Three point amplitudes from the Matrix Model approach to 11 dimensional M-theory

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IAS

We will first review the BFSS matrix model for M-theory

then we will discuss what we recently did with

[JM , Herderschee](#)
[JM, Herderschee(2)](#)

(All references are hyperlinks).

You are all familiar with tree level gravity amplitudes

As you go to loops there are two problems:

-UV divergencies
-IR   divergencies

The IR divergencies require us to define smarter observables.

The UV divergencies will require us to define a better theory.

We will avoid IR divergences by going to higher dimensions, where this issue is not present.

UV → better theory

``Theory of gravity'' = theory that reduces to Einstein gravity at low energy, but gives well defined answers for scattering amplitudes.

An example is string theory, where we get finite answers at each order in perturbation theory.

We will mainly discuss “M-theory”.

It is supposed to be an 11 dimensional theory that reduces to 11 d supergravity at low energies.

We do not have an explicit definition of this theory.

M-theory can be obtained via various limits:

- Strong coupling limit of IIA string theory → M-theory on S^1

- Via the BFSS conjecture

- Large N limit of $AdS_4 x S^7$ via the dual field theory, which is explicitly defined (it is a 3d Chern Simons + matter theory).

Chester, Pufu, X. Yin ; Binder, Chester, Pufu

The BFSS conjecture

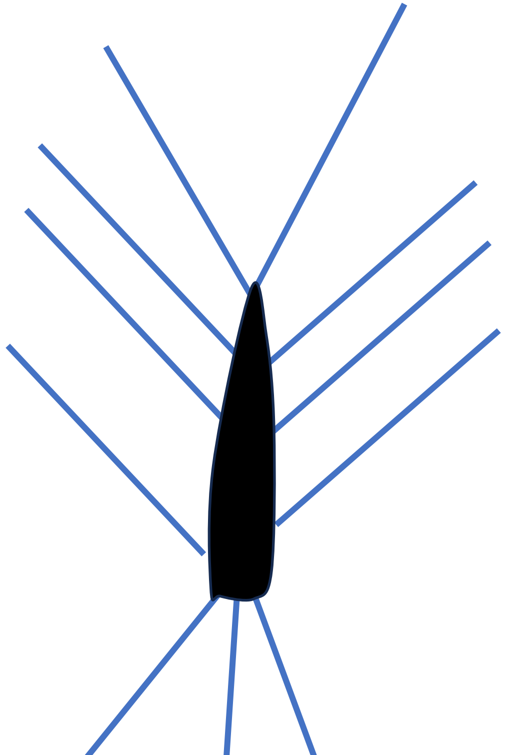
M-theory scattering amplitudes

=

the large N limit of a scattering problem in a matrix quantum mechanics.

Banks, Fischler, Shenker, Susskind 1996

The main observables are the scattering amplitudes of massless particles, namely the ones in the graviton supermultiplet.



This includes processes involving black hole formation and evaporation, etc.

We are talking about the full non perturbative amplitudes.

For now the statement is that we expect them to be well defined, even if we can compute them.

Light front, or light-cone, coordinates

$$ds^2 = -2dx^+dx^- + d\vec{y}^2 \ , \qquad \vec{y} \in R^9$$

$$e^{ip_\mu x^\mu} \ , \qquad p_\mu = -i\frac{\partial}{\partial x^\mu}$$

$$-p_- \geq 0 \quad \text{and} \quad -p_+ \geq 0$$ positive

$$-p_+ = \frac{\vec{p}^2}{2(-p_-)}$$ For a massless particle.

View x^+ as time and $-p_+$ as the Hamiltonian →

looks like a non-relativistic particle of mass $M = -p_-$

Now we will do a drastic operation.

Compactify the x^- direction

$$x^- \sim x^- + 2\pi R\ , \qquad \rightarrow \qquad -p_- = \frac{N}{R}$$

$$N \geq 0$$

The null compactification

Susskind, Sen, Seiberg

- A null circle has zero proper length.
- We view it as as the zero size limit of a spacelike circle.
- Small spacelike circle → IIA string theory at weak coupling.
- N units of momentum along the small circle → N D0 branes

Carefully working out the limit one finds:

Amplitudes in compactified theory = low energy limit of D0 brane scattering → computed by a matrix model

Polchinski , Witten

$$\mathcal{A}_{\text{compactified}}(p_1, \cdots, p_n) = \mathcal{A}_{\text{Matrix Model}}(N_1, \vec{p}_1; \cdots; N_n, \vec{p}_n)$$

(We will define the matrix model in more detail later.)

