AdS string amplitudes and their high energy limit

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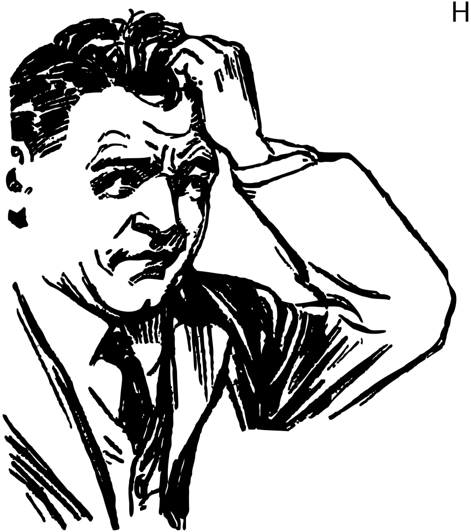
Based on work with Luis F. Alday[a], Shai Chester[c], Maria Nocchi[n], João Silva[s], De-liang Zhong[z]

AdS Virasoro-Shapiro: 2204.07542[a,s], 2209.06223[a,s], 2305.03593[a,s], 2306.12786[a]

AdS Veneziano: 2403.13877[a,c,z], 2404.16084[a]

High energy limit: 2312.02261[a,n]

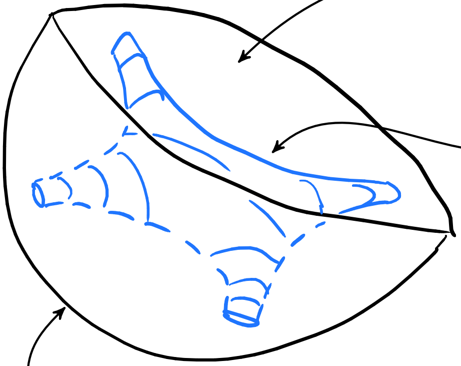


How to formulate string theory on curved spacetime?

At least for AdS_5/CFT_4?

WWVD? – Fix the amplitude first!

1 process - 3 descriptions



5d bulk of AdS:
IIB string theory on $AdS_5 \times S^5$

2d string worldsheet:
2d CFT???

4d boundary of AdS:
$\mathcal{N} = 4$ super Yang Mills theory

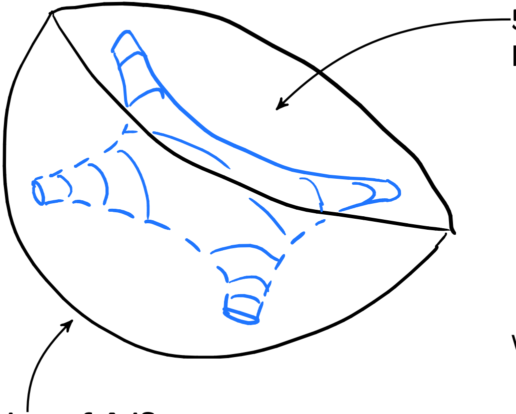
This talk:

1. Find the amplitude without quantizing the string.

2. Compare high energy limit to a classical worldsheet computation.

1. The AdS Virasoro-Shapiro amplitude

Parameters



5d bulk of AdS:
IIb string theory on $AdS_5 \times S^5$

- AdS radius R
- string length $L_s = \sqrt{\alpha'}$
- string coupling g_s

4d boundary of AdS:
$\mathcal{N} = 4$ super Yang Mills theory

- $SU(N)$ gauge group
- coupling $\sqrt{\lambda} = \frac{R^2}{\alpha'}$

Weakly coupled strings:

$$g_s \ll 1 \quad \Leftrightarrow \quad N \gg 1$$

Expansion around flat space:

$$\frac{R^2}{\alpha'} \gg 1 \quad \Leftrightarrow \quad \sqrt{\lambda} \gg 1$$

The AdS amplitude

CFT stress-tensor correlator
$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$

