{51,61} + \omega^{52,62} + \omega^{51,62} + \omega^{52,61}$$

$$\omega^{51,61} = \frac{\rho_{25}\,\rho_{26}}{d_5\,d_6}\left(1 + \frac{\rho_{15}}{d_{5621}} + \frac{\rho_{16}}{d_{5612}}\right) \qquad \omega^{51,62} = \frac{\rho_{25}\,\rho_{16}\,\rho_{56}}{d_5\,d_6\,d_{5612}}$$

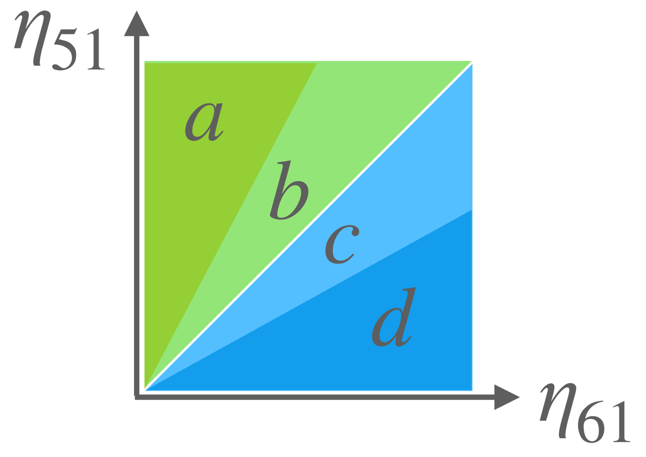
$$\omega^{52,62} = \frac{\rho_{15}\,\rho_{16}}{d_5\,d_6}\left(1 + \frac{\rho_{25}}{d_{5621}} + \frac{\rho_{26}}{d_{5612}}\right) \qquad \omega^{52,61} = \frac{\rho_{15}\,\rho_{26}\,\rho_{56}}{d_5\,d_6\,d_{5621}}$$

$$\rho_{ab} = 1 - \cos\vartheta_{ab}\,,\ \eta_{ab} = \rho_{ab}/2$$

$$d_{i=5,6} = \rho_{1i} + \rho_{2i} = 2$$

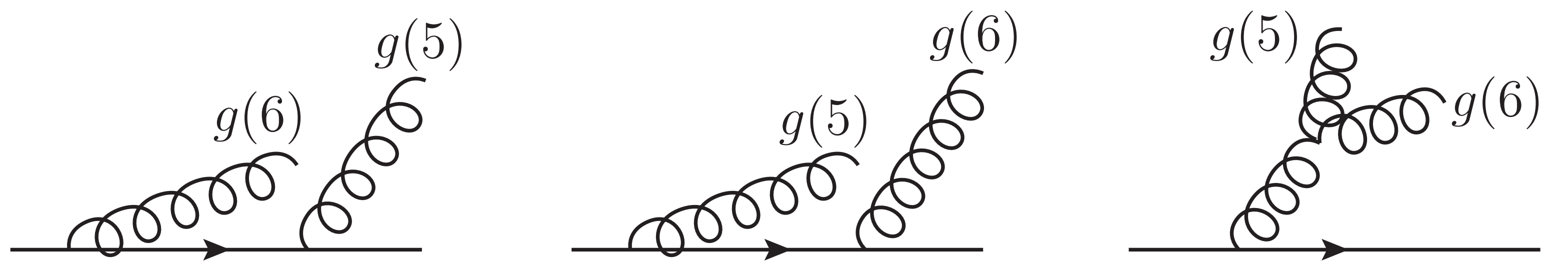
$$d_{5621} = \rho_{56} + \rho_{52} + \rho_{61}$$

$$d_{5612} = \rho_{56} + \rho_{51} + \rho_{62}$$



$$1 = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{61} < \eta_{51}\right) + \theta\left(\eta_{51} < \frac{\eta_{61}}{2}\right) + \theta\left(\frac{\eta_{61}}{2} < \eta_{51} < \eta_{61}\right)$$

$$= \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}$$



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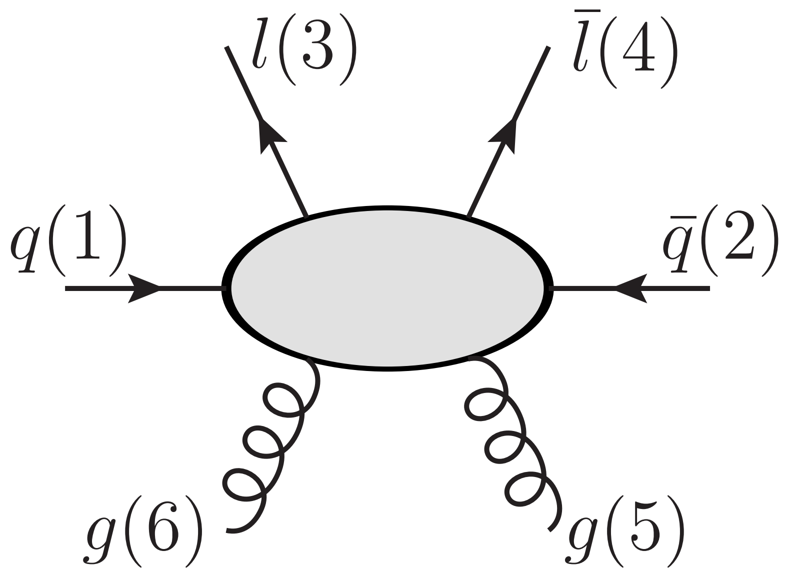


