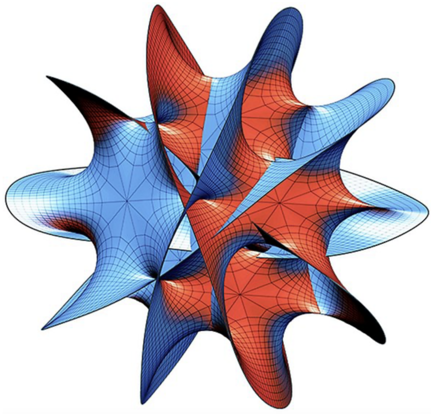

Amplitudes 2024

IAS Princeton

Recent developments in string amplitudes

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& Centre for Geometry and Physics)

June 13th 2024

I. Why string amplitudes?

Prominent role within string theory (starting with **[Veneziano '68]**)

- already in flat spacetime: low-energy eff. actions $\sim \text{tr}(D^k F^n)$, $D^k R^n$
  $\Longrightarrow$ testing / exploiting string dualities (primarily S-duality of type IIB)
- string amplitudes in AdS $\Rightarrow$ gauge/gravity duality, holography,
  bootstrap & recent crosstalk with (integrated) correlators in $\mathcal{N} = 4$
  [Hansen's talk]

Rich source of inspiration and input for other fields

- closed vs. open strings: BCJ duality & gravitational double copy
  [KLT '86, . . ., reviews 1909.01358, 2203.13013, 2203.13017, 2204.06547, 2210.14241]
- function spaces for precision calculations in particle physics / gravity
  [reviews 2203.07088, 2203.09099, 2203.13014, 2203.13021, 2208.07242]

I. Why string amplitudes?

Numerous formalisms in amplitudes are in close contact with string theory:

- since 90's: worldline formalisms (Bern-Kosower, . . ., WQFT)
  [rewiew: Schubert 0101036; Uhre Jakobsen, Mogull, Plefka, Steinhoff '20, '21]
- since 2013: CHY formalism and ambitwistor strings
  [Cachazo, He, Yuan 1307.2199, 1309.0885; review: Mason, Geyer 2203.13017]
- tropical geometry: $\alpha' \to 0$ limit of string amplitudes
  [Tourkine 1309.3551; Lam 2405.17332]

  ... tropical moduli spaces of Feynman graphs $\leftrightarrow$ graph complexes
  [Borinsky, Brown, Munch, Tellander, Vermaseren, Vogtmann '21 to '24; Borinsky's lecture series at amplitudes summer school next week]
- curve integral formalism
  [Arkani-Hamed, Cao, De, Dong, Figueiredo, Frost, He, Pokraka, Plamondon, Salvatori, Skowronek, Spradlin, Thomas, Volovich; Figueiredo's & Spradlin's talk]
- intersection theory
  [Mizera 1706.08527, 1711.00469]

Outline

I. Why string amplitudes? ✓

II. KLT and intersection theory at genus one

[Bhardwaj's gong-show talk; Bhardwaj, Pokraka, Ren, Rodriguez 2312.02148]

[Stieberger 2212.06816, 2310.07755; Mazloumi, Stieberger 2403.05208]

III. Evaluating string amplitudes from convergent integrals

[Eberhardt, Mizera 2208.12233, 2302.12733, 2403.07051]

[Banerjee, Eberhardt, Mizera 2403.07064]

IV. Integration on higher-genus surfaces

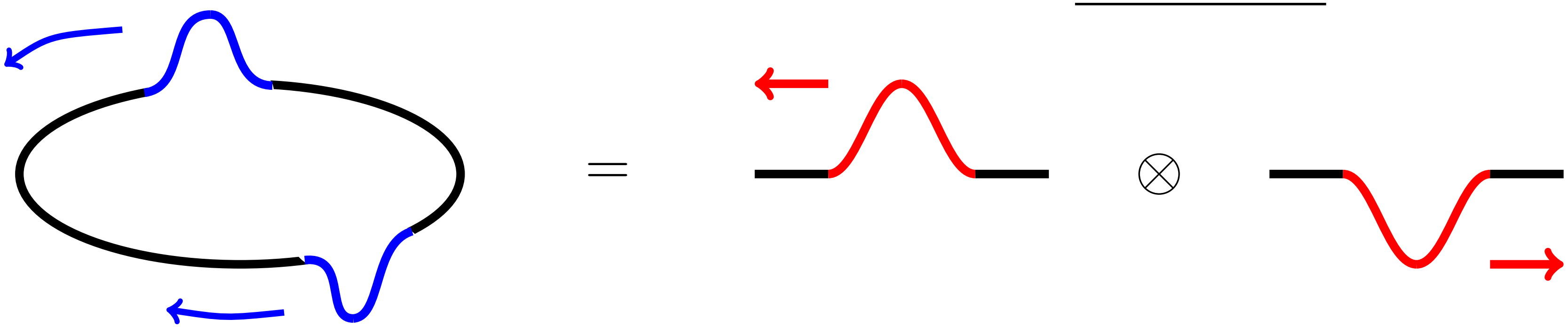
[D'Hoker, Hidding, OS 2306.08644 & 2308.05044; Enriquez 1112.0864]

V. Alternative double copy for single-valued periods

[Frost, Hidding, Kamlesh, Rodriguez, OS, Verbeek 2312.00697; Dorigoni, Doroudiani, Drewitt, Hidding, Kleinschmidt, OS, Schneps, Verbeek 2403.14816, 2406.05099]

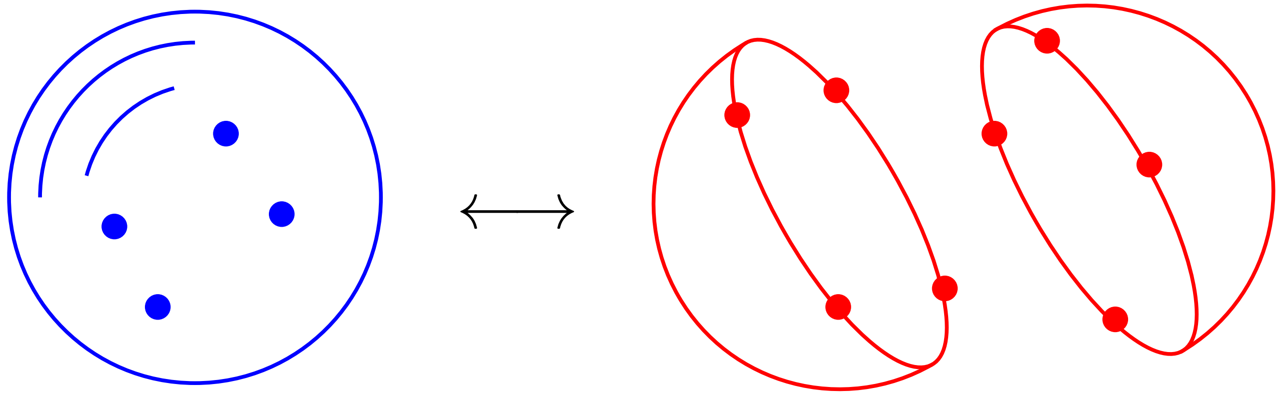
II. KLT and intersection theory at genus one

Goal: (closed strings) as (open string)$^{\otimes 2}$ for integrated amplitudes



At tree level, done deal by KLT relations [Kawai, Lewellen, Tye '86]

closed strings: sphere



open strings: disk

e.g. $$\underbrace{M^{\rm tree}_{\rm closed}(4\,{\rm pt};\alpha')}_{\int_{\mathbb{C}} \frac{{\rm d}^2 z\, |z|^{2s}|1-z|^{2t}}{z(1-\bar z)}} = \underbrace{A^{\rm tree}_{\rm open}(1,2,3,4;\alpha')}_{\int_0^1 \frac{{\rm d}z\, z^s(1-z)^t}{z}} \sin(\pi s) \underbrace{\tilde A^{\rm tree}_{\rm open}(1,2,4,3;\alpha')}_{\int_{-\infty}^0 \frac{{\rm d}\bar z\, (-\bar z)^s(1-\bar z)^t}{1-\bar z}}$$

II. 1 Tree-level KLT from intersection theory

Genus-0 integrands $\ni$ multivalued $u(z)=\prod_{i<j}(z_i-z_j)^{s_{ij}}$, their $|\cdot|$ & cc's

$\longrightarrow$ use intersection theory: "dealing with multivalued integrands"

[Aomoto, Cho, Kita, Matsumoto, Mimachi, Yoshida et al: '80s / '90's]

- increasingly relevant for Feynman-integral computations

[talks of Lee and Tancredi; see e.g. 2002.10476, 2203.13011 for reviews]

- open-string integrals $\leftrightarrow$ pairing twisted cycle $|\gamma\otimes u_\gamma]$ & rational form $\langle\varphi_{\rm L}|$

$$\left\langle \frac{{\rm d}z}{z}\right|\{0<z<1\}\otimes z^s(1-z)^t\Big] = \int_0^1 z^s(1-z)^t\,\frac{{\rm d}z}{z}$$

- closed-string integrals $\leftrightarrow$ pairing two "twisted cocycles" (with cc $|\varphi^\vee_{\rm R}\rangle$)

$$\left\langle \frac{{\rm d}z}{z}\right|\left(\frac{{\rm d}z}{1-z}\right)^\vee\Big\rangle = \int_{\mathbb{C}} \frac{{\rm d}^2 z\,|z|^{2s}|1-z|^{2t}}{z\,(1-\bar z)}$$

II. 1 Tree-level KLT from intersection theory

- open-string integrals $\leftrightarrow$ pairing twisted cycle $|\gamma \otimes u_\gamma]$ & rational form $\langle \varphi_{\mathrm{L}}|$

$$\left\langle \frac{\mathrm{d}z}{z} \middle| \{0<z<1\} \otimes z^s(1-z)^t \right] = \int_0^1 z^s(1-z)^t \frac{\mathrm{d}z}{z}$$

- closed-string integrals $\leftrightarrow$ pairing two "twisted cocycles" (with cc $|\varphi_{\mathrm{R}}^\vee\rangle$)

$$\left\langle \frac{\mathrm{d}z}{z} \middle| \left(\frac{\mathrm{d}z}{1-z}\right)^\vee \right\rangle = \int_{\mathbb{C}} \frac{\mathrm{d}^2 z\, |z|^{2s} |1-z|^{2t}}{z\,(1-\bar{z})}$$

- pairing two twisted cycles $\leftrightarrow$ regularized intersection number

$$\left[\{-\infty<\bar{z}<0\} \otimes (-\bar{z})^s(1-\bar{z})^t \middle| \{0<z<1\} \otimes z^s(1-z)^t \right] = \frac{1}{2i\,\sin(\pi s)}$$

- 4pt KLT involves <u>inverse</u> intersection number

$$\int_{\mathbb{C}} \frac{\mathrm{d}^2 z\, |z|^{2s} |1-z|^{2t}}{z\,(1-\bar{z})} = \int_0^1 \frac{\mathrm{d}z}{z} z^s(1-z)^t \ \sin(\pi s) \int_{-\infty}^0 \frac{\mathrm{d}\bar{z}}{1-\bar{z}} (-\bar{z})^s(1-\bar{z})^t$$

$$\langle \varphi_{\mathrm{L}} | \varphi_{\mathrm{R}}^\vee \rangle = \langle \varphi_{\mathrm{L}} | \gamma_{\mathrm{L}} \otimes u_{\gamma_{\mathrm{L}}}] \, [\gamma_{\mathrm{L}} \otimes u_{\gamma_{\mathrm{L}}} | \gamma_{\mathrm{R}} \otimes u^\vee_{\gamma_{\mathrm{R}}}]^{-1} \, [\gamma_{\mathrm{R}} \otimes u^\vee_{\gamma_{\mathrm{R}}} | \varphi_{\mathrm{R}}^\vee \rangle$$

[Mizera 1706.08527, 1711.00469]

II. 1 Tree-level KLT from intersection theory

KLT at $n \geq 5$ points: $\exists$ $(n-3)!$ basis permutations $\rho_a \in S_{n-3}$ of ...