This follows simply from the M-theory/IIA relationship.

This is not yet the BFSS conjecture.

Note that:

$$\mathcal{A}_{\text{compactified}}(p_1, \cdots, p_n) \neq \mathcal{A}_{\text{un-compactified}}(p_1, \cdots, p_n)$$

$$\mathcal{A}_{\text{compactified}}(p_1, \cdots, p_n) = \mathcal{A}_{\text{Matrix Model}}(N_1, \vec{p}_1; \cdots; N_n, \vec{p}_n)$$

In other words,

This follows almost by definition of the compactified theory.

The BFSS conjecture is the following

What we were interested in

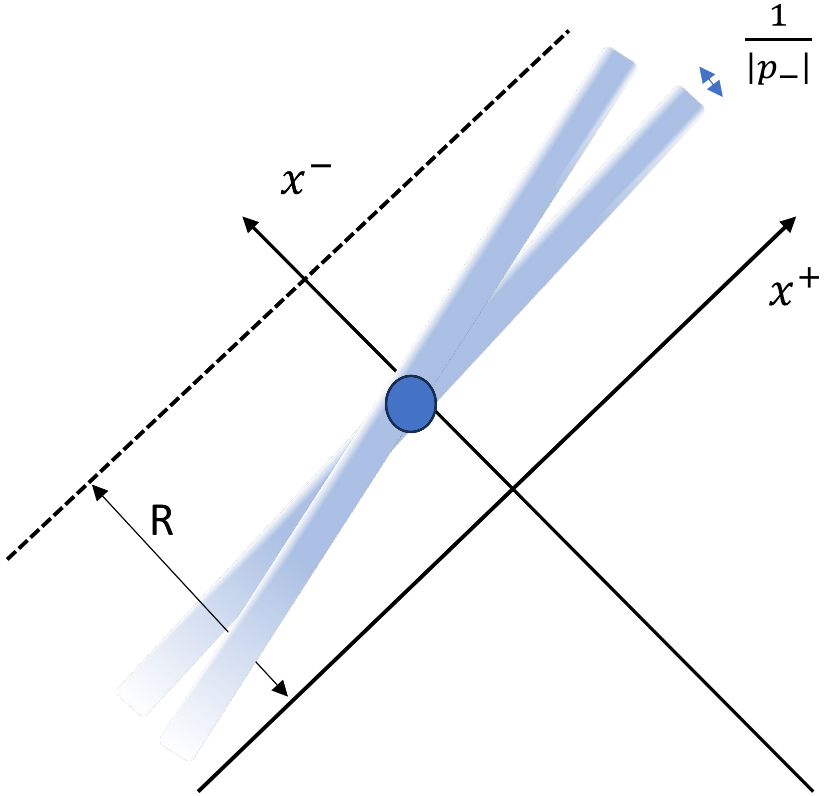
$$\lim_{N^i\to\infty,\ R\to\infty}\left[\mathcal{A}_{\text{Matrix Model}}(N_1,\vec{p}_1;\cdots;N_n,\vec{p}_n)\right]=(2\pi R)^{1-\frac{n}{2}}\mathcal{A}_{\text{un}-\text{compactified}}(p_1,\cdots,p_n)$$

