Categorical Symmetries & $1+1$d Scattering Amplitudes

Christian Copetti
(Oxford)

Based on 2403.04835 and 2406.XXXXX with L. Cordova and S. Komatsu

Science and Technology Facilities Council

What is this talk about?

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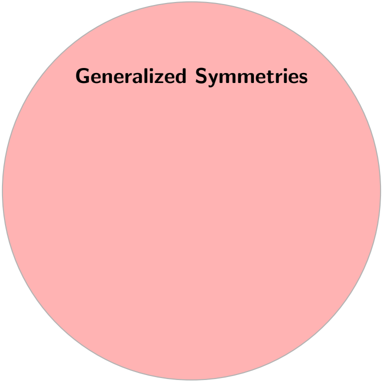

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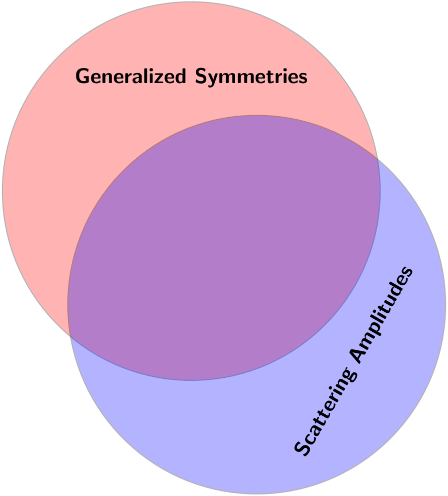

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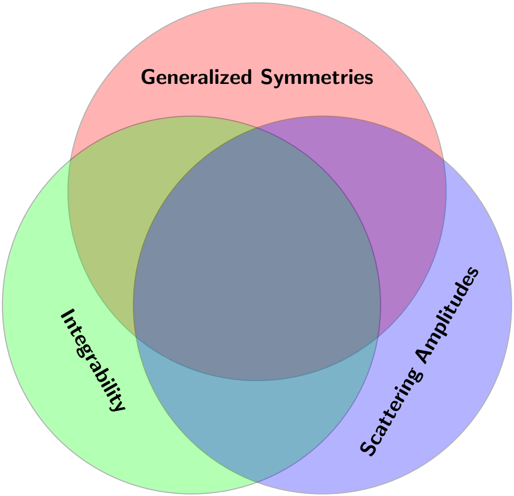

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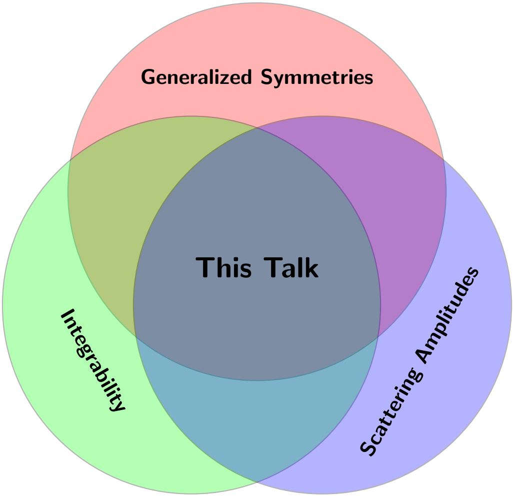

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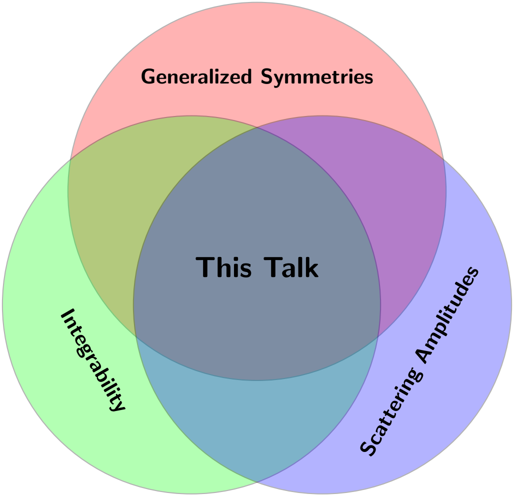

Aim: Discuss action of **Generalized Symmetries** on **S-matrix** and derive physical consequences (**Integrable** examples, but conclusions more general!).