... twisted cycles $|\gamma_a] := \Big|\{\rho_a(0<z_1<\ldots<z_{n-3}<1)\}\otimes\prod_{i<j}\rho_a\{(z_j-z_i)^{s_{ij}}\}\Big]$

[Plahte '70; Bjerrum-Bohr, Damgaard, Vanhove 0907.1425; Stieberger 0907.2211]

... twisted cocycles (e.g. Parke-Taylor) $\langle\varphi_b| = \Big\langle \prod_{j=1}^{n-3} \frac{\mathrm{d}z_j}{z_j-z_{j+1}}\Big|$ by IBP

[Aomoto '87; Mafra, OS, Stieberger 1106.2645; Broedel, OS, Stieberger 1304.7267]

Typical open-string integrals $(z_{pq} := z_p-z_q)$

$$\langle\varphi_{b=1}|\gamma_a] = \int_{0<z_{\rho_a(i)}<z_{\rho_a(i+1)}<1} \frac{\mathrm{d}z_1\mathrm{d}z_2\ldots\mathrm{d}z_{n-3}}{z_{12}z_{23}\cdots z_{n-3,n-2}} \prod_{1\leq i<j}^{n-1} |z_i-z_j|^{s_{ij}}$$

[review: Mafra, OS 2210.14241; talks of Figueiredo and Sturmfels]

II. 1 Tree-level KLT from intersection theory

KLT at $n \geq 5$ points: $\exists$ $(n{-}3)!$ basis permutations $\rho_a \in S_{n-3}$ of ...

... twisted cycles $|\gamma_a] := \Big|\{\rho_a(0{<}z_1{<}\ldots{<}z_{n-3}{<}1)\}\otimes\prod_{i<j}\rho_a\{(z_j{-}z_i)^{s_{ij}}\}\Big]$

[Plahte '70; Bjerrum-Bohr, Damgaard, Vanhove 0907.1425; Stieberger 0907.2211]

... twisted cocycles (e.g. Parke-Taylor) $\langle\varphi_b| = \big\langle \prod_{j=1}^{n-3} \frac{\mathrm{d}z_j}{z_j - z_{j+1}}\big|$ by IBP

[Aomoto '87; Mafra, OS, Stieberger 1106.2645; Broedel, OS, Stieberger 1304.7267]

$(n{-}3)! \times (n{-}3)!$ intersection matrix $\mathbf{H}_{ab} = [\gamma_a|\gamma_b^\vee] \sim \sin^{3-n}(\pi \sum_{a,b} s_{ab})$

$\Longrightarrow$ KLT formula looks like resolution of identity $\mathbb{1} = \sum_{c,d} |\gamma_c]\mathbf{H}^{-1}_{cd}[\gamma_d^\vee|$

[Mizera 1706.08527, 1711.00469]

$$M^{\rm tree}_{\rm closed}(n\,{\rm pt};\alpha') = \sum_{1\leq c,d\leq(n-3)!} A^{\rm tree}_{\rm open}(\rho_c;\alpha')\,\mathbf{H}^{-1}_{cd}\,\tilde A^{\rm tree}_{\rm open}(\rho_d;\alpha')$$

$$\langle\varphi_a|\varphi_b^\vee\rangle = \sum_{1\leq c,d\leq(n-3)!} \langle\varphi_a|\gamma_c] \quad \mathbf{H}^{-1}_{cd} \quad [\gamma_d^\vee|\varphi_b^\vee\rangle$$

II. 1 Tree-level KLT from intersection theory

KLT formula looks like resolution of identity $\mathbb{1} = \sum_{c,d} |\gamma_c] \mathbf{H}^{-1}_{cd} [\gamma^\vee_d|$

$$\langle \varphi_a | \varphi^\vee_b \rangle = \sum_{c,d=1}^{\dim} \langle \varphi_a | \gamma_c] \quad \mathbf{H}^{-1}_{cd} \quad [\gamma^\vee_d | \varphi^\vee_b \rangle$$

... and generalizes to sphere integrals with unintegrated punctures x_i, e.g.

$$\int_{\mathbb{C}} \mathrm{d}^2 z \, |z|^{2s_0} |1-z|^{2s_1} |x-z|^{2s_x} \varphi_{\mathrm{L}}(z,x) \overline{\varphi_{\mathrm{R}}(z,x)} = \begin{pmatrix} \int_0^x \mathrm{d}z \, (x-z)^{s_x} \\ \int_x^1 \mathrm{d}z \, (z-x)^{s_x} \end{pmatrix} z^{s_0} (1-z)^{s_1} \varphi_{\mathrm{L}}(z,x)$$

$$\times \begin{pmatrix} \sin(\pi s_0) & \sin(\pi(s_0+s_x)) \\ \sin(\pi s_x) & 0 \end{pmatrix} \begin{pmatrix} \int_{-\infty}^0 \mathrm{d}\bar{z} \, (-\bar{z})^{s_0} \\ \int_0^x \mathrm{d}\bar{z} \, \bar{z}^{s_0} \end{pmatrix} (1-\bar{z})^{s_1} (\bar{x}-\bar{z})^{s_x} \overline{\varphi_{\mathrm{R}}(z,x)}$$

[Vanhove, Zerbini 1812.03018; Britto, Mizera, Rodriguez, OS 2102.06206]

Contain single-valued hypergeometric / Lauricella functions.

[Brown, Dupont 1907.06603; Duhr, Porkert 2309.12772]

II. 2 Genus-one double copy for Riemann Wirtinger integrals

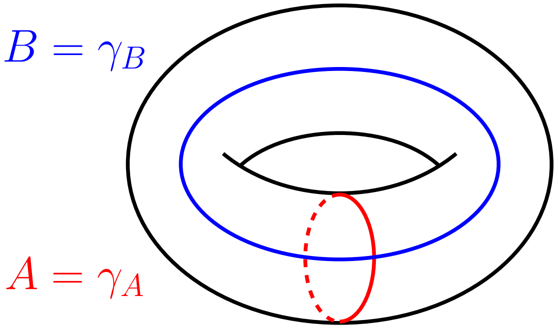
$\longrightarrow$ "warm-up" integrals towards 1-loop string amplitudes: meromorphic Riemann Wirtinger (RW) integrals on (univ. cover of) torus $T^2(\tau) = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$

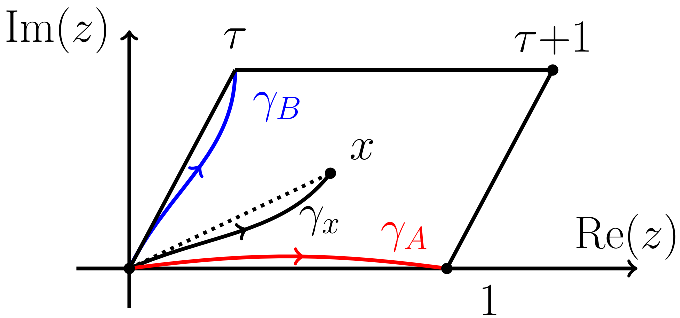
$$\langle\varphi_a|\gamma_b] = \int_0^{z_b} \mathrm{d}z\, e^{2\pi i s_A z} \left(\frac{\theta_1(z|\tau)}{\theta_1(z-x|\tau)}\right)^{s_0} F(z-z_a,\eta|\tau)$$

with $z_a, z_b \in \{1, x\}$ and constant $s_B := \tau s_A - x s_x - \eta$ and twisted cycles

$$|\gamma_b] = |\{0<z<z_b\} \otimes u_{\rm RW}\rangle \text{ where } u_{\rm RW}(z|\tau) = e^{2\pi i s_A z}\left(\frac{\theta_1(z|\tau)}{\theta_1(z-x|\tau)}\right)^{s_0}$$

[Mano '08, 09; Mano, Watanabe '12; Ghazouani, Pirio 1605.02356; Goto 2206.03177]





eliminated B-cycle : $(1-e^{2\pi i s_A})[\gamma_B| = (1-e^{2\pi i s_B})[\gamma_A| - (1-e^{-2\pi i s_0})[\gamma_x|$

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Twisted cocycle $\langle \varphi_a |$: Kronecker-Eisenstein series **[talks of Porkert, Tancredi]**

$$F(z,\eta|\tau) = \frac{\theta_1'(0|\tau)\theta_1(z{+}\eta|\tau)}{\theta_1(z|\tau)\theta_1(\eta|\tau)} = \frac{1}{\eta} + \sum_{k=1}^{\infty} \eta^{k-1} g^{(k)}(z|\tau) = F(z{+}1,\eta|\tau)$$

$$\theta_1(z{+}\tau|\tau) = -e^{-2\pi i z - i\pi\tau}\theta_1(z|\tau) \quad \Rightarrow \quad F(z{+}\tau,\eta|\tau) = e^{-2\pi i \eta} F(z,\eta|\tau)$$

2dim twisted cohomology $\varphi_a \in \mathrm{d}z\{F(z,\eta|\tau),\, F(z{-}x,\eta|\tau)\}$; note that expansion variable η is constrained to yield constant $\tau s_A - x s_x - \eta = s_B$

II. 2 Genus-one double copy for Riemann Wirtinger integrals

Complex RW integral [Ghazouani, Pirio 1906.11857] obeys genus-one KLT

$$\langle \varphi_a | \varphi_b^\vee \rangle = \int_{T^2(\tau)} \mathrm{d}^2 z\, e^{2\pi i s_A (z-\bar{z})} \left| \frac{\theta_1(z|\tau)}{\theta_1(z-x|\tau)} \right|^{2s_0} F(z-z_a,\eta|\tau)\, \overline{F(z-z_b,\eta|\tau)}$$

$$= \frac{i}{2} \frac{\sin(\pi s_0)}{\sin(\pi s_A)} \begin{pmatrix} \langle \varphi_a | \gamma_A] \\ \langle \varphi_a | \gamma_x] \end{pmatrix} \begin{pmatrix} 0 & e^{i\pi(s_0-s_A)} \\ -e^{i\pi(s_A-s_0)} & 2i \sin(\pi(s_A-s_0)) \end{pmatrix} \begin{pmatrix} [\gamma_A^\vee | \varphi_b^\vee \rangle \\ [\gamma_x^\vee | \varphi_b^\vee \rangle \end{pmatrix}$$

[Bhardwaj's gong-show talk; Bhardwaj, Pokraka, Ren, Rodriguez 2312.02148]

with $z_a, z_b \in \{0, x\}$ & reality condition $\mathrm{Im}\,\eta = s_A \mathrm{Im}\,\tau - s_x \mathrm{Im}\,x$

KLT formula amounts to $\mathbb{1} = \sum_{c,d} |\gamma_c] \mathbf{H}^{-1}_{cd} [\gamma_d^\vee|$ with intersection matrix

$$\mathbf{H} = \begin{pmatrix} [\gamma_A|\gamma_A^\vee] & [\gamma_A|\gamma_x^\vee] \\ [\gamma_x|\gamma_A^\vee] & [\gamma_x|\gamma_x^\vee] \end{pmatrix} = \frac{\sin(\pi s_A)}{\sin(\pi s_0)} \begin{pmatrix} 2i\sin(\pi(s_A-s_0)) & -e^{i\pi(s_0-s_A)} \\ e^{i\pi(s_A-s_0)} & 0 \end{pmatrix}$$

Generalizes to any #(unintegrated $x_1, x_2, \ldots$), but only for 1 integrated z.