Advantages:

1. Simple definition
2. Structure of collinear singularities fully defined
3. Same strategy holds for NNLO mixed QCDxEW processes
4. **Minimum number of sector**

Disadvantages:

1. Partition based on angular ordering -> Lorentz invariance not preserved
   -> angles defined in a given reference frame
2. Theta function

3. Solve the PS integrals

The problem is now well defined:

A. **Singular kernels** and their nested limits have to be **subtracted from the double real correction** to get integrable object

$$\int d\Phi_{n+2}\, RR_{n+2} = \int d\Phi_{n+2} \left[RR_{n+2} - K_{n+2}\right] + \int d\Phi_{n+2}\, K_{n+2}$$

$$K_{n+2} \supset C_{ij},\, C_{kl},\, S_i,\, S_{ij},\, C_{ijk}$$

B. **Counterterms** have to be **integrated over the unresolved phase space**

$$I = \int \mathrm{PS}_{\text{unres.}} \otimes \text{Limit} \otimes \text{Constraints}$$

The ‘Limit’ component is universal and known. The phase space is well defined. Constraints may vary depending on the scheme.

Several kinematic structures have to be integrated **analytically** over a 6-dim PS.

Different approximations and techniques can be applied: the results assume different form depending on the adopted strategy

Two main structure are the most complicated ones and affect most of the physical processes:

- **Double soft**
- **Triple collinear**

Kernels integration

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ g\, g$ [Caola, Delto, Frellesvig, Melnikov 1807.05835, Delto, Melnikov 1901.05213]

Two soft parton (5,6) and two hard massless radiator (1,2): arbitrary relative angle between the three-momenta of the radiators

$$I_{12}^{(gg)(56)} = \frac{(1-\epsilon)(s_{51}s_{62} + s_{52}s_{61}) - 2s_{56}s_{12}}{s_{56}^2(s_{51} + s_{61})(s_{52} + s_{62})} + s_{12} \frac{s_{51}s_{62} + s_{52}s_{61} - s_{56}s_{12}}{s_{56}s_{51}s_{62}s_{52}s_{61}} \left[1 - \frac{1}{2} \frac{s_{51}s_{62} + s_{52}s_{61}}{(s_{51} + s_{61})(s_{52} + s_{62})}\right]$$

$$I_{S_{56}}^{(gg)} = \int [\mathrm{d}k_5]\,[\mathrm{d}k_6]\,\theta(E_{\max} - E_5)\,\theta(E_5 - E_6)\,I_{12}^{(gg)(56)}\left(k_1, k_2, k_5, k_6\right)$$

$$[\,\mathrm{d}f_i\,] = \frac{\mathrm{d}^d k_i}{(2\pi)^d}\,(2\pi)\,\delta_+(k_i^2)$$

$$E_5 = E_{\max}\,\xi \qquad E_6 = E_{\max}\,\xi\,z \qquad 0 < \xi < 1\,,\, 0 < z < 1$$

Reverse unitarity: **map phase space integrals onto loop integrals** [Anastasiou, Melnikov 0207004]
after defining integral families, integration-by-part identities. Differential equations w.r.t. the ratio of energies of emitted gluons at fixed angle.
Boundary conditions for z=0, and arbitrary angle

Ingredients for higher-order corrections and main difficulties

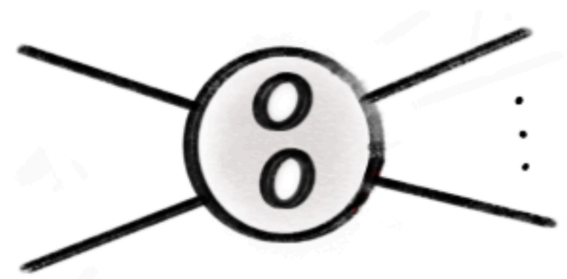
$$\frac{d\sigma}{dX} = \frac{d\sigma_{\rm LO}}{dX} + \alpha_s \frac{d\sigma_{\rm NLO}}{dX} + \alpha_s^2 \frac{d\sigma_{\rm N^2LO}}{dX} + \alpha_s^3 \frac{d\sigma_{\rm N^3LO}}{dX} + \ldots$$