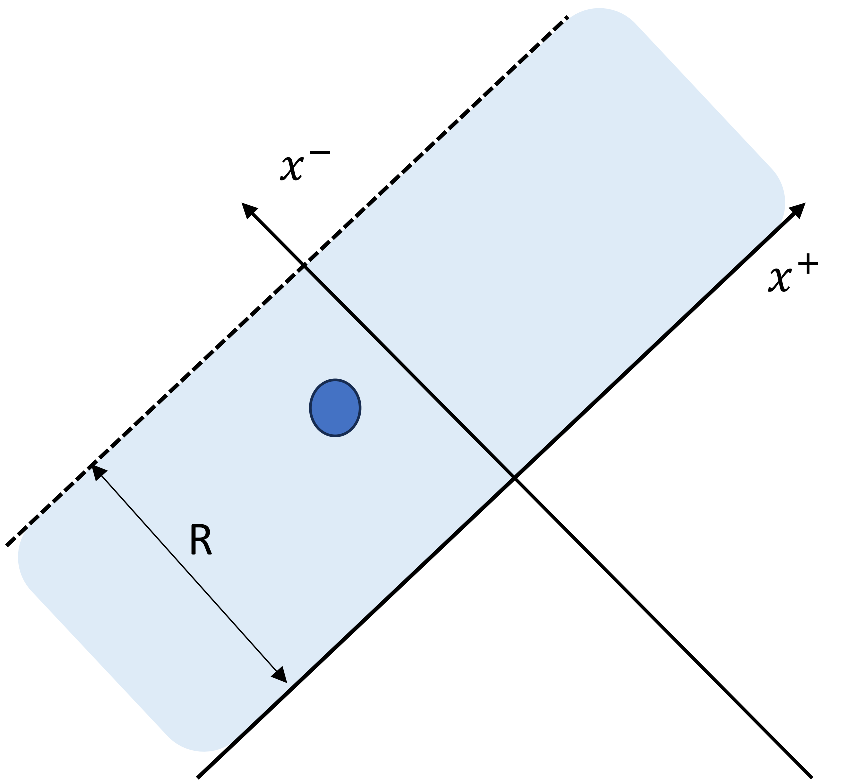
$$\text{with}\quad \frac{N^i}{R}=-p^i_-=\text{fixed}$$

Banks, Fischler, Shenker, Susskind 1996

Intuition

Banks, Fischler, Shenker, Susskind 1996





Let us now define more carefully the matrix model

The matrix model

$$S = \int dt Tr \left\{ \sum_I \frac{(D_t X^I)^2}{2R} + \frac{R}{4(2\pi)^2 l_p^6} \sum_{IJ} [X^I, X^J]^2 + \psi_\alpha D_t \psi_\alpha + i \frac{R}{(2\pi) l_p^3} \psi_\alpha \gamma^I_{\alpha\beta} [\psi_\beta, X^I] \psi_b \right\}$$

$X^I, \ \psi_\alpha$ are NxN Hermitian matrices.

$G_{N,11} = 16\pi^7 l_p^9$

I,J = 1,…, 9

$\alpha, \beta = 1, \cdots, 16$, $\gamma^I_{\alpha,\beta}$ = nine real symmetric traceless gamma matrices

SO(9) symmetry + 16 supersymmetries.

$D_0 Y = \partial_t Y + [A_0, Y]$

Gauged: A_0 → Imposes the U(N) singlet constraint.

The U(1) sector decouples

$$S = N \int dt \sum_I \frac{(\dot{X}^I)^2}{2R} + \psi_\alpha \dot{\psi}_\alpha$$

If $E_{SU(N)} = 0$,

Superparticle action in light cone gauge

$$S = N \int dt \sum_I \frac{(\dot{X}^I)^2}{2R} + \psi_\alpha \dot{\psi}_\alpha$$

Fill out the 256 = 2^8 states of the massless supergraviton multiplet

$$H = -p_+ = \vec{p}^2 \frac{1}{2} \frac{R}{N}$$

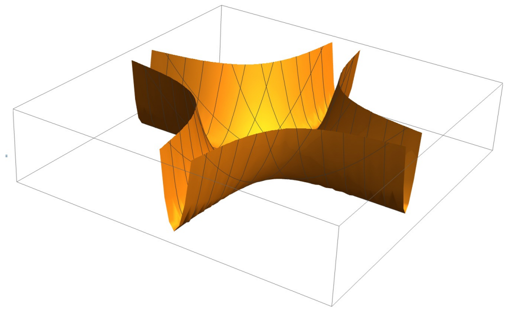
All the states of a massless particle in 11 dimensions

It is believed that the SU(N) problem has a single bound state at energy E=0.

