Minimal Cuts and Genealogical Constraints on Feynman Integrals

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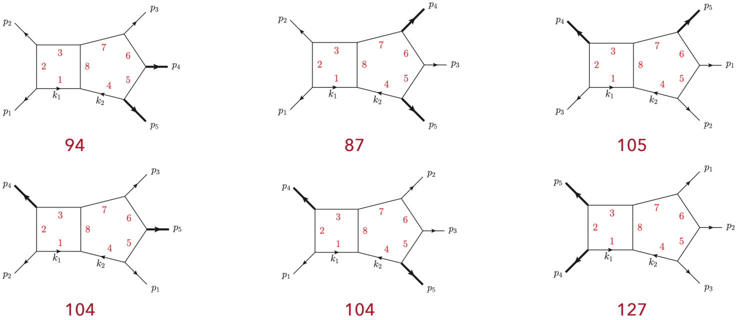
Amplitudes 2024
June 11, 2024

based on arXiv:2406.05943
with Hofie Hannesdottir, Luke Lippstreu, and Maria Polackova

Feynman Integrals

Feynman integrals are a useful computational tool, **but quickly become hard to evaluate**

(see talks by Tancredi and Abreu)



[Jiang, Liu, Xu, Yang (2024)] [Abreu, Chicherin, Sotnikov, Zoia (to appear)]

Thus, in addition to attacking the computation of Feynman integrals directly, we can ask what can be proven about their properties from first principles

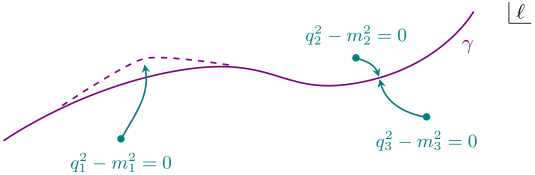
Landau Analysis

$$\mathcal{I}_G(p) = \int d^D\ell_1 \cdots d^D\ell_L \frac{1}{\prod_{e\in G}(q_e(\ell_i,p)^2 - m_e^2)}$$

- The locations where Feynman integrals can become singular and develop branch cuts are described by solutions to the **Landau Equations** [Landau (1959)]

$$\boxed{\begin{array}{ll} \alpha_e(q_e^2 - m_e^2) = 0 & \text{for every edge } e \text{ in } G \\ \displaystyle\sum_{e\in\text{loop}} \alpha_e q_e^\mu = 0 & \text{for each independent loop in } G \end{array}}$$

- In these configurations, the zeros in the denominator can pinch the integration contour γ



Landau Analysis

What happens after we start computing discontinuities?

$$\mathcal{I}_G \qquad \longrightarrow \qquad \mathrm{Disc}_\lambda \mathcal{I}_G$$

- **Can all of the singularities of the original integral still arise** in this discontinuity?

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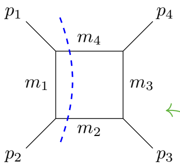
- **Can all of the singularities of the original integral still arise** in this discontinuity?

The answer to this question goes back over sixty years, and is definitively **no!**

- This can already be seen in simple examples via Cutkosky's formula [Cutkosky (1960)]



$$\longrightarrow \qquad \text{Disc}_{s-(m_2+m_4)^2} \mathcal{I}_{\square} \;\propto\; \int d^D\ell \, \frac{\delta^+(q_2^2 - m_2^2)\delta^+(q_4^2 - m_4^2)}{(q_1^2 - m_1^2)(q_3^2 - m_3^2)}$$

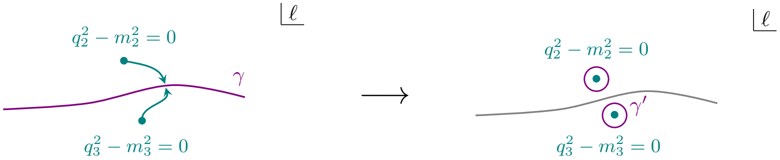
$$s = (p_1 + p_2)^2$$

a singularity in which the propagators with mass m_2 and m_4 pinch the contour

Landau Analysis

More generally, the **hierarchical principle** states that propagators that are put on shell when we compute discontinuities must stay on shell [Landshoff, Olive, Polkinghorne (1965)] [Pham (1967)] [Boyling (1968)]