X = IRC-safe, $\delta_{X_i} = \delta(X - X_i)$

Strong couplings:
$\alpha_s \sim 0.1$

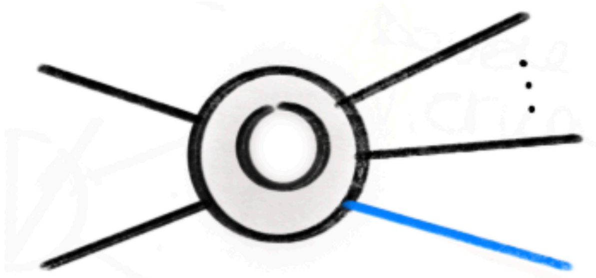
$\mathcal{O}(\alpha_s) \sim 10\,\%$ $\mathcal{O}(\alpha_s^2) \sim 1\,\%$ $\mathcal{O}(\alpha_s^3) \sim 0.1\,\%$

$$\frac{d\sigma_{\rm N^2LO}}{dX} = \int d\Phi_n\, VV\, \delta_{X_n} + \int d\Phi_{n+1}\, RV\, \delta_{X_{n+1}} + \int d\Phi_{n+2}\, RR\, \delta_{X_{n+2}}$$



Explicit poles

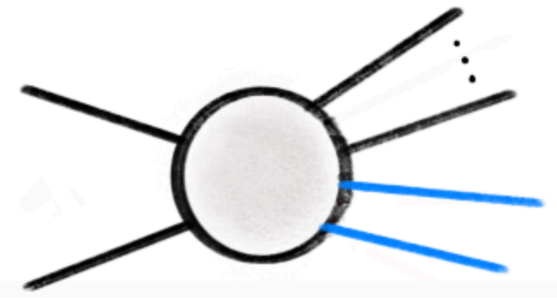
- Almost all relevant amplitudes for $2 \to 2$ massless processes
- First results for $2 \to 3$ amplitudes



Explicit poles from virtual corrections

Phase space singularities

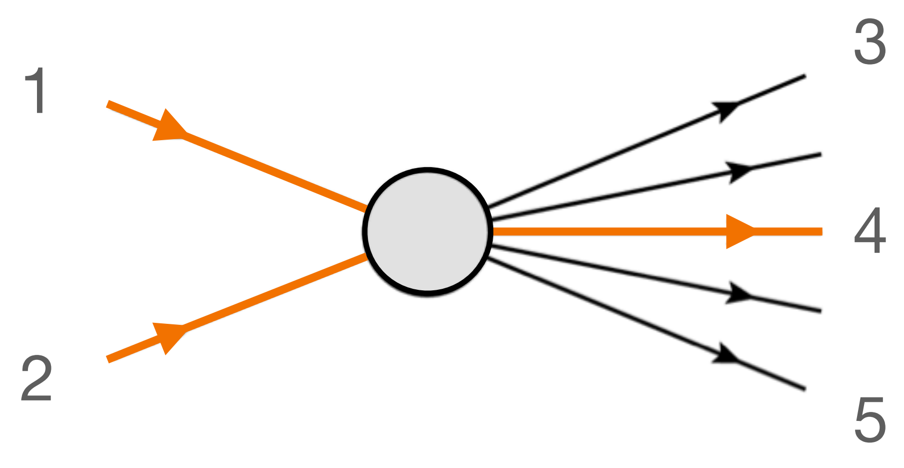
- **One-loop amplitudes in degenerate kinematics**
- OpenLoops, Recola



Well defined in the non-degenerate kinematics

- **Real emission corrections finite in the bulk of the allowed PS**
- IR singularities arise upon integration over energies and angles of emitted partons