$\longleftrightarrow$ integral transform

AdS graviton amplitude
$A(S,T)$

Small curvature expansion:

$$A(S,T) = A^{(0)}(S,T) + \frac{\alpha'}{R^2}A^{(1)}(S,T) + \left(\frac{\alpha'}{R^2}\right)^2 A^{(2)}(S,T) + \dots$$

$$A^{(0)}(S,T) = -\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$

$R =$ AdS radius

$\frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}}$ ← t'Hooft coupling

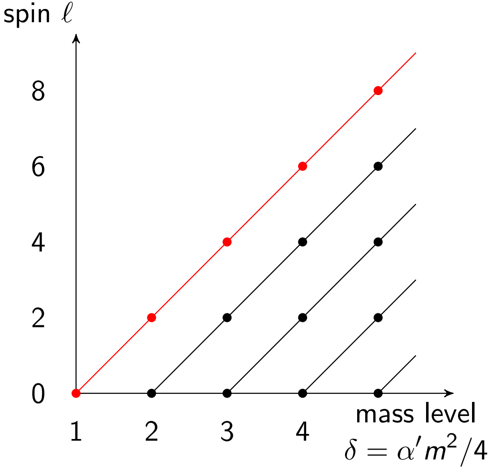
Properties of string amplitudes

Partial wave expansion

Flat space:

resonances $=$ massive string modes

$$\lim_{T\to\delta} A^{(0)}(S,T) = \sum_{\ell} \frac{a_{\delta,\ell} P_\ell(\cos\theta)}{T-\delta}$$



AdS/CFT:

conformal partial wave expansion (OPE)

$$\Delta = Rm + \ldots = R\sqrt{\frac{4\delta}{\alpha'}}\left(1 + O\left(\frac{\alpha'}{R^2}\right)\right)$$

non-critical string theory

[Gubser,Klebanov,Polyakov;1998]

integrability

[Gromov,Serban,Shenderovich,Volin;2011]

...

[Gromov,Hegedus,Julius,Sokolova;2023]

[Ekhammar,Gromov,Ryan;2024]

Pole structure from the OPE

We can expand $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$ into conformal blocks using:

Operator product expansion (OPE)

$$\mathcal{O}(x)\mathcal{O}(0) = \sum_{\mathcal{O}_{\Delta,\ell}\ \text{primaries}} C_{\Delta,\ell}\ c_{\Delta,\ell}(x,\partial_y)\mathcal{O}_{\Delta,\ell}(y)\Big|_{y=0}$$

OPE data

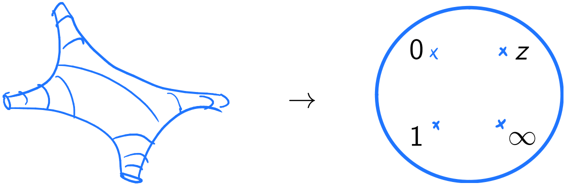
- Δ = dimension
- ℓ = spin
- $C_{\Delta,\ell}$ = OPE coefficients

This fixes the pole structure of the AdS amplitude:

$$A^{(k)}(S,T) = \frac{R^{(k)}_{3k+1}(T, \text{OPE data})}{(S-\delta)^{3k+1}} + \ldots + \frac{R^{(k)}_{1}(T, \text{OPE data})}{S-\delta} + O((S-\delta)^0)$$

The numerator functions are known explicitly.

Worldsheet integral



Flat space:

$$A^{(0)}(S,T) = \frac{1}{(S+T)^2} \int d^2z \; |z|^{-2S-2} |1-z|^{-2T-2}$$

Curvature corrections:

$$A^{(k)}(S,T) = \frac{1}{(S+T)^2} \int d^2z \; |z|^{-2S-2} |1-z|^{-2T-2} G^{(k)}_{\text{tot}}(S,T,z)$$

Consider non-linear σ-model for AdS in a small curvature expansion
$\rightarrow$ flat space string amplitudes with extra soft gravitons
$\rightarrow$ $G^{(k)}_{\text{tot}}(S,T,z) = \sum$ single-valued multiple polylogs of weight $3k$

Plan of attack

We attack the problem from 2 sides:

Poles in terms of OPE data

Single-valued worldsheet ansatz

Both have unfixed data.
Equating the two expressions fixes the answer!