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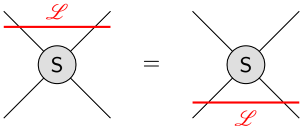

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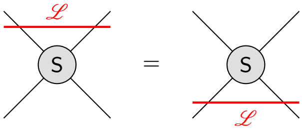

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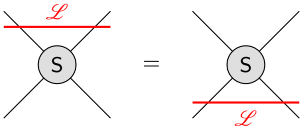

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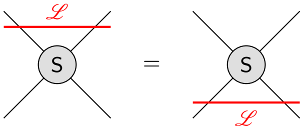

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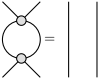

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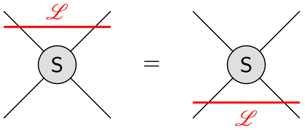

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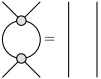

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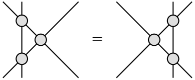

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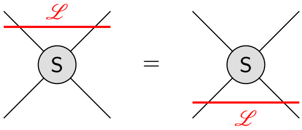

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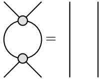

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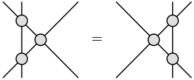

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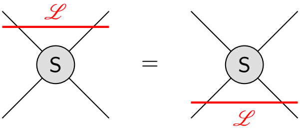

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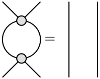

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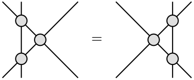

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$$S^{ab}_{dc}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}}\, S^{bc}_{ad}(i\pi - \theta)$$

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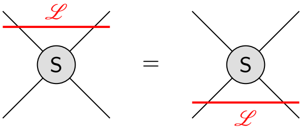

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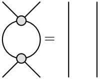

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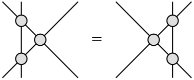

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○ Categorical symmetries can be used efficiently in the **Bootstrap** program. (See Lucia's Lectures!)

Philosophy 

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IR $\longrightarrow$ UV

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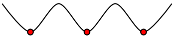

IR Vacua

Symmetric TQFT $\mathscr{M}$

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Massive Kinks

K_{ab}

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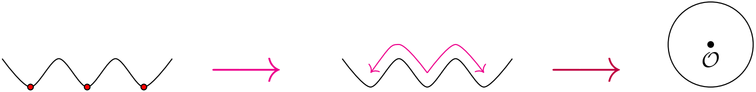

IR Vacua | Massive Kinks | UV CFT

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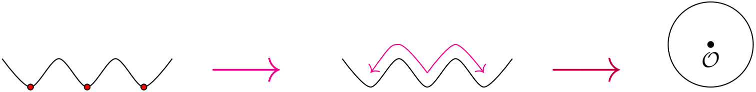

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Symmetry $\mathcal{C}$ is present at all steps.

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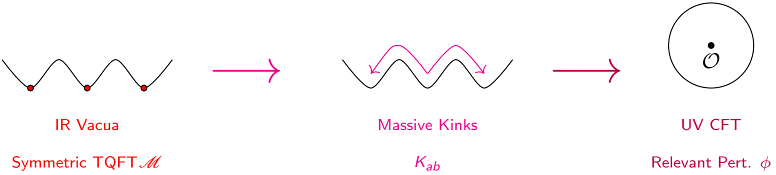

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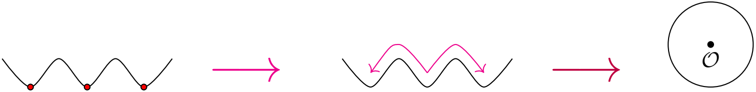