- This follows from Picard-Lefschetz theory, which describes how discontinuities can be expressed as integrals in which the integration contour is localized to the pinched surface



- This suggests that a **modified set of Landau equations** should apply to $\mathrm{Disc}_\lambda \mathcal{I}_G$:

$q_e^2 - m_e^2 = 0$	for every edge e involved in pinching the contour when $\lambda = 0$
$\alpha_e(q_e^2 - m_e^2) = 0$	for every edge e that did not participate in the pinch
$\sum_{e \in \text{loop}} \alpha_e q_e^\mu = 0$	for each independent loop in G

Landau Analysis

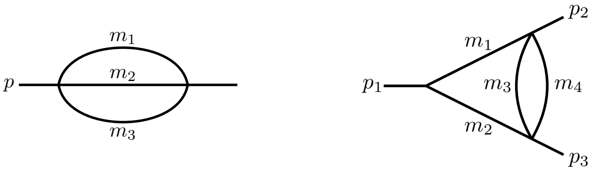
- Thus, one should be able to derive restrictions of the form

$$\mathrm{Disc}_{\lambda'}\mathrm{Disc}_{\lambda}\mathcal{I}_G = 0$$

  by comparing the solutions of these equations to those of the original Landau equations

- In practice, however, doing this can be extremely subtle, as it **requires ensuring we have found all solutions** to the (blown up versions of) these equations

- For this reason, these constraints have only been worked out in a few examples
  [Landshoff, Olive, Polkinghorne (1965)] [Pham (1967)] [Berghoff, Panzer (2022)]



Hierarchical Constraints

In this talk, I will describe how **these difficulties can be sidestepped for a large class of hierarchical constraints**

The result will be **a practical method** for deriving constraints of the form

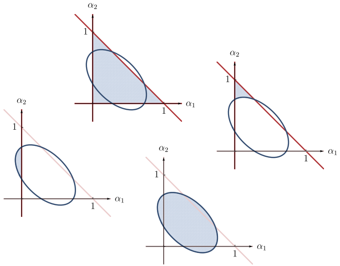
$$\text{Disc}_{\lambda'} \cdots \text{Disc}_{\lambda} \cdots \mathcal{I}_G = 0$$

that are **tailored to individual Feynman integrals**

- Does not require explicitly working out any algebraic blowups
- Can be applied to **any configuration of massive or massless particles**
- The resulting constraints **hold to all orders in dimensional regularization**

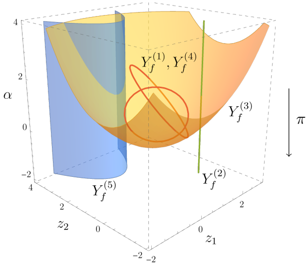
Recent Advances in Landau Analysis

- This method will build on two recent advances in particular:



An improved understanding of cuts in Feynman parameter space

[Berghoff, Panzer (2022)] [Britto (2023)]



New topological methods for probing the singularity structure of Feynman integrals (along with an implementation in the PLD code)

[Fevola, Mizera, Telen (2023)]

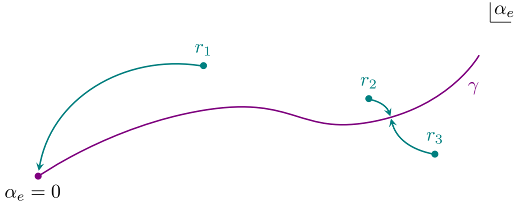
Feynman Parameter Space

The first thing we do is move to Feynman parameter space

$$\mathcal{I}_G = \int_0^\infty \frac{d\alpha_1 \cdots d\alpha_E}{GL(1)} \frac{\mathcal{U}^{E-(L+1)D/2}}{\mathcal{F}^{E-LD/2}}$$

- In this representation we encounter **endpoint singularities as well as pinch singularities**

$$\mathcal{F} = (\alpha_e - r_1)(\alpha_e - r_2)\cdots(\alpha_e - r_n)$$



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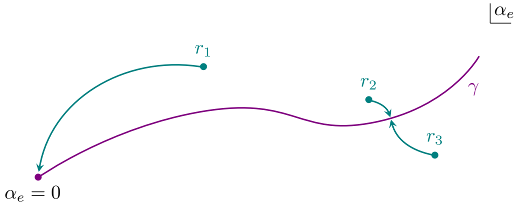
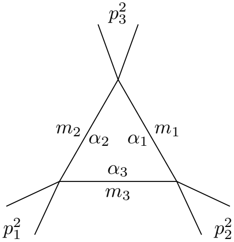