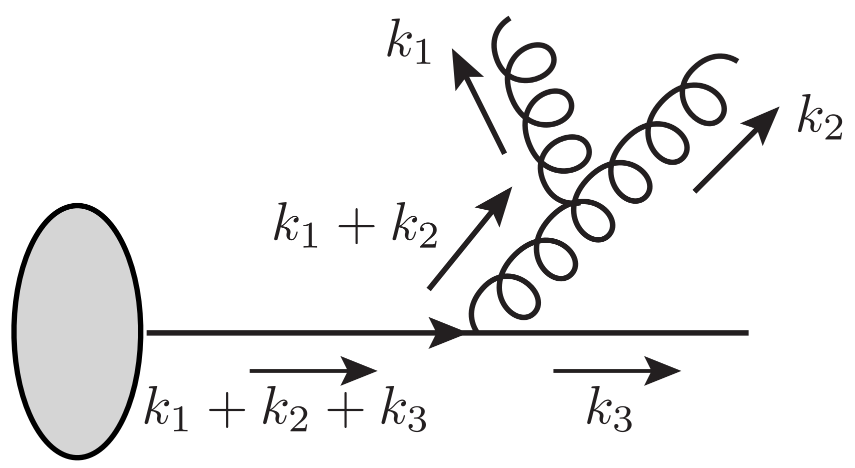
Summary of the talk



- A **subtraction scheme** based of **FKS** was proposed.
- Singular **kernels** for initial- and final-state emission are **known**. **Integration** of the most complicated **double-unresolved limits performed for arbitrary kinematics**.
- **Application to simple processes** worked out straightforwardly.
- **In principle**, **general formulas for subtraction terms and fully-resolved components** for an **arbitrary number of partons are available**.
- This can be done because we know how to deal with **multiple radiators** [**partitioning**, **energy ordering**]
- However, for **non-trivial processes** (e.g. V+j) **several difficulties** arise: partitioning, energy ordering and Casimir operators **obscure simplifications that are suggested by the simple structure of Catani's operator**.
- This suggests that we may need to take **some steps back**.

Nested soft-collinear subtraction at NNLO: generalities [Caola, Melnikov, Röntsch 1702.01352]

Extension of **FKS subtraction** [Frixione, Kunst, Signer 9512328] **to NNLO and inspired by STRIPPER** [Czakon 1005.0274]



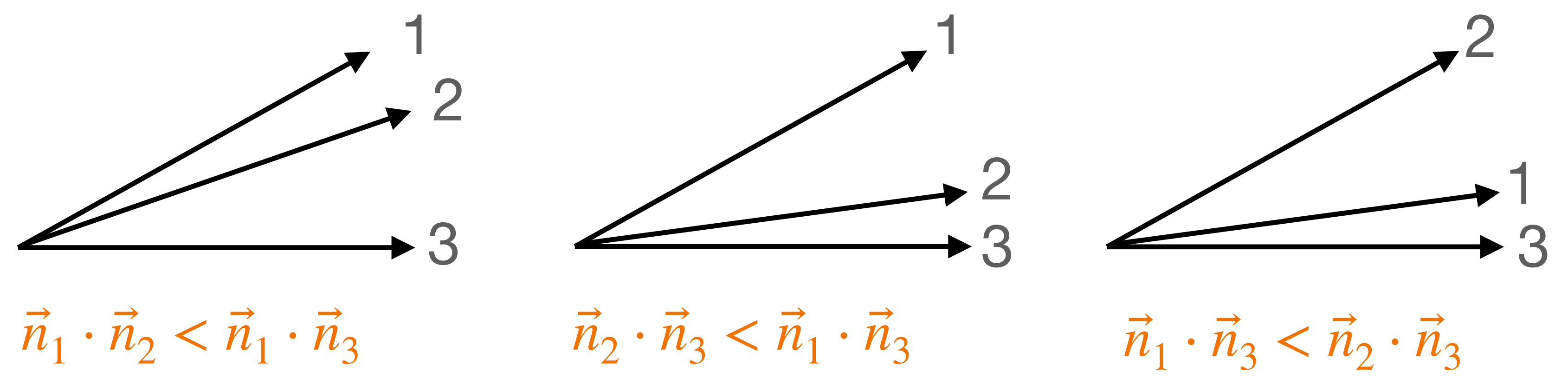
$$\sim \frac{1}{E_1 E_2 (1-\vec{n}_1\cdot\vec{n}_2)} \frac{1}{E_1 E_2(1-\vec{n}_1\cdot\vec{n}_2) + E_1 E_3(1-\vec{n}_1\cdot\vec{n}_3) + E_2 E_3(1-\vec{n}_2\cdot\vec{n}_3)}$$

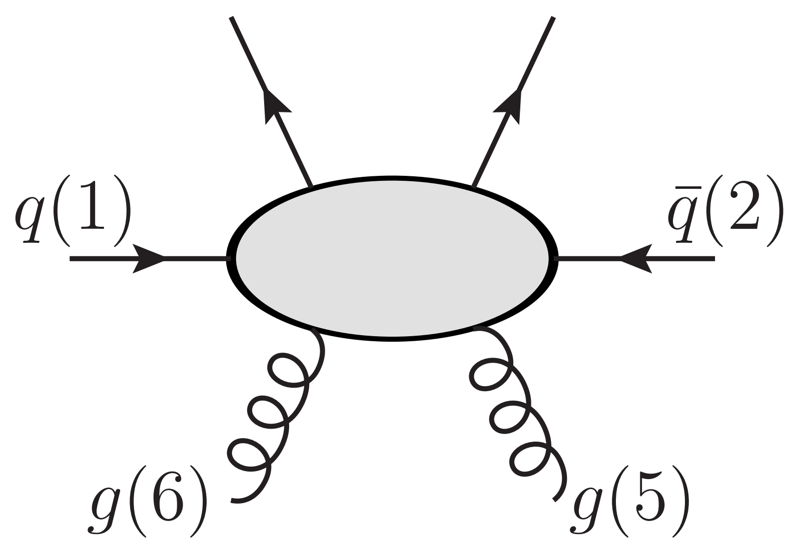
$E_1 \to 0 \quad E_2 \to 0 \quad E_1, E_2 \to 0$