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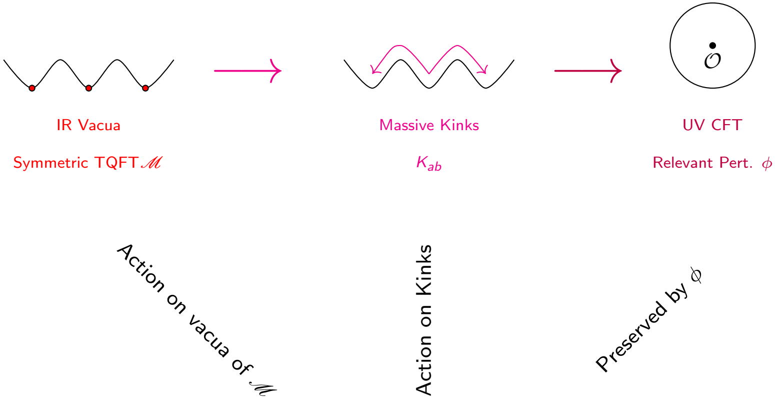

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Categorical symmetries (Review)

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Implemented by topological lines:

[Petkova, Zuber '02; Gaiotto, Kapustin, Seiberg, Willett '14; Chang, Lin, Shao, Wang, Yin '18; ...]

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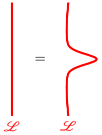

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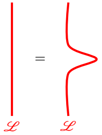
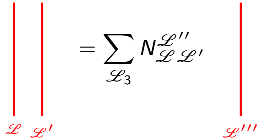


Fusion structure



$$N_{\mathscr{L}\mathscr{L}'}^{\mathscr{L}''} \in \mathbb{N}$$

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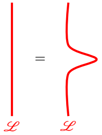

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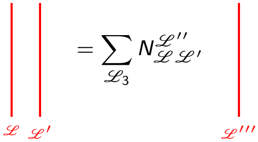

$$= \sum_{\mathscr{L}_3} N_{\mathscr{L}\mathscr{L}'}^{\mathscr{L}''}$$

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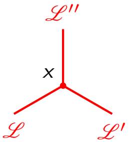
Topological junctions (vector space):

$x \in V_{\mathscr{L}\mathscr{L}'}^{\mathscr{L}''}$:



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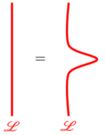

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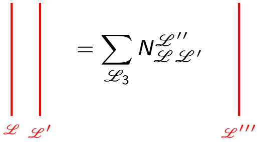

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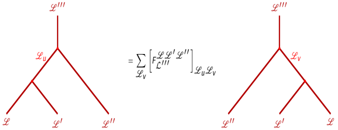
Associativity (F-symbols):



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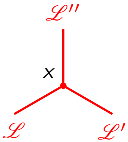

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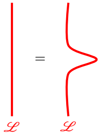

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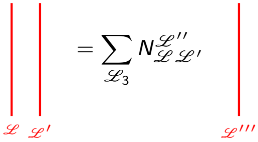

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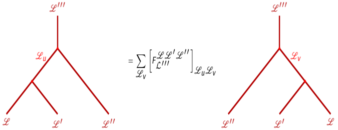
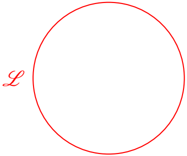


Quantum dimension:



$$= d_{\mathscr{L}} \geq 1$$

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Consider the 1+1d Ising phase diagram:

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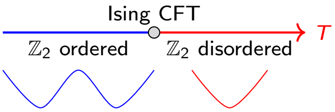

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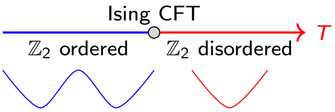

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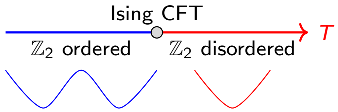

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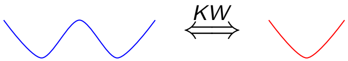

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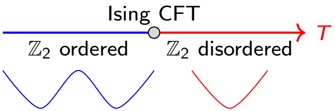

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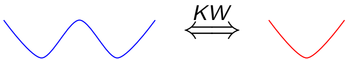

At critical point ○ this becomes a symmetry $\mathcal{N}$.