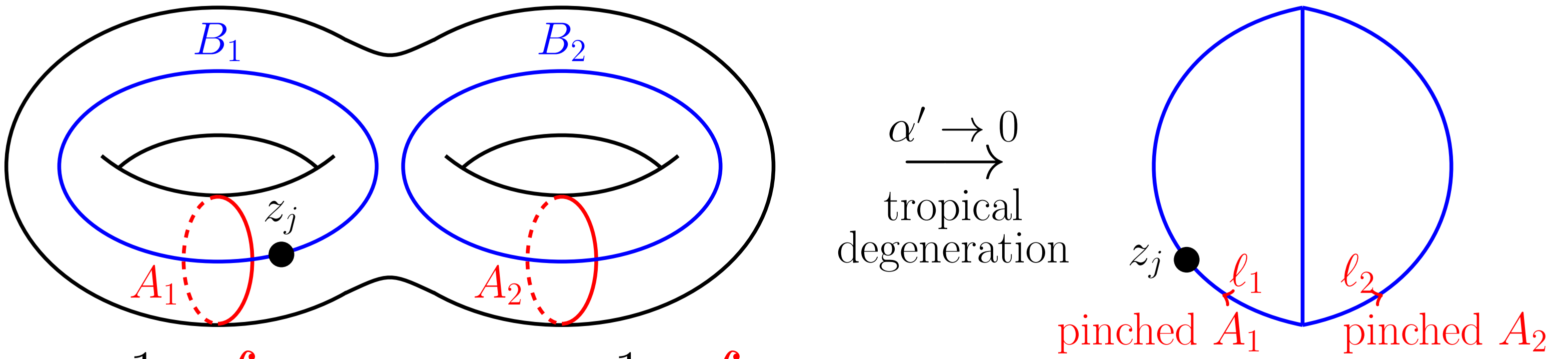
II. 3 Crashcourse in chiral splitting

Closed-string loop integrands factorize before $\int \mathrm{d}^D \ell$ — chiral amplitude

$$M^{\text{1-loop}}_{\text{closed}}(n\,\mathrm{pt}) = \int \mathrm{d}^D \ell \int_{\mathfrak{M}_1} \mathrm{d}^2\tau \left(\prod_{j=2}^{n} \int_{T^2(\tau)} \mathrm{d}^2 z_j \right) \underbrace{\mathcal{F}_n(\epsilon, k, \ell | z, \tau)}_{\text{meromorphic in } z_i, \tau} \overline{\mathcal{F}_n(\tilde\epsilon, k, \ell | z, \tau)}$$

$\longrightarrow$ "chiral splitting" **[Verlinde, Verlinde '87; D'Hoker, Phong '88, '89]**

Loop momenta ℓ_I in string theory = zero modes w.r.t. A_I cycles



$$\ell_I^m = \frac{1}{2\pi} \oint_{A_I} \partial_z X^m = \frac{1}{2\pi} \oint_{A_I} \partial_{\bar z} X^m \,, \qquad \text{shared between L \& R}$$

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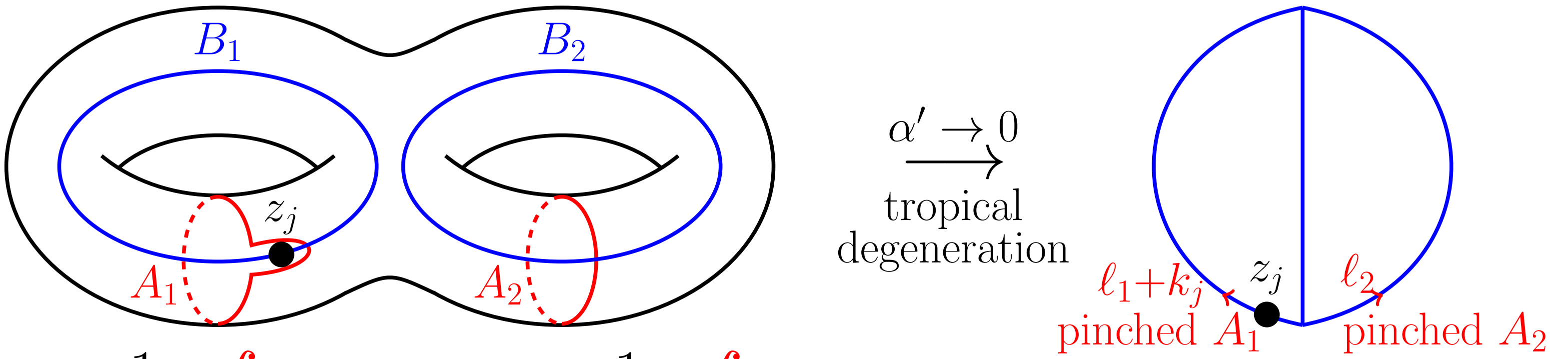

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Loop momentum jumps when transporting punctures around B_I cycles

$$z_j \to z_j + B_1 \quad \Longrightarrow \quad A_1 \to A_1 + \odot_{z_j} \quad \Longrightarrow \quad \ell_1 \to \ell_1 + k_j$$

II. 3 Crashcourse in chiral splitting

Closed-string loop integrands factorize before $\int \mathrm{d}^D\ell$ — chiral amplitude

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$\longrightarrow$ "chiral splitting" **[Verlinde, Verlinde '87; D'Hoker, Phong '88, '89]**

Chiral amplitude $\mathcal{F}_n \ni$ universal Koba-Nielsen factor $u_{\rm ST}$

$$u_{\rm ST}(z|\tau) \;=\; \exp\left(\frac{i\pi\alpha'}{2}\tau\ell^2 - i\pi\alpha' \sum_{j=2}^{n} (\ell\cdot k_j) z_j \right) \prod_{1\leq i<j}^{n} \theta_1(z_i - z_j|\tau)^{s_{ij}}$$

B-monodromy compensated by loop mom. shift $z_j \rightarrow z_j + \tau \;\Rightarrow\; \ell \rightarrow \ell + k_j$

II. 3 Crashcourse in chiral splitting

Closed-string loop integrands factorize before $\int \mathrm{d}^D\ell$ — chiral amplitude

$$M^{\text{1-loop}}_{\text{closed}}(n\,\text{pt}) = \int \mathrm{d}^D\ell \int_{\mathfrak{M}_1} \mathrm{d}^2\tau \left(\prod_{j=2}^{n} \int_{T^2(\tau)} \mathrm{d}^2 z_j \right) \underbrace{\mathcal{F}_n(\epsilon, k, \ell | z, \tau)}_{\text{meromorphic in } z_i, \tau} \overline{\mathcal{F}_n(\tilde{\epsilon}, k, \ell | z, \tau)}$$

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Chiral amplitude $\mathcal{F}_n \ni$ universal Koba-Nielsen factor u_{ST}

$$u_{\text{ST}}(z|\tau) = \exp\left(\frac{i\pi\alpha'}{2}\tau\ell^2 - i\pi\alpha' \sum_{j=2}^{n} (\ell\cdot k_j) z_j \right) \prod_{1\leq i<j}^{n} \theta_1(z_i - z_j|\tau)^{s_{ij}}$$

B-monodromy compensated by loop mom. shift $z_j \to z_j + \tau \;\Rightarrow\; \ell \to \ell + k_j$

Compare with Riemann-Wirtinger integral at $s_A = -\frac{\alpha'}{2}\ell\cdot k$

$$u_{\text{RW}}(z|\tau) = e^{2\pi i s_A z} \theta_1(z|\tau)^{s_0} \prod_{j\geq 1} \theta_1(z - x_j|\tau)^{s_j}\,, \qquad s_0 + \sum_{j\geq 1} s_j = 0$$

overall B-monodromies $F(z+\tau,\eta) u_{\text{RW}}(z+\tau) = e^{2\pi i s_B} F(z,\eta) u_{\text{RW}}(z)$.

II. 4 Genus-one double copy for string amplitudes

For rectangular tori Re $(\tau)=0$, contour def's $\Rightarrow$ factorize $\int \mathrm{d}^2 z = \int \mathrm{d}\xi \mathrm{d}\chi$

$$M^{\text{1-loop}}_{\text{closed}}(n\,\text{pt})\,\Big|_{\mathrm{Re}\,(\tau)=0} = \int \mathrm{d}^D\ell \sum_{\rho,\sigma\in S_{n-1}} \mathcal{S}_{\alpha'}(\rho|\sigma) \int_{0<\xi_{\sigma_i}<\xi_{\sigma_{i+1}}<1} \mathrm{d}\xi_2\ldots\mathrm{d}\xi_n\,\mathcal{F}^{\text{op}}_n(\epsilon,k,\ell|\xi,\tau)$$
$$\times \int_{0<\chi_{\rho_i}<\chi_{\rho_{i+1}}<1} \mathrm{d}\chi_2\ldots\mathrm{d}\chi_n\,\overline{\mathcal{F}^{\text{op}}_n(\tilde\epsilon,k,\ell|\chi,\tau)} \prod_{j=2}^n \Psi(\xi_j,\chi_j,\ell\cdot k_j)$$

[**Stieberger 2212.06816, 2310.07755**]

- get exactly the open-string incarnation of chiral amplitudes,

$$\mathcal{F}^{\text{op}}_n(\epsilon,k,\ell|\xi,\tau) = \exp\Big(\frac{i\pi\alpha'}{2}\tau\ell^2 - i\pi\alpha'\sum_{j=2}^n(\ell\cdot k_j)\xi_j\Big)\prod_{1\leq i<j}^n\big|\theta_1(\xi_i-\xi_j|\tau)\big|^{s_{ij}}\,\overbrace{Q_n(\epsilon,k,\ell|\xi,\tau)}^{\text{thy-dependent}\ \&\ \xi\to\xi+1\ \text{inv.}}$$