$\vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3$

$\vec{n}_1 \parallel \vec{n}_2$

Strongly-ordered configurations have also to be included: $E_1 \ll E_2\,, \quad E_2 \ll E_1$

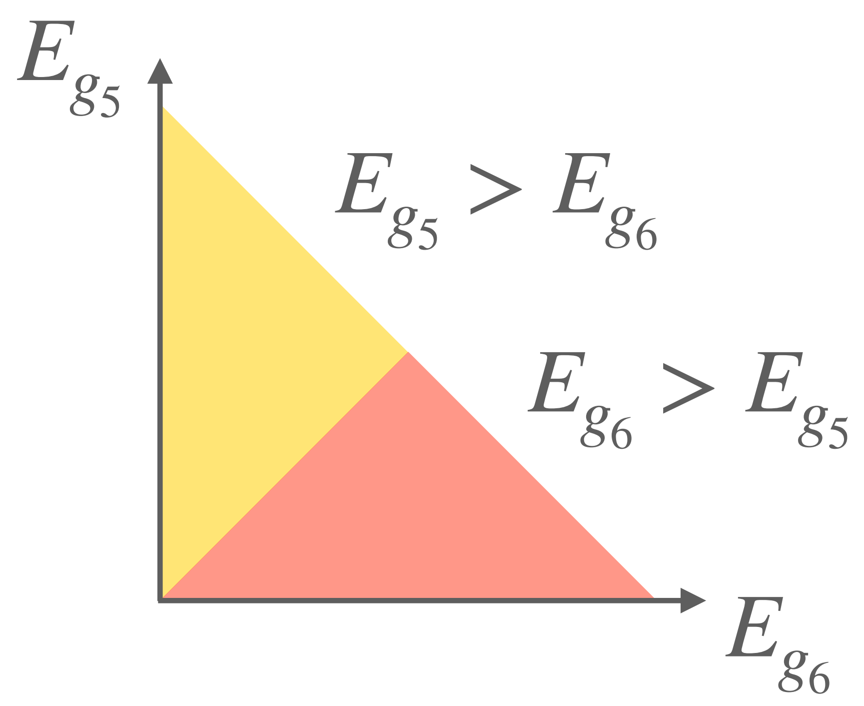




Soft limits:

- Non-trivial structure of double-soft eikonal
- Strongly-ordered limits to disentangle

$$1 = \theta\big(E_{g_5} - E_{g_6}\big) + \theta\big(E_{g_6} - E_{g_5}\big)$$



Nested soft-collinear subtraction at NNLO: generalities *[Caola, Melnikov, Röntsch 1702.01352]*

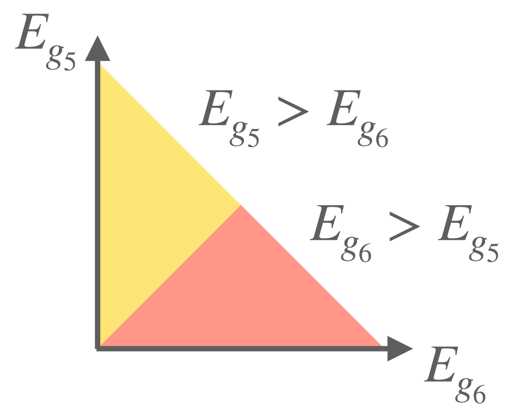
Extension of **FKS subtraction** *[Frixione, Kunst, Signer 9512328]* **to NNLO and inspired by STRIPPER** *[Czakon 1005.0274]*

- Exploit colour-coherence to discard interplay between soft and collinear

  → subtract soft limits first, then collinear

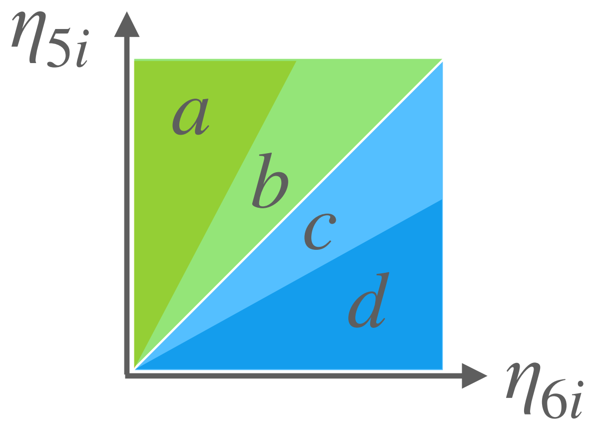
"nested approach"