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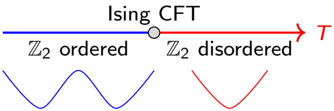

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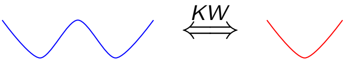

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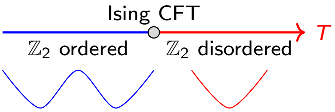

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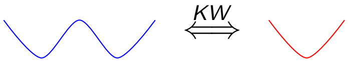

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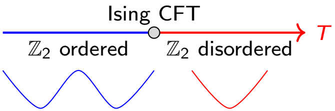

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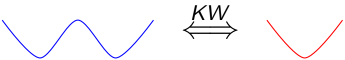

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Fusion algebra:

$$\eta^2 = 1\,, \quad \eta\,\mathcal{N} = \mathcal{N}\,\eta = \mathcal{N}$$
$$\mathcal{N}^2 = 1 + \eta$$

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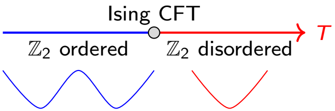

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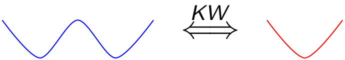

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Ising Symmetry:

$$\text{Ising} = \{1,\, \eta,\, \mathcal{N}\}$$

Fusion algebra:

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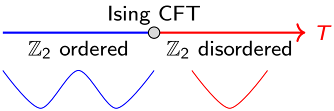
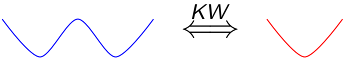
Example: Ising Symmetry

Consider the 1+1d Ising phase diagram:



KW duality exchanges high and low T



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The KW defect line $\mathcal{N}$ is **non-invertible**!

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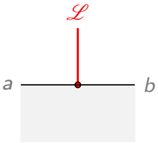

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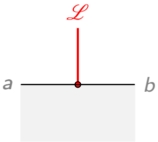

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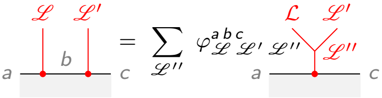

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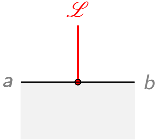

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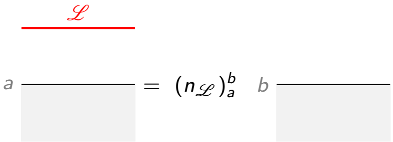

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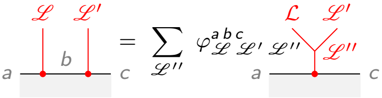

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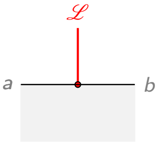

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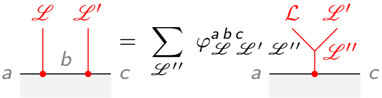

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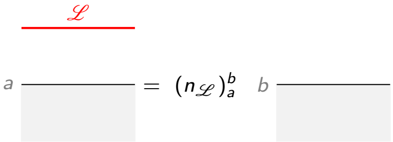

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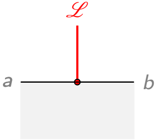

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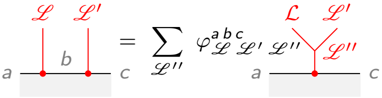

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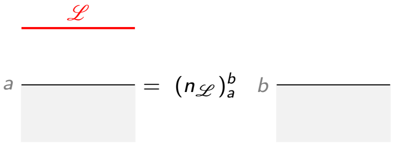

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This endows $\mathscr{M}$ with the mathematical structure of a module category over $\mathcal{C}$.