- splitting fct. obstructs (open string)$^{\otimes 2}$ factorization of ξ_j and χ_j-integrals

$$\Psi(\xi_j,\chi_j,\ell\cdot k_j) = \frac{1+e^{-i\pi\alpha'\ell\cdot k_j}}{1-e^{-i\pi\alpha'\ell\cdot k_j}}\,e^{i\pi\alpha'\ell\cdot k_j\Theta[\chi_j-\xi_j]}\,, \qquad (\text{Heaviside } \Theta)$$

- KLT kernel $\mathcal{S}_{\alpha'}(\rho|\sigma)$ is inverse of twisted intersection matrix à la Goto

[**Mazloumi, Stieberger 2403.05208**]

II. 4 Genus-one double copy for string amplitudes

For rectangular tori Re$(\tau)=0$, contour def's $\Rightarrow$ factorize $\int \mathrm{d}^2 z = \int \mathrm{d}\xi \mathrm{d}\chi$

$$M^{\text{1-loop}}_{\text{closed}}(n\,\text{pt})\,\big|_{\text{Re}\,(\tau)=0} = \int \mathrm{d}^D \ell \sum_{\rho,\sigma \in S_{n-1}} [\gamma_\sigma|\gamma^\vee_\rho]^{-1} \int\limits_{0<\xi_{\sigma_i}<\xi_{\sigma_{i+1}}<1} \mathrm{d}\xi_2 \ldots \mathrm{d}\xi_n \, \mathcal{F}^{\text{op}}_n(\epsilon,k,\ell|\xi,\tau)$$

$$\times \int\limits_{0<\chi_{\rho_i}<\chi_{\rho_{i+1}}<1} \mathrm{d}\chi_2 \ldots \mathrm{d}\chi_n \, \overline{\mathcal{F}^{\text{op}}_n(\tilde{\epsilon},k,\ell|\chi,\tau)} \prod_{j=2}^{n} \Psi(\xi_j,\chi_j,\ell\cdot k_j)$$

[Stieberger 2212.06816, 2310.07755]

splitting fct. obstructs (open string)$^{\otimes 2}$ factorization of ξ_j and χ_j-integrals

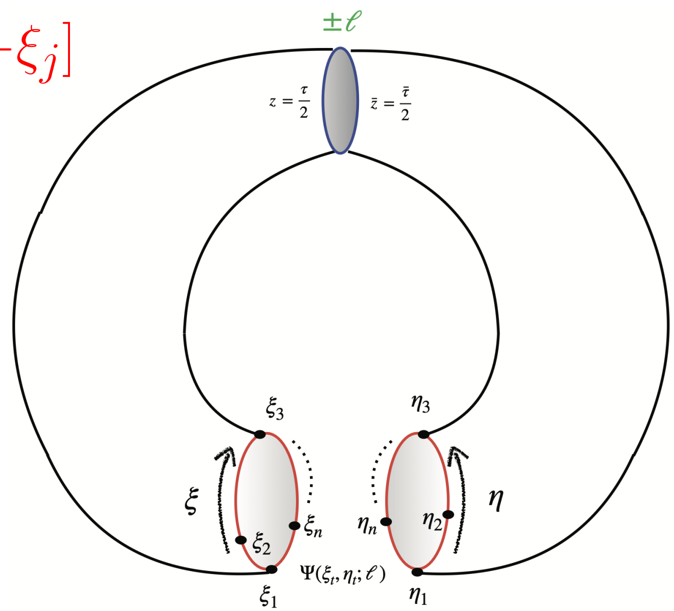
$$\Psi(\xi_j,\chi_j,\ell\cdot k_j) = \frac{1+e^{-i\pi\alpha'\ell\cdot k_j}}{1-e^{-i\pi\alpha'\ell\cdot k_j}}\, e^{i\pi\alpha'\ell\cdot k_j \Theta[\chi_j-\xi_j]}$$

... but imposes level matching

... & admits interpretation as non-planar

cylinder with closed-string bulk insertion

[figure taken from Stieberger 2212.06816]



II. 4 Genus-one double copy for string amplitudes

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$$\times \int_{0<\chi_{\rho_i}<\chi_{\rho_{i+1}}<1} d\chi_2 \ldots d\chi_n\, \overline{\mathcal{F}^{\text{op}}_n(\tilde{\epsilon},k,\ell|\chi,\tau)} \prod_{j=2}^n \Psi(\xi_j,\chi_j,\ell\cdot k_j)$$

[Stieberger 2212.06816, 2310.07755]

From the degeneration limit $\tau\to i\infty$, recover ...

... via $\alpha'\to 0$, the one-loop KLT formula for supergravity ℓ-integrands

[He, OS 1612.00417; He, OS, Zhang 1706.00640]

... upon α'-expansion, the KLT formula for $D^{2k}R^n$ 1-loop matrix elements

[Edison, Guillen, Johansson, OS, Teng 2107.08009]

$\longrightarrow$ maybe find a way around the linearized Feynman propagators

$(\ell+K)^2\to 2\ell\cdot K+K^2$ in the (effective) field-theory KLTs from '16–'21?

II. 5 Discussion of genus-one double copies

Have seen two flavors of one-loop double copy formulae

- cplx. Riemann Wirtinger integral with double copy "$\mathbb{1} = \sum_{c,d} |\gamma_c] \mathbf{H}^{-1}_{cd} [\gamma^\vee_d|$"

$$\int_{T^2(\tau)} \mathrm{d}^2 z \, e^{2\pi i s_A (z-\bar{z})} \left| \frac{\theta_1(z|\tau)}{\theta_1(z-x|\tau)} \right|^{2s_0} F(z-z_a,\eta|\tau) \, \overline{F(z-z_b,\eta|\tau)}$$

[Bhardwaj, Pokraka, Ren, Rodriguez 2312.02148]

  - ∗ so far, only for one integrated puncture z
  - ∗ reality constraint $\mathrm{Im}\,\eta = s_A \mathrm{Im}\,\tau - s_x \mathrm{Im}\,x$

- closed-string n-point one-loop amplitudes (integrand w.r.t. ℓ and τ)

[Stieberger 2212.06816, 2310.07755; Mazloumi, Stieberger 2403.05208]

  - ∗ so far, only for rectangular tori $\mathrm{Re}\,(\tau) = 0$
  - ∗ splitting fct's $\Psi(\xi_j, \chi_j, \ell \cdot k_j)$ interlocking $\int$'s over open-string ξ_j, χ_j's

Rewarding to study both approaches in tandem & combine their strengths!

III. Evaluating string amplitudes from convergent integrals

Recent progress in overcoming the following concerns on traditional integration contours over moduli space $\mathfrak{M}_{g,n}$ in string amplitudes

[Eberhardt, Mizera 2208.12233, 2302.12733, 2403.07051]

- tension between Lorentzian spacetime and Euclidean worldsheet

[Witten 1307.5124]

- integrals don't converge for phys. kinematics (e.g. $\int_0^1 \frac{\mathrm{d}z}{z} |z|^s$ @ $\mathrm{Re}\, s \leq 0$)
- traditional formulae for loop amplitudes are manifestly real whereas optical theorem requires imaginary part for the discontinuities in s

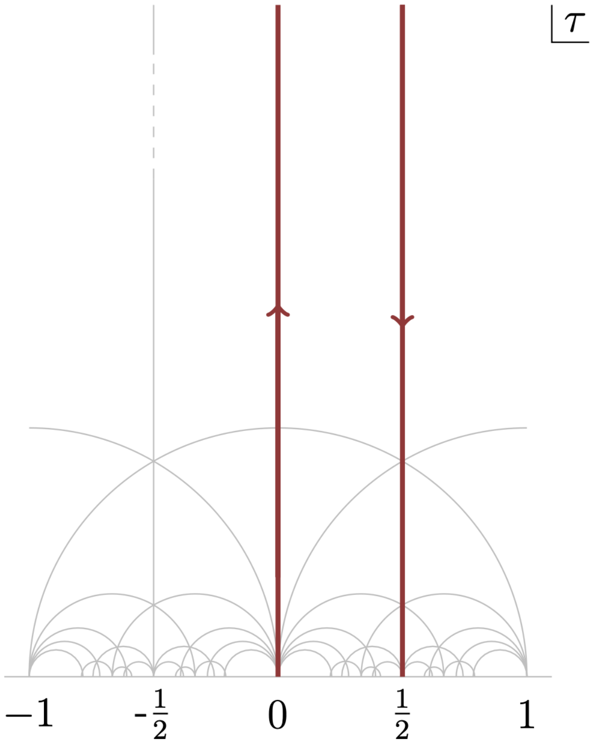
Why did we rarely hear about these concerns?

- marked points z_i more forgiving than cplx. structure moduli τ_j
- no problem in α'-expansion, only finite α' requires new contours

III. 1 New amplitude prescription at one loop

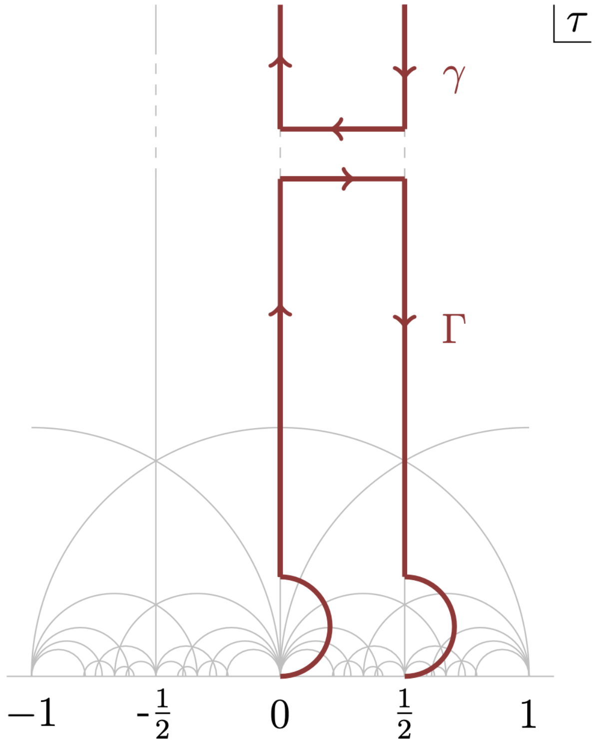
Consider planar 1-loop 4pt amplitude of open superstring: gauge grp. $SO(32)$

$\Longrightarrow$ cylinder ($\tau \in i\mathbb{R}^+$) & Möbius strip ($\tau \in \frac{1}{2}+i\mathbb{R}^+$) @ relative factor -1



contour in [textbooks]

$\Rightarrow \ \text{Im}\, A^{\text{1-loop}}_{\text{open}} = 0$



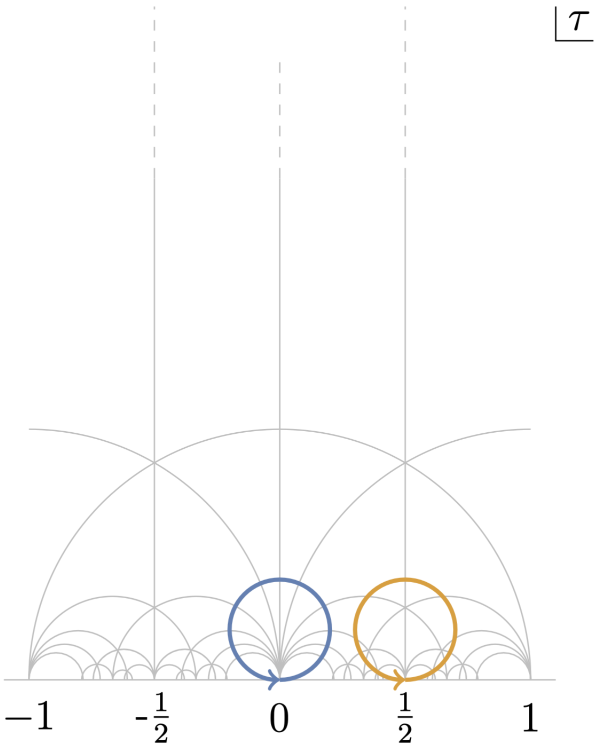
new amplitude prescription [2302.12733]

NOT related by contour deformation!