Single zero energy bound state

The potential has zero energy valleys, so the quantum mechanics will have a continuous spectrum (more on that later).

This is a truly normalizable zero energy state.



Evidence: Index arguments.

Piljin Yi ; Sethi, Stern; Moore, Nekrasov, Shatashvili ; Konechny ; Porrati, Rozenberg ;Sethi, Stern ;

Low energy states of the matrix model.

The potential vanishes when the matrices commute: $[X^I, X^J] = 0 \rightarrow$ diagonalize them

$$X^i = \begin{pmatrix} x_1 \mathbf{1}_{N_1} & & & \\ & x_2 \mathbf{1}_{N_2} & & \\ & & x_3 \mathbf{1}_{N_3} & \\ & & & x_4 \mathbf{1}_{N_4} \end{pmatrix}$$

Consider $|x_i - x_j| \gg$ large. And add a bound state wavefunction in each sub-block

→ we have 4 gravitons with momenta N_i.

Asymptotic states of the matrix model.

In general the N x N matrix will separate into n submatrices, where the center of mass coordinate of each will be very far away from the others.

With each group of size N_i forming a bound state.

This gives an asymptotic n graviton state.

Scattering in the matrix model

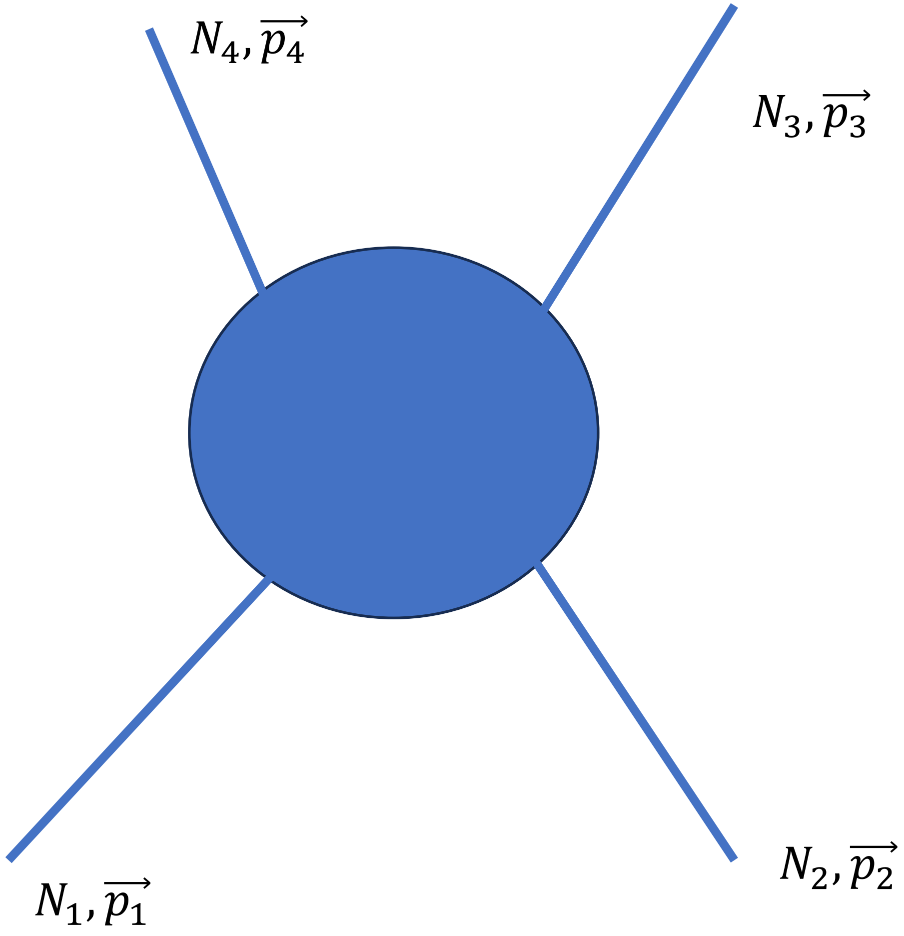
$$\begin{pmatrix} x_3 \mathbf{1}_{N_3} & 0 \\ 0 & x_4 \mathbf{1}_{N_4} \end{pmatrix}$$