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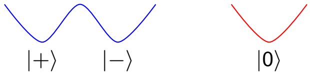

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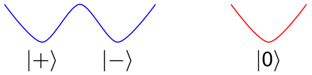

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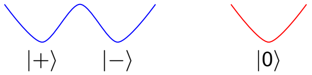

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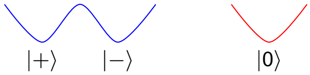

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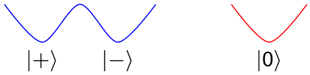

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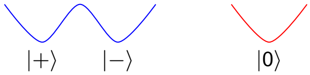

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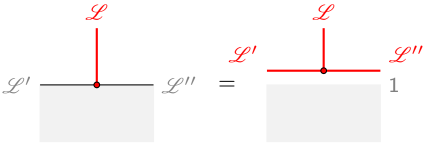

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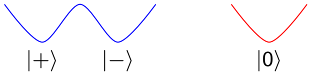

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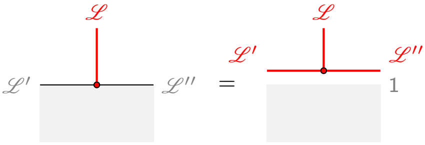

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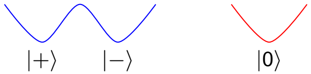

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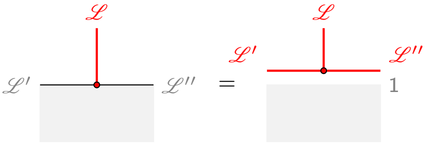

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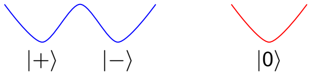

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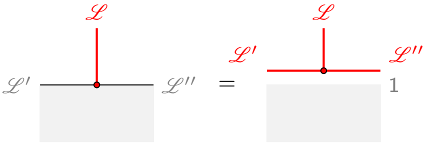

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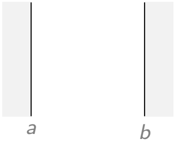

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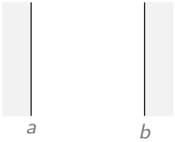
$\mathcal{H}_{a,b} \simeq$ 

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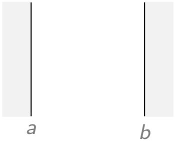
$\mathcal{H}_{a,b} \simeq$ 

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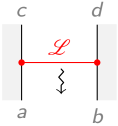

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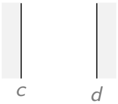

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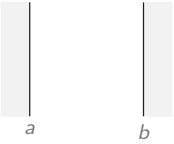
$\mathcal{H}_{a,b} \simeq$ 

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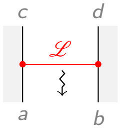
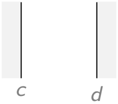
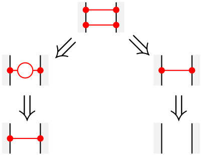


$= \sqrt{d_{\mathscr{L}}}\,[\mathscr{L}]_{ab}^{cd}$ 

Composition of two lines $\mathscr{L}$, $\mathscr{L}'$



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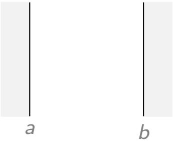

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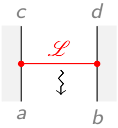

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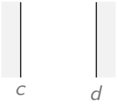

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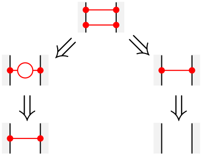

Gives the algebra:

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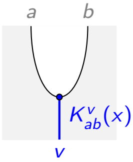
... and Topological Lines

The irreducible representations of this algebra are labelled by lines $v \in \mathcal{C}^*_{\mathcal{M}}$.