[**Eberhardt, Mizera 2302.12733; figures taken from the reference**]

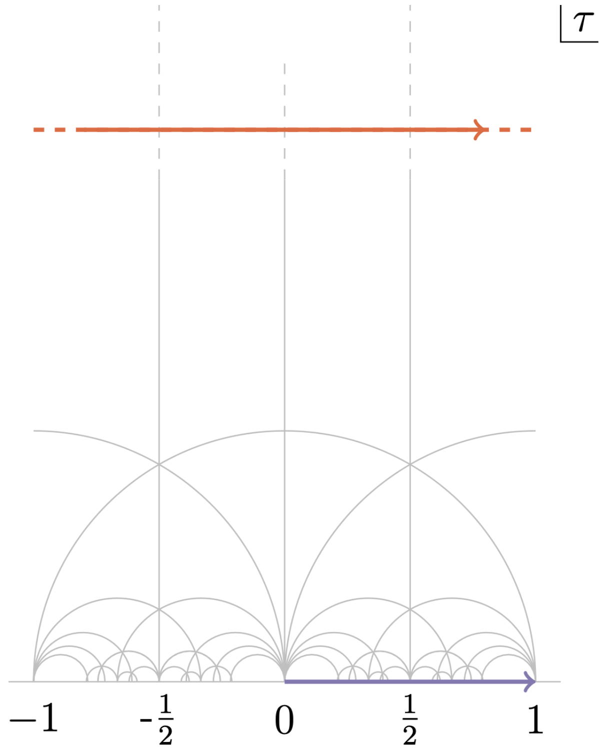
III. 1 New amplitude prescription at one loop

New prescription yields non-zero Im $A^{\text{1-loop}}_{\text{open}}$ and Im $M^{\text{1-loop}}_{\text{closed}}$ localizing on

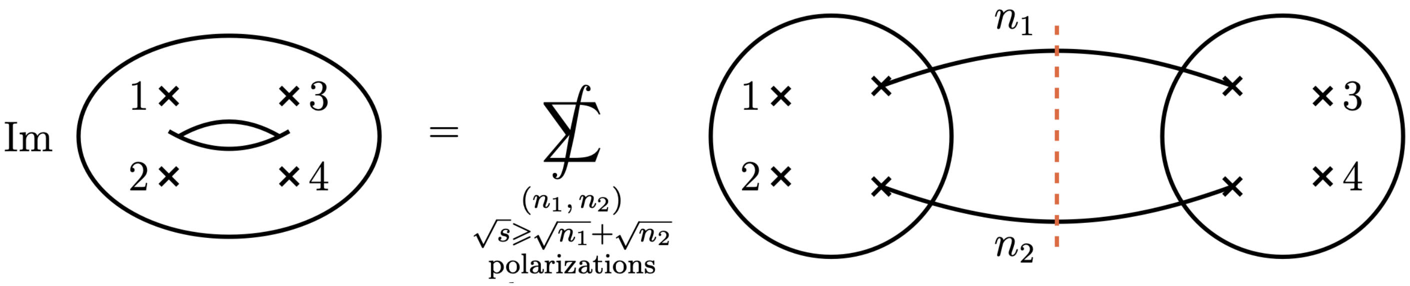


open string: cylinder and Möbius strip contribution

closed string: contour for Re τ and Im τ



consistent with unitarity cuts!

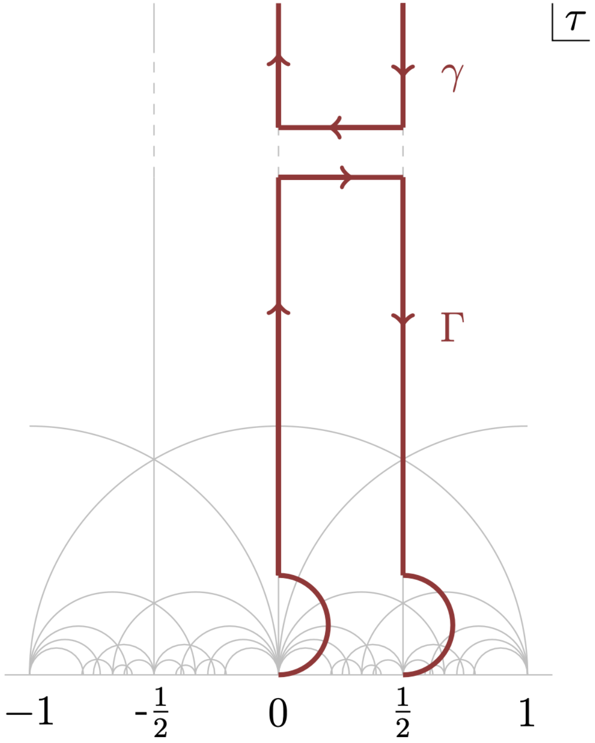


[Eberhardt, Mizera 2208.12233; figures taken from the reference]

III. 1 New amplitude prescription at one loop

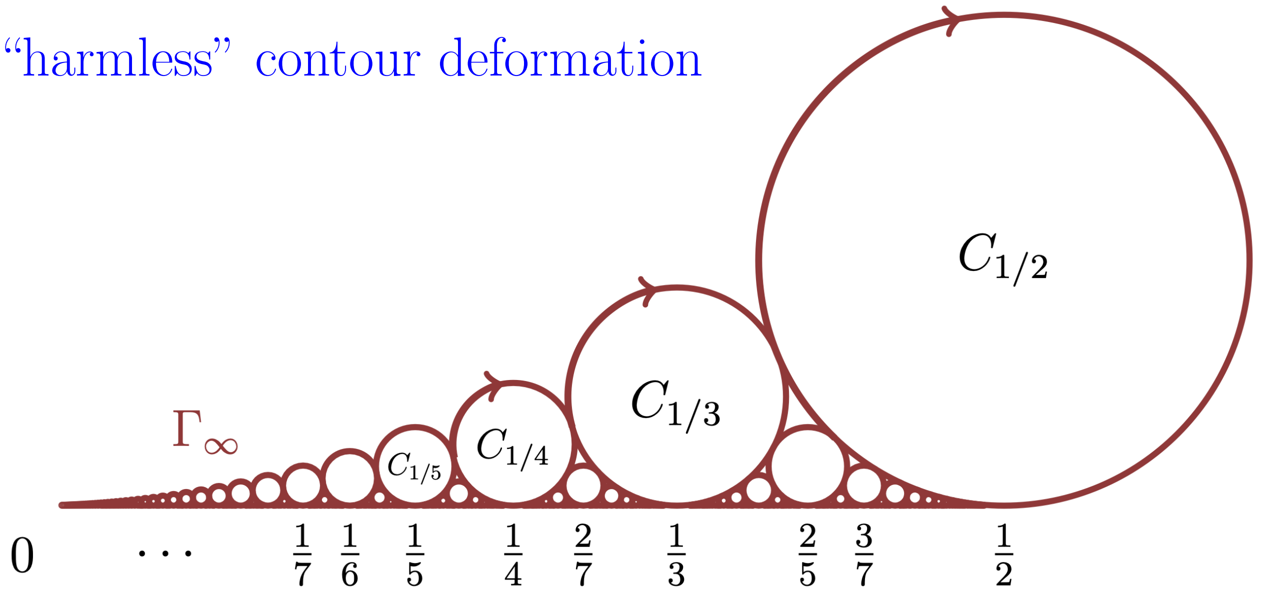
For open strings, full $A^{\text{1-loop}}_{\text{open}}$ (both Re & Im) most conveniently evaluated

on "Rademacher contour" Γ_∞ (∞ collection of circles at $\mathbb{Q}+i\mathbb{Q}$ centers)



this time, two pictures related by

"harmless" contour deformation

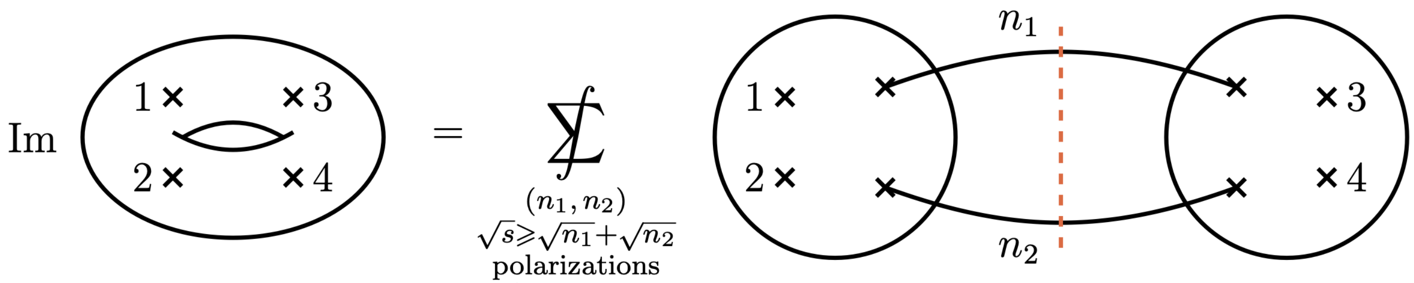


$\longrightarrow$ analytical checks & numerical control at finite α', say $(k_1+k_2)^2 \sim \frac{10}{\alpha'}$

[Eberhardt, Mizera 2302.12733; figures taken from the reference]

III. 2 Checks and applications

Simplified formulae for imaginary parts of loop amplitudes at all energies



[Eberhardt, Mizera 2208.12233; figure taken from the reference]

- low-energy expansions of Im $M^{\text{1-loop}}_{\text{closed}}$ & Im $A^{\text{1-loop}}_{\text{open}}$ match "log(s)-part" of

[D'Hoker, Green 1906.01652; Edison, Guillen, Johansson, OS, Teng 2107.08009]

- at high-energies, Regge limit sometimes dominated by imaginary part

[Banerjee, Eberhardt, Mizera 2403.07064]

Similarly: new integration contours identified for n-point tree amplitudes

$\Longrightarrow$ convergent integral representations, numerical control at finite α'

[Eberhardt, Mizera 2403.07051]