- Define subtraction terms in 3 steps:
  - Globally remove double soft singularities
  - Globally remove single soft singularities [using energy ordering]
  - FKS partition and sectoring to treat the minimum number of collinear singularities at a time



$$1 = \sum_{i,j} \omega^{i5,j6}$$

$$\omega^{5i,6i} = \omega^{5i,6i}\left(\theta_a + \theta_b + \theta_c + \theta_d\right)$$



- Integrate subtraction terms analytically using Reverse Unitarity *[Anastasiou, Melnikov '02]*:
  map phase space integrals onto loop integrals *[Caola, Delto, Frellesvig, Melnikov '18, '19]*

Double virtual contribution

Universal structure, regulated by Catani's operator, valid for any number of external coloured partons *[Catani '98]* . Features a single structure with color-correlations

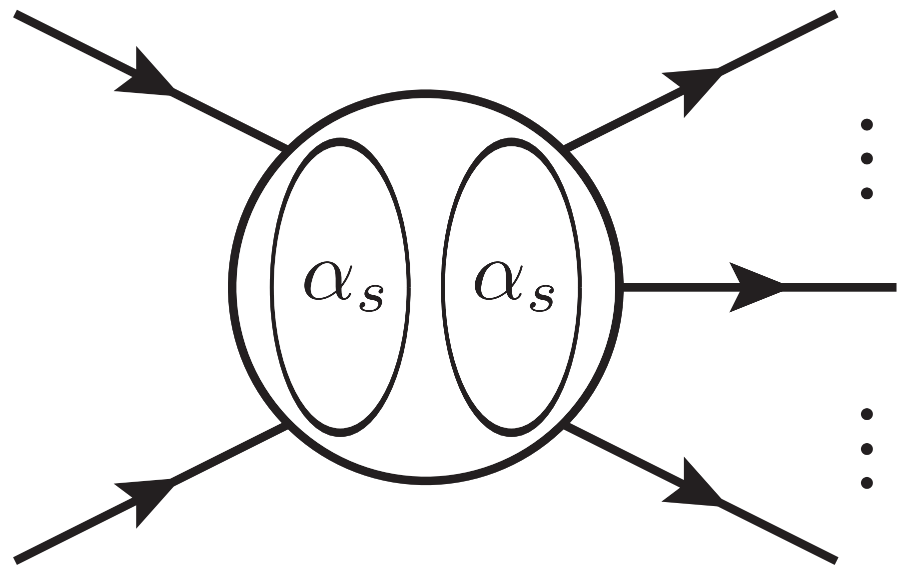
$$\langle F_{\mathrm{LVV}} \rangle = \left(\frac{\alpha_s}{2\pi}\right)^2 \Big\langle \frac{1}{2}\left[2\Re(\mathcal{I}_1(\epsilon))\right]^2 F_{\mathrm{LM}} - \frac{\beta_0}{\epsilon} 2\Re(\mathcal{I}_1(\epsilon)) F_{\mathrm{LM}}$$
$$+ \frac{e^{-\epsilon\gamma_E}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \frac{\beta_0}{\epsilon} 2\Re(\mathcal{I}_1(2\epsilon)) F_{\mathrm{LM}} + \frac{e^{-\epsilon\gamma_E}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} K\, 2\Re(\mathcal{I}_1(2\epsilon)) F_{\mathrm{LM}}$$
$$+ 2\, \underbrace{\frac{e^{\epsilon\gamma_E}}{4\epsilon\,\Gamma(1-\epsilon)} \mathcal{H}_2(\epsilon)}_{\text{Process-dependent}} F_{\mathrm{LM}} + 2\Re(\mathcal{I}_1(\epsilon)) F_{\mathrm{LV}}^{\mathrm{fin}} + \underbrace{F_{\mathrm{LVV}}^{\mathrm{fin}} + F_{\mathrm{LV^2}}^{\mathrm{fin}}}_{\text{Finite remainders from 2-loop and (1-loop)}^2\text{ amplitudes}} \Big\rangle ,$$