Checks:

- Massive string dimensions (vs integrability)
- OPE coefficient for Konishi (vs numerical bootstrap)
- Low energy expansion (vs localization)
- High energy limit (vs classical scattering)

Multiple polylogarithms

Definition ($|z_1 \dots z_r| = r =$ weight)

$$L_{z_1 \dots z_r}(z) = \int_{0 \le t_r \le \dots \le t_1 \le z} \frac{dt_1}{t_1 - z_1} \cdots \frac{dt_r}{t_r - z_r}$$

Properties:

- $\partial_z L_{z_i w}(z) = \dfrac{1}{z - z_i} L_w(z)$
- multi-valued
- holomorphic
- $L_w(1) =$ multiple zeta values

Examples:

- $L_{1^p}(z) = \dfrac{1}{p!} \log^p(1 - z)$
- $L_{0^p 1}(z) = -\mathrm{Li}_{p+1}(z)$

Single-valued multiple polylogarithms

SVMPLs [Brown;2004]

$$\mathcal{L}_w(z) = \sum_{|w_1|+|w_2|=|w|} c_{w_1 w_2} L_{w_1}(z) L_{w_2}(\bar{z})$$

Properties:

- $\partial_z \mathcal{L}_{z_i w}(z) = \frac{1}{z - z_i} \mathcal{L}_w(z)$
- single-valued
- non-holomorphic
- $\mathcal{L}_w(1) \equiv$ single-valued multiple zeta values

Examples:

- $\mathcal{L}_{1^p}(z) = \frac{1}{p!} \log^p |1 - z|^2$
- $\mathcal{L}_{01}(z) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log(1 - \bar{z}) \log |z|^2$

Anti-holomorphic derivative:

- $\partial_{\bar{z}} \mathcal{L}_{w z_i}(z) = \frac{1}{\bar{z} - z_i} \mathcal{L}_w(z) +$ other terms

AdS Virasoro-Shapiro amplitude

$$A^{(k)}(S,T) = \frac{1}{(S+T)^2}\int d^2z|z|^{-2S-2}|1-z|^{-2T-2}G^{(k)}_{\text{tot}}(S,T,z)$$
$$= B^{(k)}(S,T) + B^{(k)}(U,T) + B^{(k)}(S,U)$$

$$B^{(k)}(S,T) = \frac{1}{(S+T)^2}\int d^2z|z|^{-2S-2}|1-z|^{-2T-2}G^{(k)}(S,T,z)$$

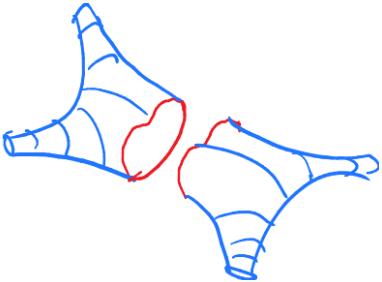
Worldsheet integrand

$$G^{(0)}_{\text{tot}}(S,T,z) = 1$$

$$G^{(1)}(S,T,z) = (S+T)^2\left(-\frac{1}{6}\mathcal{L}^+_{000}(z) + 0\mathcal{L}^+_{001}(z) - \frac{1}{4}\mathcal{L}^+_{010}(z) + 2\zeta(3)\right)$$
$$+ (S^2-T^2)\left(-\frac{1}{6}\mathcal{L}^-_{000}(z) + \frac{1}{3}\mathcal{L}^-_{001}(z) + \frac{1}{6}\mathcal{L}^-_{010}(z)\right)$$

$$G^{(2)}(S,T,z) = \frac{1}{18}(S+T)^2(ST-S^2-T^2)\mathcal{L}^+_{000000}(z) + \text{ 44 more terms}$$

$$\mathcal{L}^\pm_w(z) = \mathcal{L}_w(z) \pm \mathcal{L}_w(1-z) + \mathcal{L}_w(\bar{z}) \pm \mathcal{L}_w(1-\bar{z})$$



We extract the OPE data:

		$A^{(0)}$ data		$A^{(1)}$ data		$A^{(2)}$ data	
$\Delta_{\delta,\ell}$	$=$	$2\sqrt{\delta}\lambda^{\frac{1}{4}}$	$+$	$\lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell}$	$+$	$\lambda^{-\frac{3}{4}}\Delta^{(2)}_{\delta,\ell}$	$+\dots$
$C^2_{\delta,\ell}$	$=$	$C^{2(0)}_{\delta,\ell}$	$+$	$\lambda^{-\frac{1}{2}}C^{2(1)}_{\delta,\ell}$	$+$	$\lambda^{-1}C^{2(2)}_{\delta,\ell}$	$+\dots$

Leading Regge trajectory ($\delta = 1$ is Konishi):

$$\Delta = 2\sqrt{\delta}\lambda^{\frac{1}{4}}\left(1+\left(\frac{3\delta}{4}+\frac{1}{2\delta}-\frac{1}{4}\right)\frac{1}{\sqrt{\lambda}}-\left(\frac{21\delta^2}{32}+\frac{1}{8\delta^2}-\frac{(3-12\zeta(3))\delta}{8}-\frac{1}{8\delta}-\frac{17}{32}\right)\frac{1}{\lambda}+\dots\right)$$

Agrees with integrability result!
[Gromov,Serban,Shenderovich,Volin;2011],[Basso;2011],[Gromov,Valatka;2011]

OPE coefficient for Konishi agrees with numerical bootstrap! [Zahra's talk]

Check 2: Low energy expansion

Relates to low energy effective action (SUGRA + derivative interactions)

$$A(S,T) = \text{SUGRA} + \sum_{a,b,k=0}^{\infty} \frac{\sigma_2^a \sigma_3^b}{\sqrt{\lambda}^k} \alpha_{a,b}^{(k)}, \qquad \sigma_2 = S^2 + T^2 + U^2, \sigma_3 = STU$$

$$= \text{SUGRA} + \underset{R^4}{\alpha_{0,0}^{(0)}} + \underset{D^2R^4}{\frac{\alpha_{0,0}^{(1)}}{\sqrt{\lambda}}} + \underbrace{\sigma_2\, \alpha_{1,0}^{(0)} + \frac{\alpha_{0,0}^{(2)}}{\lambda}}_{D^4R^4} + \underbrace{\sigma_3\, \alpha_{0,1}^{(0)} + \frac{\sigma_2\, \alpha_{1,0}^{(1)}}{\sqrt{\lambda}} + \frac{\alpha_{0,0}^{(3)}}{\sqrt{\lambda}^3}}_{D^6R^6} + \ldots$$

$\alpha_{a,b}^{(0)}$ = flat space, we fix all $\alpha_{a,b}^{(1)}$ and $\alpha_{a,b}^{(2)}$, in particular:

$$\alpha_{0,0}^{(1)} = 0\,, \quad \alpha_{1,0}^{(1)} = -\frac{22}{3}\zeta(3)^2\,, \quad \alpha_{0,0}^{(2)} = \frac{49}{4}\zeta(5)\,, \quad \alpha_{1,0}^{(2)} = \frac{4091}{16}\zeta(7)$$

Agrees with localisation result! Altogether we fully fix D^8R^4 and $D^{10}R^4$.
[Binder,Chester,Pufu,Wang;2019],[Chester,Pufu;2020],[Alday,TH,Silva;2022]

AdS Veneziano amplitude

What about open strings?

type IIB string theory in $AdS_5 \times S^5$ with 7-branes $\leftrightarrow$ $4d\ \mathcal{N} = 2$ SCFT

We fixed $G^{(1)}_{\text{open}}$ and $G^{(2)}_{\text{open}}$ in the color-ordered gluon amplitude ($G^{(0)}_{\text{open}} = 1$):

$$A_{\text{open}}(S,T) = \frac{1}{S+T} \int_0^1 dx\, x^{-S-1}(1-x)^{-T-1} \sum_{k=0}^{\infty} \left(\frac{\alpha'}{R^2}\right)^k G^{(k)}_{\text{open}}(S,T,x)$$

$G^{(k)}_{\text{open}}$ contains SVMPLs of weights $\leq 3k$.