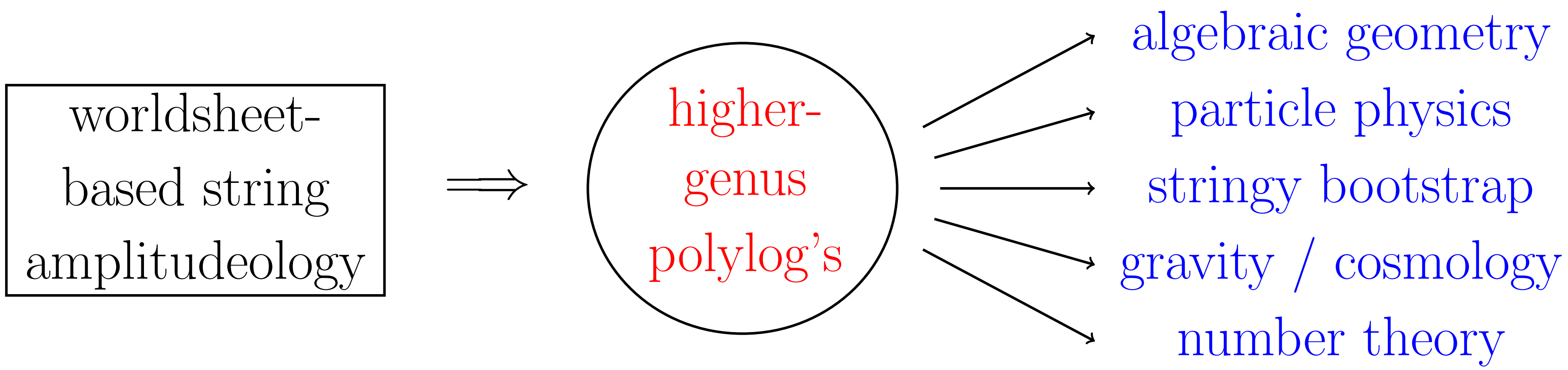
IV. Integration on higher-genus surfaces

Strong symbiosis between two questions:

Q1: What is a good set of integration kernels on Riemann surfaces such that their iterated integrals close under taking primitives?

Q2: What is an integration-friendly function space for integrands of multiloop string amplitudes, universal to type I & II / het / bos theories?

Example that string-theoretic objectives / techniques are useful for other fields



[e.g. talks of Bern, Hansen, McLeod, Lee, Porkert, Sturmfels, Tancredi]

IV. Integration on higher-genus surfaces

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Q2: What is an integration-friendly function space for integrands of multiloop string amplitudes, universal to type I & II / het / bos theories?

$\longrightarrow$ genus zero: $\mathrm{d}\log(z-a)$ kernels of multiple polylogarithms resonate with Parke-Taylor basis for string tree amplitudes in arbitrary theories

$\longrightarrow$ genus one: Kronecker-Eisenstein kernels $g^{(k)}(z|\tau)$ or $f^{(k)}(z|\tau)$ are unified language for elliptic polylogs, modular forms, 1-loop string amp's

$\longrightarrow$ now: higher-genus generalization of $f^{(k)}$ **[D'Hoker, Hidding, OS 2306.08644]**

IV. 1 Double-life of Kronecker-Eisenstein kernel

2×periodic but non-mero' kernels $f^{(k)}(z|\tau) = f^{(k)}(z{+}1|\tau) = f^{(k)}(z{+}\tau|\tau)$

instead of meromorphic / multivalued $g^{(k)}(z|\tau)$ generated by

$$\exp\left(2\pi i\eta\,\frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau}\right)\frac{\theta_1'(0|\tau)\theta_1(z{+}\eta|\tau)}{\theta_1(z|\tau)\theta_1(\eta|\tau)} = \frac{1}{\eta} + \sum_{k=1}^{\infty}\eta^{k-1}f^{(k)}(z|\tau)$$

- backbone of elliptic polylogs in formulation of **[Brown, Levin 1110.6917]**
- function space for 1-loop string integrands (or $g^{(k)}(z|\tau)$ before $\int \mathrm{d}^D\ell$)
  [Broedel, Mafra, OS 1412.5535; Gerken, Kleinschmidt, OS 1811.02548]
- $f^{(k)}(z|\tau)$ at rational pt's $z \in \mathbb{Q} + \tau\mathbb{Q}$ $\Rightarrow$ modular forms of congruence subgroups $\Gamma(N)$ $\Rightarrow$ symbol alphabet for elliptic polylogs at rational pt's
  [Broedel, Duhr, Dulat, Penante, Tancredi 1803.10256]
- convolutions of $f^{(k)}$'s $\Rightarrow$ modular graph forms & sv elliptic polylog's
  [Gerken, Kleinschmidt, OS 1911.03476; D'Hoker, Kleinschmidt, OS 2012.09198]

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$$\exp\left(2\pi i\eta\,\frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau}\right)\frac{\theta_1'(0|\tau)\theta_1(z{+}\eta|\tau)}{\theta_1(z|\tau)\theta_1(\eta|\tau)} = \frac{1}{\eta} + \sum_{k=1}^{\infty}\eta^{k-1}f^{(k)}(z|\tau)$$

Alternatively construction via bosonic (Arakelov) Green function on $T^2(\tau)$

$$\mathcal{G}(z|\tau) = -\log\left|\frac{\theta_1(z|\tau)}{\eta(\tau)}\right|^2 + 2\pi\,\frac{(\mathrm{Im}\,z)^2}{\mathrm{Im}\,\tau}$$

- base case is derivative: $f^{(1)}(z|\tau) = -\partial_z\mathcal{G}(z|\tau) = \partial_z\log\theta_1(z|\tau){+}2\pi i\frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau}$

- higher $k \geq 2$ kernels recursively obtained from convolutions with $\mathcal{G}$

$$f^{(k)}(x|\tau) = \int_{T^2(\tau)}\frac{\mathrm{d}^2z}{\mathrm{Im}\,\tau}\,\partial_x\mathcal{G}(x{-}z|\tau)f^{(k-1)}(z|\tau)$$

IV. 2 Higher-genus generalization of $f^{(k)}$ and elliptic polylog's

Instead of θ_1-representation of $f^{(k)}$, generalize their construction from $\mathcal{G}$:

Arakelov Green function $\mathcal{G}(x,y)$ on higher-genus surface Σ depending on

2 pt's $x, y \in \Sigma$ is uniquely defined by symmetry $\mathcal{G}(x,y) = \mathcal{G}(y,x)$ and

- Laplace eq: $\partial_x \partial_{\bar{x}} \mathcal{G}(x,y) = \pi \kappa(x) - \pi \delta^2(x,y)$ "locally behaves like log"
- absence of zero mode $\int_\Sigma \mathrm{d}^2 x \, \kappa(x) \mathcal{G}(x,y) = 0$

with $\kappa(x)$ the Kähler form on Σ with unit normalization $\int_\Sigma \mathrm{d}^2 x \, \kappa(x) = 1$.

[Faltings '84; Alvarez-Gaumé, Moore, Nelson, Vafa, Bost '86]

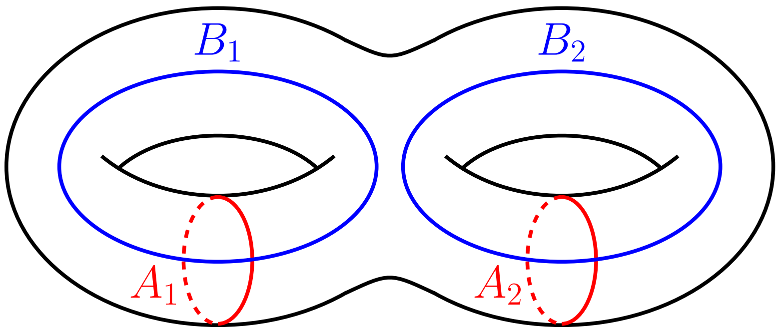
- also $\exists$ representation in terms of the "prime form" (higher-genus θ-fct's)
- separating and non-separating degenerations of $\mathcal{G}(x,y)$ well studied

[D'Hoker, Green, Pioline 1712.06135]

IV. 2 Higher-genus generalization of $f^{(k)}$ and elliptic polylog's

Instead of θ_1-representation of $f^{(k)}$, generalize their construction from $\mathcal{G}$:

Convolute with Abelian differential $\omega_{I=1,2,\ldots,h}(x)$ on genus-h surface Σ



normalization $\displaystyle\oint_{A_I} \omega_J(z)\mathrm{d}z = \delta_{IJ}$

period matrix $\displaystyle\oint_{B_I} \omega_J(z)\mathrm{d}z = \Omega_{IJ}$

with cplx. conjugates $\bar\omega^I(x) = [(\mathrm{Im}\,\Omega)^{-1}]^{IJ}\bar\omega_J(x)$ @ $I, J = 1, 2, \ldots, h$

Even though $\mathcal{G}(x,z)$ integrates to zero against $\kappa(z) = \frac{1}{h}\bar\omega^I(z)\omega_I(z)$ obtain tensorial $f^{(1)}$ kernel from remaining "traceless" $h^2{-}1$ vol. forms $\bar\omega^J(z)\omega_I(z)$

$$f^I{}_J(x,y) = \int_\Sigma \mathrm{d}^2 z\, \partial_x \mathcal{G}(x,z)\bar\omega^J(z)\omega_I(z) - \delta^I_J \partial_x \mathcal{G}(x,y)$$

IV. 2 Higher-genus generalization of $f^{(k)}$ and elliptic polylog's

Instead of θ_1-representation of $f^{(k)}$, generalize their construction from $\mathcal{G}$:

tensorial $f^{(1)}$ kernel from remaining "traceless" h^2-1 vol. forms $\bar\omega^J(z)\omega_I(z)$

$$f^I{}_J(x,y) = \int_\Sigma \mathrm{d}^2 z\, \partial_x \mathcal{G}(x,z)\bar\omega^J(z)\omega_I(z) - \delta^I_J \partial_x \mathcal{G}(x,y)$$

Higher kernels $f^{(k\geq 2)}$ with $k+1$ free indices mimic recursion of $h=1$ case

$$f^{I_1\ldots I_k}{}_J(x,y) = \int_\Sigma \mathrm{d}^2 z\, \partial_x \mathcal{G}(x,z)\, \bar\omega^{I_1}(z) f^{I_2\ldots I_k}{}_J(z,y)$$

Kernels $f^{I_1\ldots I_k}{}_J(x,y)$ at rank $k\geq 2$ are regular throughout $\Sigma\times\Sigma$,

only $k=1$ case has simple pole $f^I{}_J(x,y) = \dfrac{\delta^I{}_J}{x-y} + \mathcal{O}\big((x-y)^0\big)$

IV. 2 Higher-genus generalization of $f^{(k)}$ and elliptic polylog's

Instead of θ_1-representation of $f^{(k)}$, generalize their construction from $\mathcal{G}$:

tensorial $f^{(1)}$ kernel from remaining "traceless" $h^2{-}1$ vol. forms $\bar\omega^J(z)\omega_I(z)$

$$f^I{}_J(x,y) \;=\; \int_\Sigma \mathrm{d}^2 z\, \partial_x \mathcal{G}(x,z)\bar\omega^J(z)\omega_I(z) - \delta^I_J \partial_x \mathcal{G}(x,y)$$