Process-dependent

Finite remainders from 2-loop and $(\text{1-loop})^2$ amplitudes

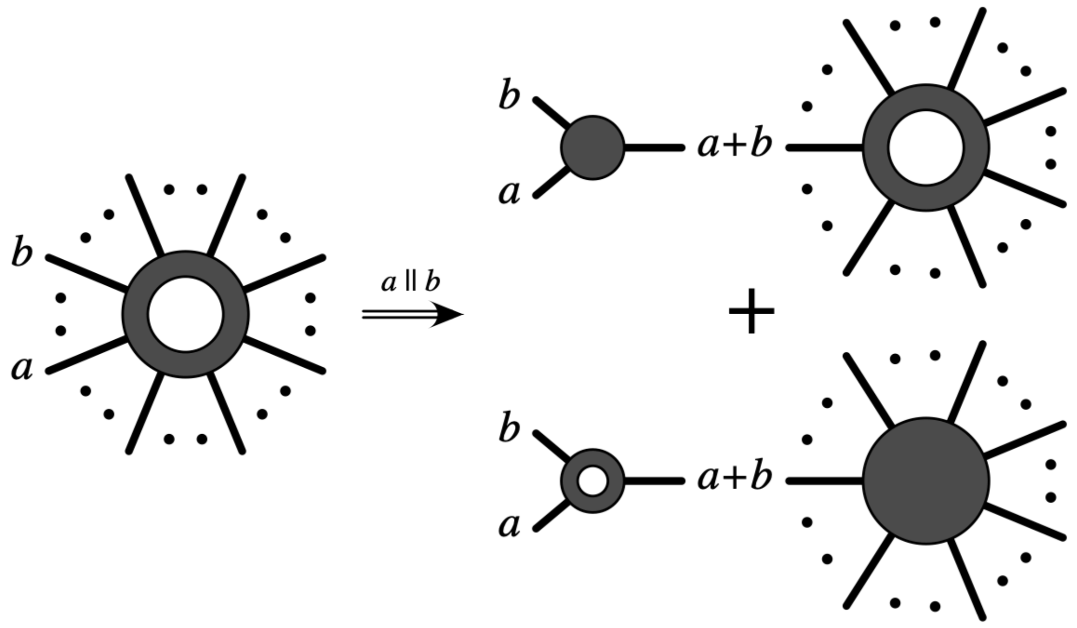
Color-correlations inside
$\mathcal{I}_1(\epsilon)$
(already encountered at NLO)

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R n_f \, .$$



Hard-collinear real-virtual and single soft RR

Also in this case the IR structure is know in full generality *[Kosower '99, Bern, Del Duca et al. '99]*.
For $q\bar{q} \to V + ggg$ the integrated contribution reads



$$\sum_{i=1}^{3} \langle (I - S_4) C_{4i}\, \Delta^{(4)}\, F_{\rm LV}(4) \rangle = [\alpha_s]^2 \left\langle I_C(\epsilon)\, 2\Re(\bar{\mathcal{I}}_1(\epsilon)) F_{\rm LM} \right\rangle + \frac{\alpha_s}{2\pi} [\alpha_s] \left\langle I_C(\epsilon) F_{\rm LV}^{\rm fin} \right\rangle$$
$$- [\alpha_s] \frac{\alpha_s}{2\pi} \frac{\beta_0}{\epsilon} \left\langle I_C(\epsilon) F_{\rm LM} + \sum_{k=1}^{2} P_{qq}^{\rm gen}(z) \otimes F_{\rm LM}^{(k)}(z) \right\rangle$$
$$+ [\alpha_s]^2 \left\langle \Gamma_g^{\rm 1loop}\, F_{\rm LM} \right\rangle + \frac{[\alpha_s]^2}{\epsilon} \sum_{i=k}^{2} \left\langle P_{qq}^{\rm 1loop} \otimes F_{\rm LM}^{(k)}(z) \right\rangle$$
$$+ [\alpha_s]^2 \sum_{k=1}^{2} \left\langle P_{qq}^{\rm gen}(z) \otimes 2{\rm Re}(\bar{\mathcal{I}}_1(z,\epsilon)) F_{\rm LM}^{(k)}(z) \right\rangle + [\alpha_s] \frac{\alpha_s}{2\pi} \sum_{k=1}^{2} \langle P_{qq}^{\rm gen}(z) \otimes F_{\rm LV}^{{\rm fin},\,(k)}(z) \rangle$$

Same structure as NLO

One-loop splitting functions, known analytically

Single soft: different subtraction terms combined → careful with the limits order

$$\sum_{i=1}^{3} \left\langle (I - S_4) C_{4i} \left[\langle S_5\, \Delta^{(45)}\, F_{\rm LM}^{4>5}(4,5) \rangle \right] + S_5\, (I - S_4)\, C_{4i}\, \Delta^{(45)}\, F_{\rm LM}^{5>4}(4,5) \right\rangle =$$
$$+ [\alpha_s]^2 \sum_{k=1}^{2} \left\langle I_{1R}(\epsilon) P_{qq}^{\rm gen}(z) \otimes F_{\rm LM}^{(k)}(z) \right\rangle + [\alpha_s]^2 \left\langle I_{1R}(\epsilon)\, I_C(\epsilon) F_{\rm LM} \right\rangle$$
$$+ \frac{[\alpha_s]^2}{\epsilon^2} N_s\, C_A \left[\sum_{k=1}^{2} \left\langle \left(\frac{2E_k}{\mu} \right)^{-2\epsilon} \tilde{P}_{qq}^{\rm gen,}(z) \otimes F_{\rm LM}^{(k)}(z) \right\rangle + \sum_{k=1}^{3} \left\langle \left(\frac{2E_k}{\mu} \right)^{-2\epsilon} \hat{\Gamma}^{(k)\,{\rm e.o.}}\, F_{\rm LM} \right\rangle \right]$$