$$\uparrow$$

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$\uparrow$$

$$\begin{pmatrix} x_1 \mathbf{1}_{N_1} & 0 \\ 0 & x_2 \mathbf{1}_{N_2} \end{pmatrix}$$



$$N_1 + N_2 = N_3 + N_4$$

Now that we defined the scattering problem in the matrix model, we will restate the BFSS conjecture

$$\lim_{N^i \to \infty,\ R \to \infty} [\mathcal{A}_{\text{Matrix Model}}(N_1, \vec{p}_1; \cdots ; N_n, \vec{p}_n)] = (2\pi R)^{1-\frac{n}{2}} \mathcal{A}_{\text{un-compactified}}(p_1, \cdots, p_n)$$

$$\text{with} \quad \frac{N^i}{R} = -p_-^i = \text{fixed}$$

Subconjectures:

- The limit exists.
- The limit defines a suitably analytic function of p_-^i
- The result is fully Lorentz invariant.
- The S-matrix is unitary in the Fock space. (e.g. the total probability that produce finite N^i gravitons goes to zero as the other N^j go to infinity).

Disputed in: Banks, Fischler

- We have all the properties we expect from M-theory: reduces to supergravity amplitudes, contains membranes, fivebranes, black holes, etc.

This is a remarkable conjecture.

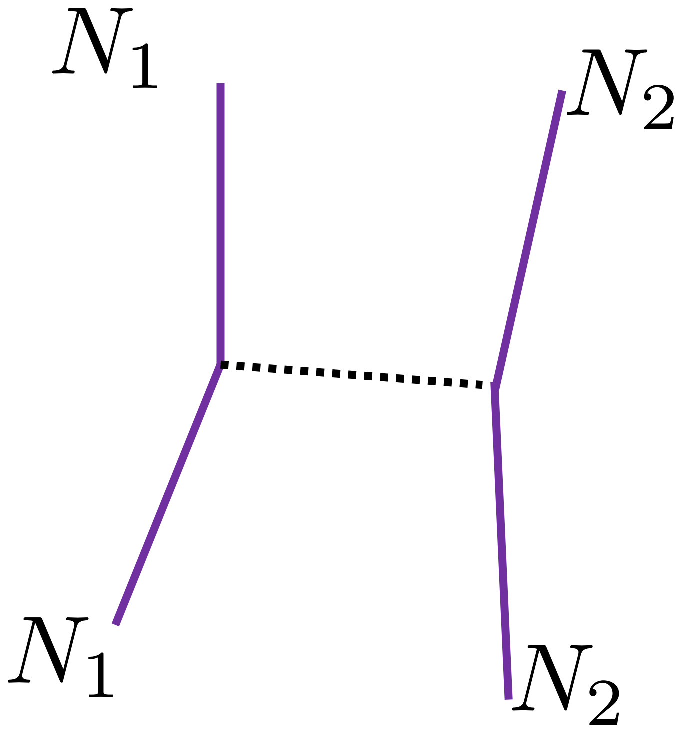
We ``just’’ have to take the large N limit of the matrix quantum mechanics.

It is hard because the matrix quantum mechanics is strongly coupled in this limit.

The only amplitude previously computed is a limit of the four point amplitude.

Low velocity expansion

Douglas, Kabat, Pouliot, Shenker



$$S = S_{\text{free}} + \int dt \frac{N_1 N_2 l_p^9}{R^3} \frac{v^4}{r^7} = S_{free} + p_-^1 p_-^2 \frac{l_p^9}{R} \int dt \frac{v^4}{r^7}$$

Can be computed because it is protected by susy.

(Also, there is no correction to the v^2 terms).

Paban, Sethi, Stern

This is a check of the conjecture

If we could compute the general 4pt amplitude, it would have a huge amount of information.

We can do something much more modest:

The three point amplitude

The three point amplitude

- Trivial in M-theory → fixed by Lorenz invariance + SUSY.
- In the matrix model we have less symmetries → non-trivial computation.