Higher kernels $f^{(k\geq 2)}$ with $k{+}1$ free indices mimic recursion of $h=1$ case

$$f^{I_1\ldots I_k}{}_J(x,y) \;=\; \int_\Sigma \mathrm{d}^2 z\, \partial_x \mathcal{G}(x,z)\, \bar\omega^{I_1}(z) f^{I_2\ldots I_k}{}_J(z,y)$$

Assembly line for higher-genus polylogarithms **[D'Hoker, Hidding, OS 2306.08644]**

- combine f's to flat connection $\mathcal{J}(z,y) \;=\; -\pi \mathrm{d}\bar z\, \bar\omega^I(z) b_I + \mathrm{d}z\, \Psi_J(z,y)\, a^J$

  where $\Psi_J(z,y) \;=\; \omega_J(z) + \mathrm{ad}_{b_I} f^I{}_J(z,y) + \mathrm{ad}_{b_{I_1}} \mathrm{ad}_{b_{I_2}} f^{I_1 I_2}{}_J(z,y) + \ldots$
- expand homotopy-inv. $\mathrm{Pexp}(\int_y^x \mathcal{J}(z,y))$ in words in non-comm. a^J, b_I

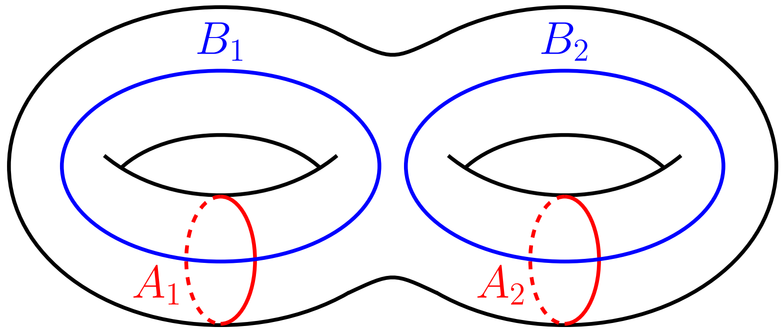
IV. 3 Applications to string-amplitude computations

Bottleneck in $h \geq 2$ loop amplitudes of RNS superstring: simplify $\prod$ of

$$S_\delta(x,y) = \frac{\theta[\delta]\big(\int_y^x \omega_I\big)}{\theta[\delta](0)E(x,y)}$$ fermion Green fct's or "Szegö kernel"

and their summation over "spin structures δ"

$\longrightarrow$ 2^{2h} configurations of $\pm$ that 2dim fermions pick up under A_I, B_J shifts



Higher-genus $f^{I_1\ldots I_k}{}_J(x,y)$-kernels completely disentangle z_i-dependence

from δ-dependence in cyclic products $S_\delta(z_1,z_2)S_\delta(z_2,z_3)\ldots S_\delta(z_n,z_1)$

- $\sum_\delta(S_\delta\text{-cycles})$ are essential parts of chiral amplitudes at $h=1,2$ loops

[D'Hoker, Phong 0501197; D'Hoker, OS 2108.01104]

- part of recent proposal for 4pt chiral amplitude at $h=3$ loops

[Geyer, Monteiro, Stark-Muchão 2106.03968]

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$$S_\delta(z_1,z_2)S_\delta(z_2,z_3)S_\delta(z_3,z_1) \; = \; F^{(3)}_{IJK}(\vec z)\, C_\delta^{IJK} + F^{(2)}_{JK}(\vec z)\, C_\delta^{JK} + F^{(0)}(\vec z)$$

z_i-independent, govern SUSY decomposition

with $F^{(3)}_{IJK}(\vec z) = \omega_I(1)\omega_J(2)\omega_K(3)$ and

$$F^{(2)}_{JK}(\vec z) \; = \; \omega_I(1) f^I{}_J(2,3)\omega_K(3) + \text{cycl}(1,2,3)$$

$$F^{(0)}(\vec z) \; = \; \big(\partial_1 \mathcal{G}(1,3) - \partial_1 \mathcal{G}(1,2)\big)\partial_2\partial_3\mathcal{G}(2,3) - \tfrac{1}{h}\omega_I(1)\partial_3 f^{IK}{}_K(2,3)$$

[D'Hoker, Hidding, OS 2308.05044]

IV. 3 Applications to string-amplitude computations

Bottleneck in $h \geq 2$ loop amplitudes of RNS superstring: simplify $\prod$ of

$$S_\delta(x,y) \ = \ \frac{\theta[\delta]\big(\int_y^x \omega_I\big)}{\theta[\delta](0)E(x,y)} \quad \text{fermion Green fct's or “Szegö kernel”}$$

Higher-genus $f^{I_1\cdots I_k}{}_J(x,y)$-kernels completely disentangle z_i-dependence

from δ-dependence in cyclic products $S_\delta(z_1,z_2)S_\delta(z_2,z_3)\ldots S_\delta(z_n,z_1)$

$$S_\delta(z_1,z_2)S_\delta(z_2,z_3)\ldots S_\delta(z_n,z_1) \ = \ F^{(0)}(\vec{z}) + \sum_{r=2}^{n} F^{(r)}_{I_1\cdots I_r}(\vec{z}) \, C_\delta^{I_1\cdots I_r}$$

with $F^{(r)}_{I_1\cdots I_r}(\vec{z})$ indep. on δ & modular tensors $C_\delta^{I_1\cdots I_r}$ indep. on z_i

[D'Hoker, Hidding, OS 2308.05044]

Next steps:

- simplify integral representations of $C_\delta^{I_1\cdots I_r}$ & rewrite via θ-fct's
- extend to open chains $S_\delta(x,z_1)S_\delta(z_1,z_2)\ldots S_\delta(z_n,y)$ at $x \neq y$

IV. 4 Fay identities

Closure of polylogs under $\int \mathrm{d}z$ requires bilinear identities among kernels

- genus zero: partial fraction $\frac{1}{(y-z)(z-x)} + \mathrm{cycl}(x,y,z) = 0$

$$\int_0^u \mathrm{d}z \, \frac{G(a_1,\ldots,a_n;z)}{(y-z)(z-x)} = \frac{1}{x-y} \int_0^u \mathrm{d}z \left[\frac{1}{z-x} - \frac{1}{z-y} \right] G(a_1,\ldots,a_n;z)$$
$$= \frac{1}{x-y} \big[G(x,a_1,\ldots,a_n;u) - G(y,a_1,\ldots,a_n;u) \big]$$

- genus one: Fay identities among Kronecker-Eisenstein kernels

$$f^{(s)}(x-z) f^{(r)}(y-z) = -(-1)^s f^{(r+s)}(y-x)$$
$$+ \sum_{\ell=0}^{s} \binom{\ell+r-1}{\ell} f^{(s-\ell)}(x-y) f^{(r+\ell)}(y-z)$$
$$+ \sum_{\ell=0}^{r} \binom{\ell+s-1}{\ell} f^{(r-\ell)}(y-x) f^{(s+\ell)}(x-z)$$

no repeated appearance

of z on right-hand side!

$\Rightarrow$ friendly to $\int \mathrm{d}z$

[Brown, Levin 1110.6917; Broedel, Mafra, Matthes, OS 1412.5535]

IV. 4 Fay identities

Closure of polylogs under $\int \mathrm{d}z$ requires bilinear identities among kernels

Higher-genus kernels $f^{I_1 \ldots I_k}{}_J(x,y)$ obey tensorial Fay identities such as

$$f^I{}_J(x,y) f^J{}_K(y,z) + f^I{}_J(y,x) f^J{}_K(x,z) - f^I{}_J(x,z) f^J{}_K(y,z)$$

$$+\omega_J(x) f^{IJ}{}_K(y,x) + \omega_J(y) f^{JI}{}_K(x,z) + \omega_J(x) f^{JI}{}_K(y,z) \ = \ 0$$

- trace w.r.t. I,K yields higher-genus uplift of partial-fraction identity

$$\underbrace{\partial_x \mathcal{G}(x,y)\partial_y \mathcal{G}(y,z) + \partial_y \mathcal{G}(y,x)\partial_x \mathcal{G}(x,z) - \partial_x \mathcal{G}(x,z)\partial_y \mathcal{G}(y,z)}_{\frac{1}{(x-y)(y-z)} + \frac{1}{(z-x)(x-y)} + \frac{1}{(y-z)(z-x)} \ + \ \text{non-singular}} + \text{non-singular} \ = \ 0$$

- at genus one, translation invariance yields cyclic form

$$f^{(1)}(x-y) f^{(1)}(y-z) + f^{(2)}(x-z) + \text{cycl}(x,y,z) = 0$$

IV. 4 Fay identities

Closure of polylogs under $\int \mathrm{d}z$ requires bilinear identities among kernels

Higher-genus kernels $f^{I_1 \ldots I_k}{}_J(x,y)$ obey tensorial Fay identities

$$
\begin{aligned}
f^{I_1\ldots I_r}{}_J(z,x) f^{P_1\ldots P_s J}{}_K(y,z) \; &= \; f^{I_1\ldots I_r}{}_J(z,x) f^{P_1\ldots P_s J}{}_K(y,x) \\
&+ \sum_{m=0}^{s} (-1)^{m-s-1} \sum_{\ell=0}^{r} f^{(P_s\cdots P_{m+1} \sqcup I_1 \cdots I_\ell)}{}_J(z,y) f^{P_1 \cdots P_m J I_{\ell+1} \cdots I_r}{}_K(y,x) \\
&+ \sum_{m=0}^{s} (-1)^{m-s-1} f^{P_1\cdots P_m}{}_J(y,x) \big[f^{(P_s \cdots P_{m+1} J \sqcup I_1 \cdots I_{r-1}) I_r}{}_K(z,x) \\
&\qquad + f^{(P_s\cdots P_{m+1} \sqcup I_1 \ldots I_r) J}{}_K(z,y) \big]
\end{aligned}
$$

no repeated z on RHS!

with shuffles such as $f^{\cdots (P \sqcup I) \cdots}{}_J(x,y) \; = \; f^{\cdots PI \cdots}{}_J(x,y) + f^{\cdots IP \cdots}{}_J(x,y)$

[D'Hoker, OS 2406.abcde]

IV. 5 Meromorphic kernels

How do mero' Kronecker-Eisenstein kernels $g^{(k)}$ generalize beyond genus 1?

$\rightarrow$ Enriquez implicitly defined meromorphic but multi-valued connection ...