Status so far

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R n_f \, .$$

$\langle F_{\rm LVV} \rangle$	$\frac{1}{2}\left[2\Re(\mathcal{I}_1(\epsilon))\right]^2$	$\frac{\beta_0}{\epsilon}\, 2\Re(\mathcal{I}_1(\epsilon))$	$K\, 2\Re(\mathcal{I}_1(2\epsilon))$	$\frac{\beta_0}{\epsilon}\, 2\Re(\mathcal{I}_1(2\epsilon))$	
$\langle S_{45} F_{\rm LM}^{4>5}(4,5) \rangle$	$\frac{1}{2}\, I_{1,R}^2(\epsilon)$		$\frac{C_A}{\epsilon^2}\, \widetilde{I}_{1,R}(2\epsilon)$	$\frac{\beta_0}{\epsilon}\, \widetilde{I}_{1,R}(2\epsilon)$	$\beta_0\, \widetilde{I}_{1,R}(2\epsilon)$
$\langle S_4\, F_{\rm LRV}(4) \rangle$	$I_{1,R}(\epsilon)\, 2\Re(\mathcal{I}_1(\epsilon))$	$\frac{\beta_0}{\epsilon}\, I_{1,R}(\epsilon)$	$C_A\, A_K\, \widetilde{I}_{1,R}(2\epsilon)$		
$\langle (I - S_4) C_{4i}\, \Delta^{(4)}\, F_{\rm LV}(4) \rangle$	$I_C(\epsilon)\, 2\Re(\bar{\mathcal{I}}_1(\epsilon))$	$\frac{\beta_0}{\epsilon}\, I_C(\epsilon)$			
$\Big\langle (I - S_4) C_{4i} \left[\langle S_5\, \Delta^{(45)}\, F_{\rm LM}^{4>5}(4,5) \rangle \right] + S_5\, (I - S_4)\, C_{4i}\, \Delta^{(45)}\, F_{\rm LM}^{5>4}(4,5) \Big\rangle$	$I_{1R}(\epsilon)\, I_C(\epsilon)$				

A term $I_C^2(\epsilon)$ needed to reconstruct $(I_1 + I_{1,R} + I_C)^2$

→ look at double-collinear

reconstruct $I_1(\epsilon) + I_{1,R}(\epsilon) + I_C(\epsilon)$ but with extra $1/\epsilon$

Clear interplay → $C_A, 2\epsilon$

non-transparent cancellation

Suggest $I_1(2\epsilon) + I_{1,R}(2\epsilon) + I_C(2\epsilon)$ but with extra $1/\epsilon$

Hard-collinear real-virtual and single soft RR

Manipulations required to reconstruct recurring structures and match, for instance, PDFs-like corrections

$$\frac{1}{2}\left\langle \sum_{i,j}(I-S_4)\,(I-S_5)\,C_{4i}C_{5j}\,\Delta^{(45)}F_{\rm LM}(4,5)\right\rangle = \Bigg\langle \frac{1}{2}[\alpha_s]^2\Big(I_C(\epsilon)\Big)^2 F_{\rm LM} + \sum_{k=1}^{2} G^{(k)}(z)\,F_{\rm LM}^{(k)}(z) + G^{(3)}\,F_{\rm LM}$$

$$+\frac{1}{2}[\alpha_s]^2\sum_{k=1}^{2}\left[P_{qq}^{\rm gen}\otimes P_{qq}^{\rm gen}(z)\right]_{\rm pdf}F_{\rm LM}^{(k)}(z) + [\alpha_s]^2\sum_{k=1}^{2}P_{qq}^{\rm gen}\otimes I_C(z,\epsilon)F_{\rm LM}^{(k)}(z)$$

$$+[\alpha_s]^2\,P_{qq}^{\rm gen}(z_1)\otimes F_{\rm LM}(z_1,z_2)\otimes P_{qq}^{\rm gen}(z_2)\Bigg\rangle$$

Cancellation of the double-color-correlated contributions

$$\frac{1}{2}\left\langle\left(\frac{\alpha_s}{2\pi}\,2\Re\big(\mathcal{I}_1(\epsilon)\big) + [\alpha_s]I_{1,R}(\epsilon) + [\alpha_s]I_C(\epsilon)\right)^2 F_{\rm LM}\right\rangle = \frac{1}{2}[\alpha_s]^2\left\langle I_{1,T}^2(\epsilon)\,F_{\rm LM}\right\rangle$$

$$\longrightarrow \ \text{finite}$$

Same combination encountered at NLO: finite, and easy to be computed.