The only reason to do it is to check the conjecture

Why can it be done?

- The three point kinematics preserves some supersymmetry.

- We can relate the amplitude to the computation of an index quantity that is independent of the coupling.

We will now give a quick outline of the computation

Kinematics of the three point amplitude

$$(p_+, p_-, p_z, p_{\bar{z}}) \ , \qquad p_z = p_9 + ip_8 \ , \qquad p_{\bar{z}} = p_9 - ip_8$$

$$p_1 = (0, -\frac{N_1}{R}, p_z, 0) \ , \qquad p_2 = (0, -\frac{N_2}{R}, -p_z, 0) \ , \qquad p_3 = (0, \frac{N_3}{R}, 0, 0)$$

With p_9 real, this implies that p_8 is purely imaginary.

(Only four of the components of the momenta are shown, the rest are zero)

The SUSY preserved by the three point amplitude

$$\not{p}_i \epsilon = 0$$

$$\Gamma^- \epsilon = \Gamma^z \epsilon = 0$$

These two conditions are compatible with each other.

Preserve ¼ of the supersymmetries.

Need one extra trick

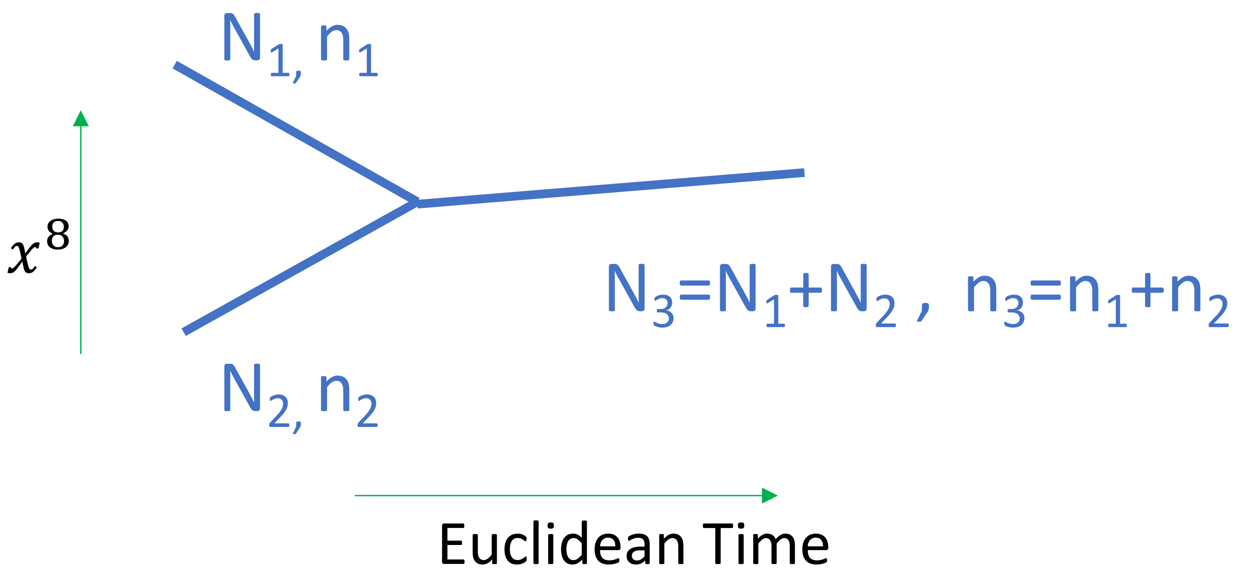
Compactifying the 9^{th} dimension

Matrix QM to 1+1 dimensional matrix gauge theory.

Momentum along the 9^{th} direction = n/R_9 $\rightarrow$

flux of the gauge field on the 1+1 dimensional worldvolume.

Similar to N D1 branes with n units of fundamental string charge. (p,q) = (N,n) strings.