Summary: AdS string amplitudes

Checks:

- OPE data for massive strings
- Low energy expansion
- High energy limit (part 2)

2. The high energy limit

Why the high energy limit?

What is the next step towards the worldsheet theory?

Flat space [Gross,Mende;1987]:

classical solution (bosonic) of the worldsheet theory → high energy limit ($S, T \to \infty$) of string amplitudes

An independent way to compute $\lim_{S,T\to\infty} A(S,T)$, agnostic to many details!

High energy limit via saddle point

The high energy limit of $A^{(0)}(S,T)$ is given by the saddle point $z = \bar{z} = \dfrac{S}{S+T}$

$$\lim_{S,T\to\infty} \int d^2z\, |z|^{-2S} |1-z|^{-2T} \sim e^{-2S \log |\frac{S}{S+T}| - 2T \log |\frac{T}{S+T}|}$$

In AdS the limit can be computed in the same way.

Goal: Compute this exponent from the string action.

Classical solution in flat space

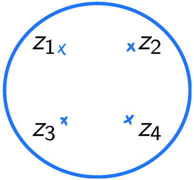
Path integral for the amplitude:

$$A_4^{\text{flat}}(S,T) \sim \int \mathcal{D}X \exp\left(-\int d^2\zeta \partial X^\mu(\zeta)\bar{\partial}X_\mu(\zeta)\right)\prod_{j=1}^{4}\int d^2 z_j\, e^{ip_j\cdot X(z_j)} = \int \mathcal{D}X\, e^{-\mathcal{S}(X^\mu)}$$

$$\Rightarrow \qquad \mathcal{S}(X^\mu) = \int d^2\zeta \left(\partial X^\mu(\zeta)\bar{\partial}X_\mu(\zeta) - i\sum_{j=1}^{4} p_j\cdot X(\zeta)\,\delta^{(2)}(\zeta - z_j)\right)$$

EOM: $\partial\bar{\partial}X^\mu = -\frac{i}{2}\sum_j p_j^\mu \delta^{(2)}(\zeta - z_j)$ Virasoro: $\partial X\cdot\partial X = \bar{\partial}X\cdot\bar{\partial}X = 0$

Solution: $X^\mu_{\text{clas}} = -i\sum_j p_j^\mu \log|\zeta - z_j|$



This classical solution gives the correct high energy exponent:

$$\lim_{S,T\to\infty} A_4^{\text{flat}}(S,T) \sim e^{-\mathcal{S}(X^\mu_{\text{clas}})}\Big|_{z=\frac{S}{S+T}} = e^{-2S\log|\frac{S}{S+T}|-2T\log|\frac{T}{S+T}|}$$

The AdS path integral

The action for AdS:

$$\mathcal{S}(X,\Lambda) = \int d^2\zeta \left(\partial X^M \bar{\partial} X_M + \Lambda(X^M X_M + R^2) - i \sum_{j=1}^{4} P_j^M X_M \delta^{(2)}(\zeta - z_j) \right)$$

AdS_d is embedded in $\mathbb{R}^{2,d-1} \ni X^M$

$$-R^2 = X^M X_M = -X^0 X^0 + X^\mu X_\mu$$

Eliminate X^0 and expand X^μ around flat space:

$$X^\mu = X_0^\mu + \frac{1}{R^2} X_1^\mu + \dots \qquad X_0^\mu = -i \sum_j p_j^\mu \log \left| 1 - \frac{\zeta}{z_j} \right|$$

Equation of motion for X_1^μ:

$$\partial\bar{\partial} X_1^\mu = \partial X_0 \cdot \bar{\partial} X_0 \, X_0^\mu = \frac{i}{4} \sum_{i,j,k} \frac{p_i \cdot p_j}{(\zeta - z_i)(\bar{\zeta} - z_j)} p_k^\mu \log \left| 1 - \frac{\zeta}{z_k} \right|$$

Classical solution in AdS

Equation of motion for X_1^μ:

$$\partial\bar{\partial}X_1^\mu = \partial X_0 \cdot \bar{\partial} X_0\, X_0^\mu = \frac{i}{8}\sum_{i,j,k}\frac{p_i \cdot p_j}{(\zeta - z_i)(\bar{\zeta} - z_j)} p_k^\mu \mathcal{L}_{z_k}(\zeta)$$