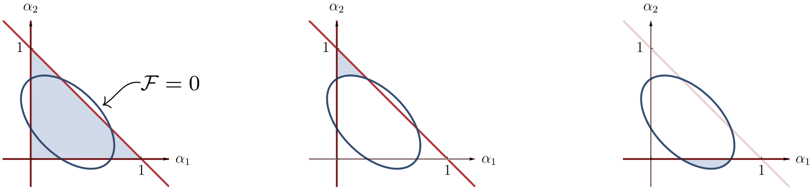
- A version of the hierarchical principle also exists in Feynman parameter space [Berghoff, Panzer (2022)]
- Just as in momentum space, discontinuities localize the integration contour to the pinched surface — here, **we will start to forget the integration boundaries at $\alpha_e = 0$**

Feynman Parameter Cuts

- This phenomenon was nicely illustrated for the massive triangle in [Britto (2023)]



$$\alpha_3 = 1 - \alpha_1 - \alpha_2$$



$\mathcal{I}$ $\quad$ $\mathrm{Disc}_{m_2^2}\mathcal{I}$ $\quad$ $\mathrm{Disc}_{p_1^2-(m_2+m_3)^2}\mathcal{I}$

Two Practical Hurdles

- There are two practical questions that need to be answered if we want to use this observation about Feynman parameter boundaries to derive hierarchical constraints

(1) Which $\alpha_e = 0$ boundaries are dropped when we compute the discontinuity Disc_λ?

(2) Once we have decided which boundaries to drop, how can we reliably rule out whether a kinematic singularity at $\lambda' = 0$ can still exist?

- Both questions are made difficult by the possible existence of further solutions to the Landau equations that we haven't found

Two Practical Hurdles

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(1) Which $\alpha_e = 0$ boundaries are dropped when we compute the discontinuity Disc_λ?

- We address this by introducing **minimal cuts**

(2) Once we have decided which boundaries to drop, how can we reliably rule out whether a kinematic singularity at $\lambda' = 0$ can still exist?

- This is solved by the **Euler characteristic test** [Fevola, Mizera, Telen (2023)]

- Both questions are made difficult by the possible existence of further solutions to the Landau equations that we haven't found

The Euler Characteristic Test

- Singularities only arise in Feynman integrals when the space on which the integration contour is defined degenerates [Pham (1967)]

$$Y = \mathbb{C}^{E-1} \Big\backslash \left(\mathcal{F} = 0 \cup \mathcal{U} = 0 \bigcup_{e=1}^{E} \alpha_e = 0 \right)$$

- One way to probe when this can happens is by computing the Euler characteristic of this space—this number should change if something topologically interesting happens [Fevola, Mizera, Telen (2023)]

- This **Euler characteristic test** has been implemented in the PLD code, using the fact that $|\chi(Y)|$ can be computed as the number of solutions to the system of equations

$$\frac{\mu_1}{\mathcal{F}} \frac{\partial \mathcal{F}}{\partial \alpha_e} + \frac{\mu_2}{\mathcal{U}} \frac{\partial \mathcal{U}}{\partial \alpha_e} + \frac{\nu_e}{\alpha_e} = 0 \qquad \text{for } e \in \{1, 2, \ldots, E\}$$

  for generic μ_i and ν_e [Huh (2013)]

The Euler Characteristic Test

- If we know that specific $\alpha_e = 0$ boundaries no longer appear in our integral, however, a clear modification of this test suggests itselfs

$$Y = \mathbb{C}^{E-1} \Big\backslash \left(\mathcal{F} = 0 \cup \mathcal{U} = 0 \bigcup_{e=1}^{E} \alpha_e = 0 \right)$$

$$\downarrow$$

$$Y_{i,\ldots,j} = \mathbb{C}^{E-1} \Big\backslash \left(\mathcal{F} = 0 \cup \mathcal{U} = 0 \bigcup_{e \notin \{i,\ldots,j\}} \alpha_e = 0 \right)$$