... with mero' coefficients $\omega^{I_1 \ldots I_k}{}_J(x,y)$ multiplying $\mathrm{ad}_{b_{I_1}} \ldots \mathrm{ad}_{b_{I_k}} a^J$

... with monodromies $\omega^{I_1 \ldots I_k}{}_J(x{+}B_L,y) = \sum_{\ell=0}^{k} \frac{1}{\ell!} \delta_L^{I_1} \ldots \delta_L^{I_k} \omega^{I_{\ell+1} \cdots I_k}{}_J(x,y)$

generalizing $g^{(k)}(x{+}\tau) = \sum_{\ell=0}^{k} \frac{1}{\ell!}(-2\pi i)^\ell g^{(k-\ell)}(x)$ to arbitrary genus

... including $\omega_J(x) = \omega^{\emptyset}{}_J(x,y)$ as $k=0$ instance **[Enriquez 1112.0864]**

- in chiral splitting / before $\prod_{J=1}^{h} \int \mathrm{d}^D \ell_J$, expect $\omega^{I_1 \ldots I_k}{}_J(x,y)$ to be suitable function space for chiral amplitudes $\mathcal{F}_n(\epsilon, k, \ell | z, \Omega)$

- expressing $\omega^{I_1 \ldots I_k}{}_J(x,y)$ in terms of $f^{I_1 \ldots I_k}{}_J(x,y)$: under investigation

[D'Hoker, Enriquez, OS, Zerbini: work in progress]

IV. 5 Meromorphic kernels

Conjecture: Fay id's of $f^{I_1\ldots I_k}{}_J(x,y)$ hold in identical form for $\omega^{I_1\ldots I_k}{}_J(x,y)$

$$f^I{}_J(x,y)f^J{}_K(y,z)+f^I{}_J(y,x)f^J{}_K(x,z)-f^I{}_J(x,z)f^J{}_K(y,z)$$

$$+\omega_J(x)f^{IJ}{}_K(y,x)+\omega_J(y)f^{JI}{}_K(x,z)+\omega_J(x)f^{JI}{}_K(y,z) = 0$$

$$\omega^I{}_J(x,y)\omega^J{}_K(y,z)+\omega^I{}_J(y,x)\omega^J{}_K(x,z)-\omega^I{}_J(x,z)\omega^J{}_K(y,z)$$

$$+\omega_J(x)\omega^{IJ}{}_K(y,x)+\omega_J(y)\omega^{JI}{}_K(x,z)+\omega_J(x)\omega^{JI}{}_K(y,z) = 0$$

[D'Hoker, OS 2406.abcde; proof under discussion with Enriquez, Zerbini]

IV. 5 Meromorphic kernels

Conjecture: Fay id's of $f^{I_1\ldots I_k}{}_J(x,y)$ hold in identical form for $\omega^{I_1\ldots I_k}{}_J(x,y)$

$$\omega^{I_1\ldots I_r}{}_J(z,x)\omega^{P_1\ldots P_s J}{}_K(y,z) = \omega^{I_1\ldots I_r}{}_J(z,x)\omega^{P_1\ldots P_s J}{}_K(y,x)$$
$$+\sum_{m=0}^{s}(-1)^{m-s-1}\sum_{\ell=0}^{r}\omega^{(P_s\cdots P_{m+1}\sqcup I_1\cdots I_\ell)}{}_J(z,y)\omega^{P_1\cdots P_m J I_{\ell+1}\cdots I_r}{}_K(y,x)$$
$$+\sum_{m=0}^{s}(-1)^{m-s-1}\omega^{P_1\cdots P_m}{}_J(y,x)\big[\omega^{(P_s\cdots P_{m+1}J\sqcup I_1\cdots I_{r-1})I_r}{}_K(z,x)$$
$$+\,\omega^{(P_s\cdots P_{m+1}\sqcup I_1\ldots I_r)J}{}_K(z,y)\big]$$

no repeated z on RHS!

[D'Hoker, OS 2406.abcde; proof under discussion with Enriquez, Zerbini]

Alternative to meromorphic & multivalued connection of **[Enriquez 1112.0864]**:

meromorphic and single-valued connection with higher poles $(x-y)^{\leq -2}$

[Enriquez, Zerbini 2110.09341, 2212.03119]

V. Alternative double copy for single-valued periods

This section: no $\sin(\pi s)$ or related trigonometric intersection numbers

- genus-0 target: single-valued polylog's $\ni$ multi-Regge kinematics of SYM

  [Dixon, Duhr, Penington 1207.0186; Broedel, Sprenger, Torres Orjuela 1606.08411, Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,Papathanasiou,Verbeek '16-19]

- genus-1 target: non-holo "modular graph forms" $\ni$ closed-strings @1loop

  [D'Hoker, Green, Gürdogan, Vanhove 1512.06779; D'Hoker, Green 1603.00839]

Both are double copies of meromorphic quantities (genus-0 polylog's or iterated Eisenstein integrals) $\times$ their complex conjugates $\times$ MZVs

Devil in the detail: the MZV part is surprisingly hard!

e.g. $$G^{\rm sv}(0,0,1,1;z) = G(0,0,1,1;z) + \overline{G(1;z)}G(0,0,1;z) + \overline{G(1,1;z)}G(0,0;z) + \overline{G(1,1,0;z)}G(0;z) + \overline{G(1,1,0,0;z)} + 2\zeta_3\,\overline{G(1;z)}$$

V. Alternative double copy for single-valued periods

This section: no $\sin(\pi s)$ or related trigonometric intersection numbers

- genus-0 target: single-valued polylog's $\ni$ multi-Regge kinematics of SYM
  [Dixon, Duhr, Penington 1207.0186; Broedel, Sprenger, Torres Orjuela 1606.08411, Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,Papathanasiou,Verbeek '16-19]
- genus-1 target: non-holo "modular graph forms" $\ni$ closed-strings @1loop
  [D'Hoker, Green, Gürdogan, Vanhove 1512.06779; D'Hoker, Green 1603.00839]

Both are double copies of meromorphic quantities (genus-0 polylog's or iterated Eisenstein integrals) $\times$ their complex conjugates $\times$ MZVs

Devil in the detail: the MZV part is surprisingly hard!

e.g. generating series $\mathbb{G}(e_0, e_1; z)$ & $\mathbb{G}^{\rm sv}(e_0, e_1; z)$ of mero' / sv polylogs

$$\mathbb{G}^{\rm sv}(e_0, e_1; z) = \overline{\mathbb{G}(e_0, \hat{e}_1; z)}^{\,t} \mathbb{G}(e_0, e_1; z) \, ,$$

non-commutative e_0, e_1

$$\hat{e}_1 = \Phi^{\rm sv}(e_0, e_1) e_1 \Phi^{\rm sv}(e_0, e_1)^{-1}$$

sv Drinfeld associator

[Brown '04]

V. 1 Zeta generators

Reformulated construction of sv polylogs in **[Brown '04]** (and multi-variable generalizations **[1606.08807]**) via "zeta generators" σ_{2k+1} with Lie brackets

$$[\sigma_{2k+1}, e_0] = 0\,, \qquad [\sigma_3, e_1] = [[[e_1, e_0], e_0{+}e_1], e_1]\,, \quad \text{etc.}$$

[Ihara '92; Furusho 0011261]

[Frost, Hidding, Kamlesh, Rodriguez, OS, Verbeek 2312.00697]

Clue: Reformulated construction smoothly extends beyond genus zero!

Above $\sigma_{2k+1} = \sigma^{(g=0)}_{2k+1}$ acting on braid operators e_0, e_1 have organic uplift to zeta generators $\sigma^{(g=1)}_{2k+1}$ at genus one acting on non-comm. variables ϵ_k dual to holomorphic Eisenstein series $\mathrm{G}_k(\tau)$ at $k = 0, 2, 4, \ldots$

[Tsuongai '95; Enriquez 1003.1012; Brown 1504.04737; Schneps 1506.09050; Hain-Matsumoto 1512.03975; Dorigoni, Doroudiani, Drewitt, Hidding, Kleinschmidt, OS, Schneps, Verbeek (DDDHKSSV) 2406.05099]

V. 1 Zeta generators

Combine $\sigma^{(g=0)}_{2k+1}$ and $\sigma^{(g=1)}_{2k+1}$ into genus-agnostic generating series

$$\mathbb{M}^{\rm sv}(\sigma^{(g)}_{2k+1}) = 1 + 2\sum_{k=1}^{\infty} \zeta_{2k+1}\,\sigma^{(g)}_{2k+1} + 2\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \zeta_{2k+1}\zeta_{2\ell+1}\sigma^{(g)}_{2k+1}\,\sigma^{(g)}_{2\ell+1} + \text{higher depth}$$

$$= 1 + \sum_{r=1}^{\infty}\sum_{k_1,\ldots,k_r=1}^{\infty} \underbrace{\phi^{-1}\,{\rm sv}(f_{2k_1+1}\ldots f_{2k_r+1})}_{\text{all single-valued MZVs}}\,\sigma^{(g)}_{2k_1+1}\ldots\sigma^{(g)}_{2k_r+1}$$

Then, obtain universal form for generating series of

- single-valued polylogs: zeta generators $\sigma^{(g=0)}_{2k+1}$ acting on e_0, e_1

$$\mathbb{G}^{\rm SV}(e_0,e_1;z) \;=\; \mathbb{M}^{\rm SV}(\sigma^{(0)}_{2k+1})^{-1}\,\overline{\mathbb{G}(e_0,e_1;z)}^{\,t}\,\mathbb{M}^{\rm SV}(\sigma^{(0)}_{2k+1})\,\mathbb{G}(e_0,e_1;z)$$

- single-valued iterated Eisenstein integrals / modular graph forms:

zeta generators $\sigma^{(g=1)}_{2k+1}$ acting on $\epsilon_k \;\leftrightarrow\; \int {\rm G}_k(\tau)$ in mero' series $\mathbb{I}(\epsilon_k;\tau)$

$$\mathbb{I}^{\rm SV}(\epsilon_k;\tau) \;=\; \mathbb{M}^{\rm SV}(\sigma^{(1)}_{2k+1})^{-1}\,\overline{\mathbb{I}(\epsilon_k;\tau)}^{\,t}\,\mathbb{M}^{\rm SV}(\sigma^{(1)}_{2k+1})\,\mathbb{I}(\epsilon_k;\tau)$$

[DDDHKSSV 2403.14816]

V. 1 Zeta generators

- single-valued iterated Eisenstein integrals / modular graph forms:

  zeta generators $\sigma_{2k+1}^{(g=1)}$ acting on $\epsilon_k \leftrightarrow \int \mathrm{G}_k(\tau)$ in mero' series $\mathbb{I}(\epsilon_k;\tau)$

$$\mathbb{I}^{\rm SV}(\epsilon_k;\tau) \ = \ \mathbb{M}^{\rm SV}(\sigma^{(1)}_{2k+1})^{-1}\, \overline{\mathbb{I}(\epsilon_k;\tau)}^{t}\, \mathbb{M}^{\rm SV}(\sigma^{(1)}_{2k+1})\, \mathbb{I}(\epsilon_k;\tau)$$

[DDDHKSSV 2403.14816]

- concrete genus-one realization of general theory of single-valued periods

[Brown, Dupont 1810.07682]

- makes Brown's equivariant iterated Eisenstein integrals fully explicit

[Brown 1707.01230, 1708.03354]

- inspires $\sigma^{(1)}_{2k+1}$-based proposal for motivic coaction of elliptic MZVs

[Kleinschmidt, Porkert, OS: in progress]

Conclusion & Outlook

- 2 flavors of one-loop double copy formulae à la KLT from intersection theory with complementary strenghts and (? temporary ?) limitations
- new $\int$ contours for string amplitudes $\Rightarrow$ unprecedented control @ finite α'
- progress on construction & properties of integration kernels for higher-genus polylogarithms; $\exists$ first links with Enriquez' meromorphic kernels
- zeta generators $\Rightarrow$ generating series for sv periods in genus-agnostic form

Thank you for your attention !