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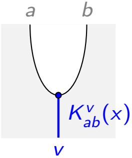

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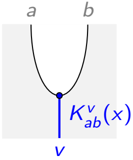

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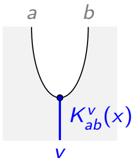

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The fusion algebra $v \times v' = \sum_{v''} \widetilde{N}^{v''}_{vv'}\, v''$ encodes the tensor product decomposition of irreps $\longrightarrow$ **kink bound states!**

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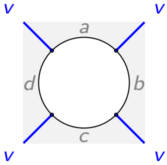

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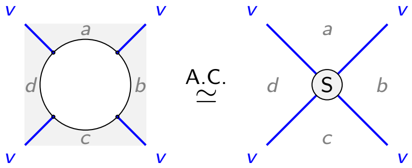

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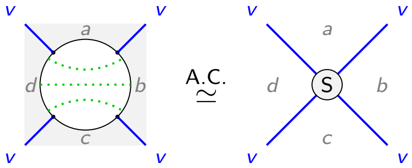

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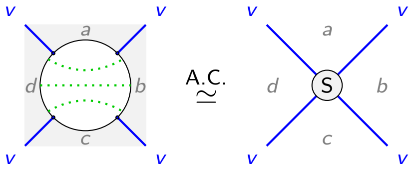

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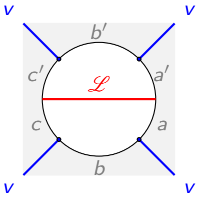

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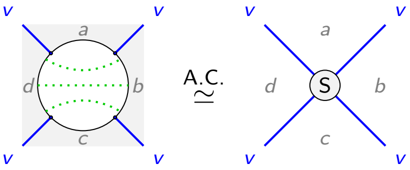

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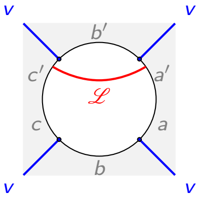

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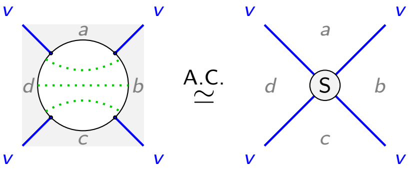

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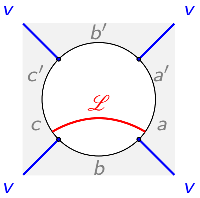

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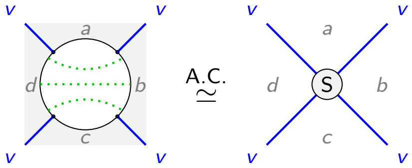

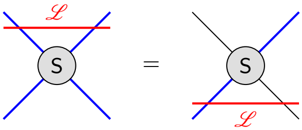

$$\sum_e [\mathcal{L}; v]_{cb}^{c'e} [\mathcal{L}; v]_{ba}^{ea'} \sqrt{\frac{d_a}{d_c}}\, S_{a'e}^{c'b'}(\theta) = \sum_{e'} [\mathcal{L}; v]_{b'c'}^{e'c} [\mathcal{L}; v]_{a'b'}^{ae'} \sqrt{\frac{d_{a'}}{d_{c'}}}\, S_{ab}^{ce'}(\theta)\,.$$

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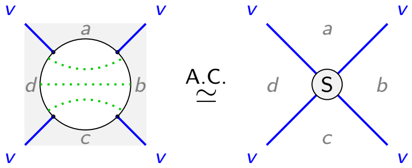

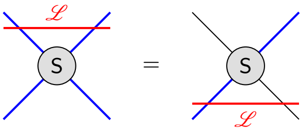

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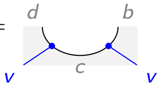
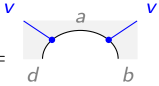
$|\psi\rangle_{in} =$ 

$\langle\psi|_{out} =$ 

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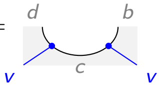
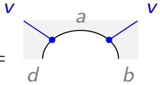
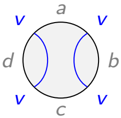
$|\psi\rangle_{in} =$ 