- Specifically, to probe whether our integral can become singular at $\lambda = 0$ *after* the $\alpha_e = 0$ boundaries have been dropped for $e \in \{i, \ldots, j\}$, we ask

$$|\chi(Y_{i,\ldots,j}|_{\lambda=0})| \overset{?}{<} |\chi(Y_{i,\ldots,j})|$$

- **If these numbers are equal, there can be no singularity at $\lambda = 0$**

Minimal Cuts

- The Euler characteristic test provides us with a reliable way to determine which singularities can still arise after a given set of $\alpha_e = 0$ boundaries have been dropped

- This leaves the question of how we decide which $\alpha_e = 0$ boundaries are dropped when we compute a discontinuity

Rather than try to determine which $\alpha_e = 0$ boundaries have *actually* been dropped, our strategy will be to identify **the minimum set of boundaries must that must be dropped**

Minimal Cuts

Minimal Cut: Given a Feynman integral $\mathcal{I}_G$ and a singular kinematic surface $\lambda(\{s_{i\dots j}\}, \{m_k^2\}) = 0$, we refer to a set of cut propagators as a *minimal cut* if:

(i) the cut propagators partition the external momenta into the combinations that appear in the Mandelstam variables $\{s_{i\dots j}\}$

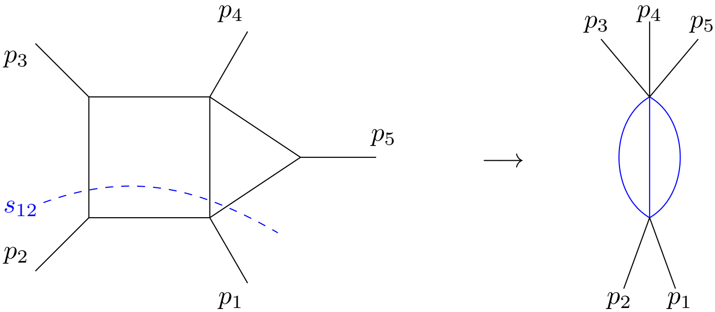
(ii) each of the internal masses in $\{m_k^2\}$ appears in at least one of the cut propagators (unless this mass has already appeared as one of the Mandelstam variables)

(iii) one of the first two properties is no longer satisfied if any of the cut propagators are taken off shell

More colloquially, minimal cuts are designed to make sure that all of the kinematic variables that appear in λ are "resolved" by the cut diagram

Minimal Cuts

As an example, consider some of the singularities of the massless triangle-box:



$$\lambda = s_{12} = (p_1 + p_2)^2 = 0$$

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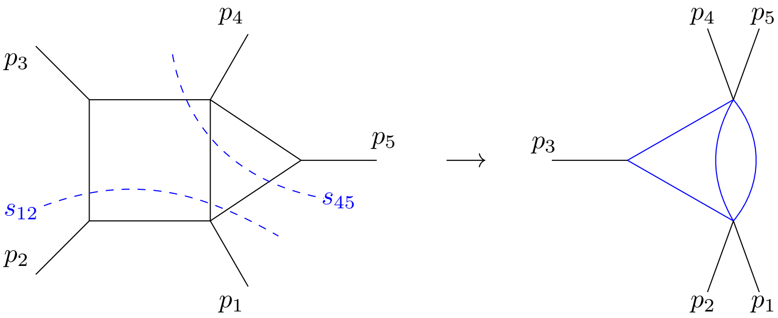

$$\lambda = s_{12} - s_{45} = 0$$

Putting Everything Together

We now have all the ingredients necessary to derive hierarchical constraints in practice

To investigate whether a given sequential discontinuity

$$\text{Disc}_{\lambda'}\text{Disc}_{\lambda}\mathcal{I}_G$$

can be nonzero, we:

- **construct the minimal cut** associated with the singularity at $\lambda = 0$
- **conclude that the $\alpha_e = 0$ boundary associated with each of the cut propagators**, labelled $\{i, \dots, j\}$, **can no longer appear in our integral**
- **compute $|\chi(Y_{i,\dots,j}|_{\lambda'=0})|$ and $|\chi(Y_{i,\dots,j})|$** (using, for instance, the PLD code)

If the first number is not smaller than the second, we conclude that

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Genealogical Constraints

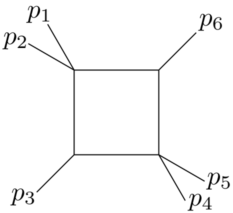
Note that this method is *not* expected to give us the full set of hierarchical constraints