We can "integrate" this using

$$\int d\zeta \frac{\mathcal{L}_w(\zeta)}{\zeta - z_i} \to \mathcal{L}_{z_i w}(\zeta)\,, \qquad \int d\bar{\zeta}\frac{\mathcal{L}_w(\zeta)}{\bar{\zeta} - z_j} \to \mathcal{L}_{w z_j}(\zeta) + \cdots$$

Result:

$$X_{1,\text{clas}}^\mu = \frac{i}{8}\sum_{i,j,k=1}^{4} p_i \cdot p_j \; p_k^\mu \left(\mathcal{L}_{z_i z_k z_j}(\zeta) + \mathcal{L}_{z_k}(z_j)\mathcal{L}_{z_i z_j}(\zeta) - \mathcal{L}_{z_j}(z_k)\mathcal{L}_{z_i z_k}(\zeta)\right)$$

More generally:

$$X_{\text{clas}}^\mu = \mathcal{L}_{|w|=1}(\zeta) + \frac{1}{R^2}\mathcal{L}_{|w|=3}(\zeta) + \frac{1}{R^4}\mathcal{L}_{|w|=5}(\zeta) + \ldots$$

Comparison with AdS Virasoro-Shapiro amplitude

$$e^{-\mathcal{S}(X^\mu_{\text{clas}})}\Big|_{z=\frac{S}{S+T}} = \exp\left(-S\mathcal{S}_1\left(\tfrac{S}{T}\right) - \frac{S^2}{R^2}\mathcal{S}_3\left(\tfrac{S}{T}\right) - \frac{S^3}{R^4}\mathcal{S}_5\left(\tfrac{S}{T}\right) - O\left(\frac{S^4}{R^6}\right)\right)$$

In the limit $S, T, R \to \infty$ with S/T and S/R fixed, $\mathcal{S}_5$ and further terms vanish!

We successfully compare with AdS Virasoro-Shapiro at the saddle point:

$$e^{-\frac{S^2}{R^2}\mathcal{S}_3\left(\frac{S}{T}\right)} = 1 + \frac{1}{R^2}G^{(1)}_{\text{tot}}(z = \tfrac{S}{S+T}) + \frac{1}{R^4}G^{(2)}_{\text{tot}}(z = \tfrac{S}{S+T}) + \ldots$$

✓

This implies

$$G^{(2)}_{\text{tot}}(z = \tfrac{S}{S+T}) = \frac{1}{2}\left(G^{(1)}_{\text{tot}}(z = \tfrac{S}{S+T})\right)^2$$

✓

Final result to all orders in S/R:

$$\lim_{S,T,R\to\infty} A_4^{\text{AdS}}(S,T) = \left(\lim_{S,T\to\infty} A_4^{\text{flat}}(S,T)\right) e^{-\frac{S^2}{R^2}\mathcal{S}_3\left(\frac{S}{T}\right)}$$

Summary: High energy limit

We compared $A(S,T)$ to classical computation a la Gross & Mende.

- Relation to worldsheet action agnostic to fermions and prefactors
- $A(S,T)$ fixed to all orders in S/R

$$\lim_{S,T,R\to\infty} A_4^{\mathrm{AdS}}(S,T) = \left(\lim_{S,T\to\infty} A_4^{\mathrm{flat}}(S,T)\right) e^{-\varepsilon_{\mathrm{open/closed}}(S,T)}$$

- The exponents (weight 3 SVMPLs) for open and closed strings satisfy the expected relation:

$$\varepsilon_{\mathrm{open}}(S,T) = \frac{1}{2}\varepsilon_{\mathrm{closed}}(4S,4T)$$

Future directions

- Relation between open and closed strings beyond the HE limit?
- Other AdS backgrounds, e.g. type IIA on $AdS_4 \times CP^3$ / ABJM
- Go beyond the small curvature expansion?
- Compute AdS string amplitudes directly from string theory?

Thank you!