$\langle\psi|_{out} =$ 

Their norms are schematically ${}_{in}\langle\psi|\psi\rangle_{in} =$  $= d_v\sqrt{d_b d_d}$

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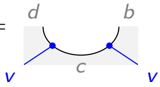
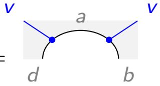
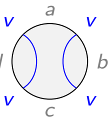
$|\psi\rangle_{in} =$ 

$\langle\psi|_{out} =$ 

Their norms are schematically $_{in}\langle\psi|\psi\rangle_{in} =$ d  b $= d_v\sqrt{d_b d_d}$

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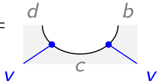
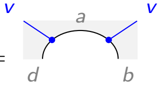
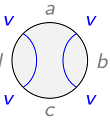
$|\psi\rangle_{in} =$ 

$\langle\psi|_{out} =$ 

Their norms are schematically $_{in}\langle\psi|\psi\rangle_{in} =$  $= d_v\sqrt{d_b d_d}$

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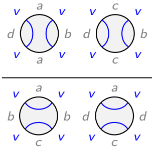

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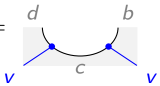
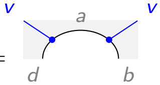
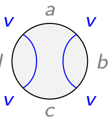
$|\psi\rangle_{in} =$ 

$\langle\psi|_{out} =$ 

Their norms are schematically ${}_{in}\langle\psi|\psi\rangle_{in} = d$  $b = d_v\sqrt{d_b d_d}$

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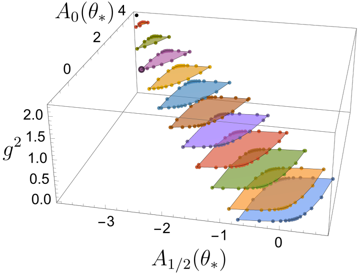

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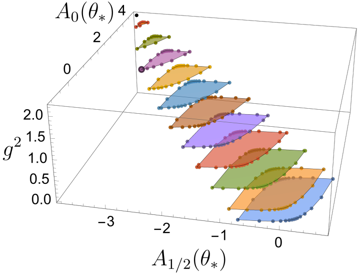

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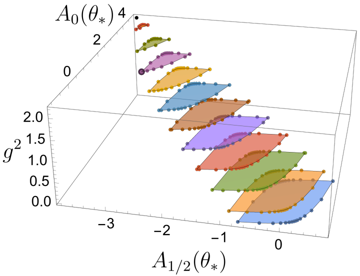

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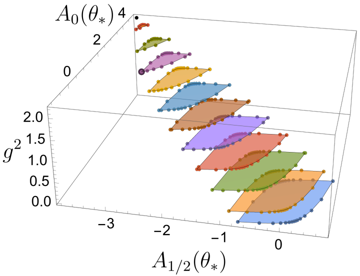

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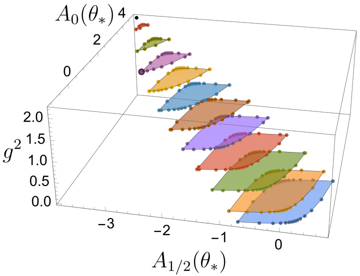

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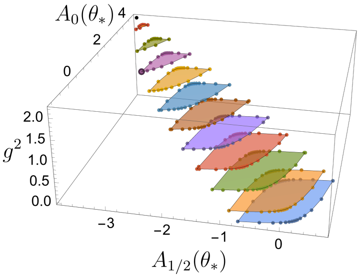

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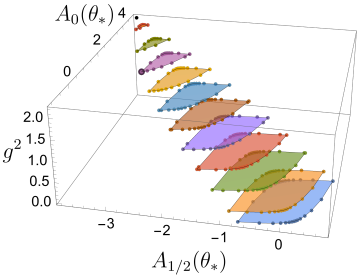

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