- **Minimal cuts provide a conservative bound** on what propagators/Feynman parameters have participated in a pinch singularity
- It thus remains possible that *more* $\alpha_e = 0$ boundaries have dropped out of $\text{Disc}_\lambda \mathcal{I}_G$
- However, this is not a problem, since our predictions can only be made stronger by dropping additional boundaries

We call the subset of hierarchical constraints that our method allows us to derive, which describe what singularities cannot descend from minimal cuts, **genealogical constraints**

- We will see in examples that **genealogical constraints account for the vast majority of possible hierarchical constraints**

One-Loop Example: the Two-Mass Easy Box



- This integral depends on four Mandelstam variables, and involves ten singular surfaces

$$s_{ij} = (p_i + p_j)^2 \qquad\qquad s_{ijk} = (p_i + p_j + p_k)^2$$

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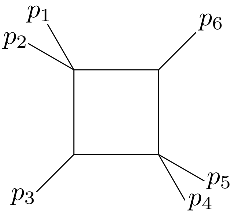

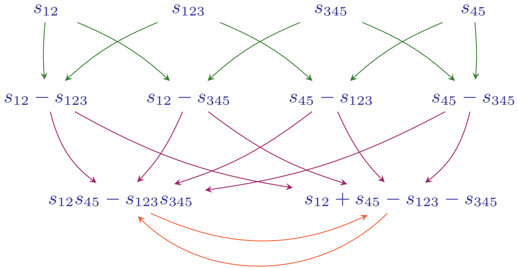

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- Applying our method, we are able to derive 64 genealogical constraints of the form

$$\mathrm{Disc}_{\lambda'} \cdots \mathrm{Disc}_{\lambda} \cdots \mathcal{I}_{\boxtimes} = 0$$

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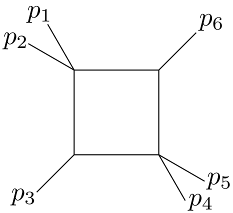

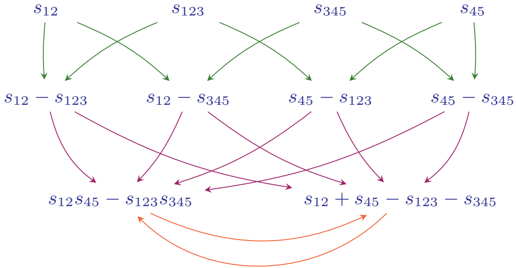

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- These constraints are indeed obeyed to all orders in dimensional regularization

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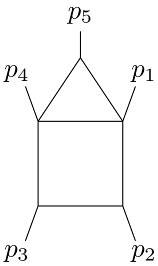

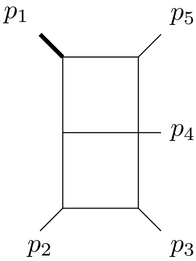

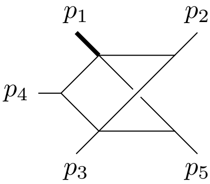

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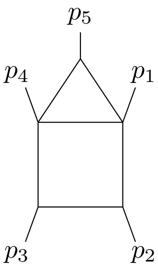

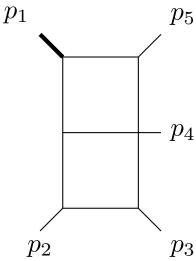

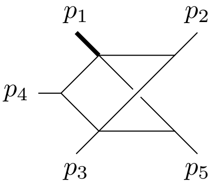

156 genealogical constraints

miss only 31 constraints

[Chicherin, Gehrmann, Henn,
Lo Presti, Mitev, Wasser (2019)]

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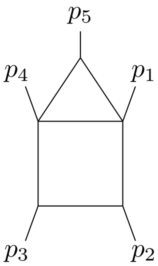

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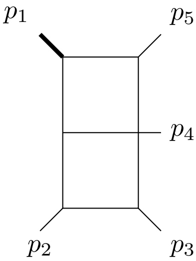

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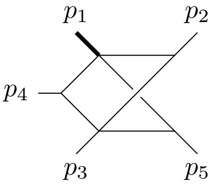
[Abreu, Ita, Moriello, Page, Tschernow, Zeng (2020)]



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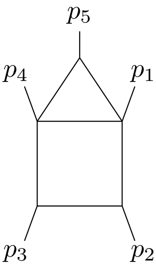

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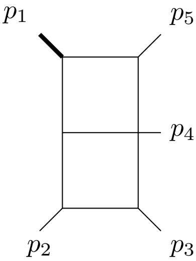

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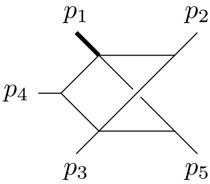

540 genealogical constraints

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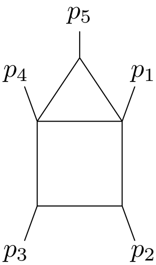
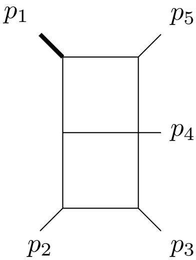
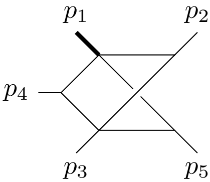
[Abreu, Ita, Page, Tschernow (2022)]

Two-Loop Examples







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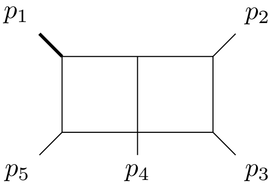
These constraints follow from restrictions on both algebraic and logarithmic discontinuities

Two-Loop Examples

How constraining are these genealogical constraints?



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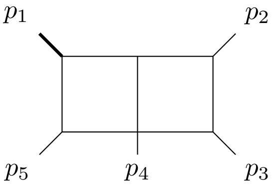

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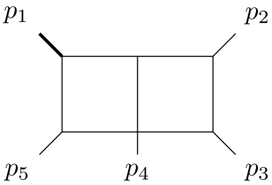
- **679 weight-four symbols** can be constructed with appropriate first entries
- Our best previous constraints—the **extended Steinmann relations**—tell us that s_{12} and s_{15} cannot appear next to each other in the symbol; this leaves **569 possible symbols**
  [Caron-Huot, Dixon, Dulat, von Hippel, AJM, Papathanasiou (2019)]

Two-Loop Examples

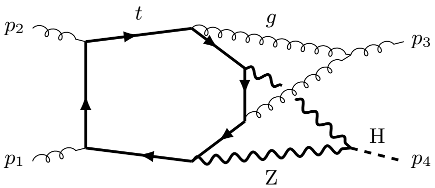
How constraining are these genealogical constraints?



- This integral is polylogarithmic, and gives rise to 37 different symbol letters
- **679 weight-four symbols** can be constructed with appropriate first entries
- Our best previous constraints—the **extended Steinmann relations**—tell us that s_{12} and s_{15} cannot appear next to each other in the symbol; this leaves **569 possible symbols**
  [Caron-Huot, Dixon, Dulat, von Hippel, AJM, Papathanasiou (2019)]
- However, our new genealogical constraints cut this space down to **264 possible symbols**

Beyond Two Loops

Note that we can use this method to derive constraints on complicated integrals, even if we don't know the full set of singularities that can arise



- For example, from just knowledge of *some* of the solutions to the Landau equations, we can already derive constraints on the above integral that contributes to $gg \to Hg$:

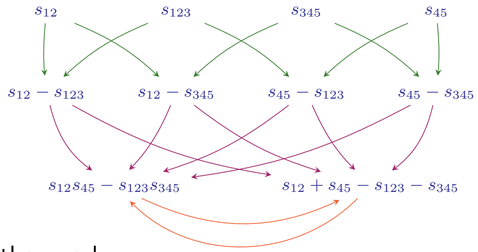
$$\text{Disc}_{s_{12}+s_{23}-m_H^2+m_Z^2} \cdots \text{Disc}_{m_H^2-4m_Z^2} \cdots \mathcal{I}_G = 0$$

Conclusions and Outlook

We have presented a practical method for deriving hierarchical constraints on Feynman integrals

- These **genealogical constraints** can be efficiently derived for integrals involving any configuration of massive or massless particles

- The constraints hold to **all orders in dimensional regularization**



There remain a number of directions that merit further work

- We have focused for now on scalar Feynman integrals, but it is also possible to include **numerators and propagators raised to higher powers**

- Can these types of predictions **help inform how we build bases of master integrals?**

- These predictions will be ideal for **applying bootstrap methods to Feynman integrals!**

Thanks!