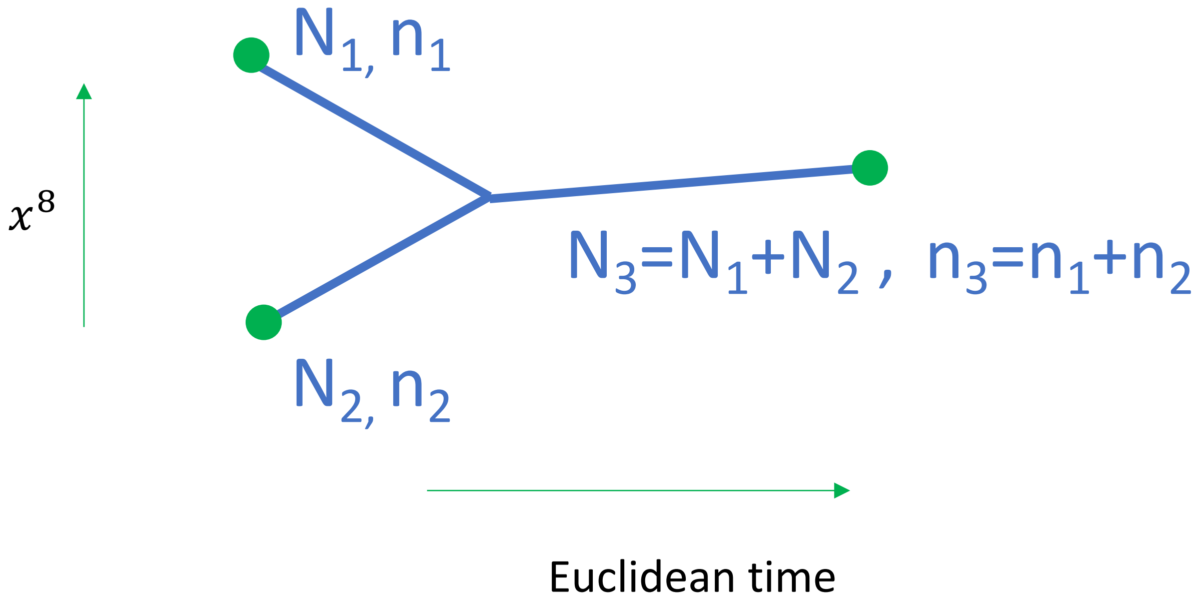


Similar to (N,n) string junction.

They are extended along the compact $\tilde{9}$ circle

Thinking of $\tilde{9}$ as the Euclidean time circle → index computation Tr[$(-1)^F$]

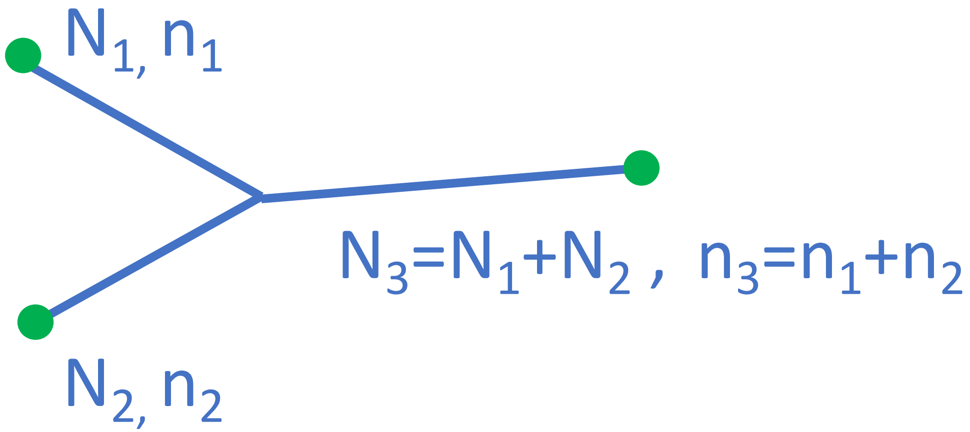
An extra step that helps the computation: Make them end on D3 branes. (still preserving SUSY).



Use a computation that Ashoke Sen did for the index.

(Related to dyons in SU(3) SYM).

Sen



$$Tr[J^6(-1)^F] = \pm(N_1 n_2 - N_2 n_1)$$

Fermion zero modes

Sen

The expected three point amplitude

To describe the multiplet it is convenient to use four dimensional notation.

$$\mathcal{A}_3 \sim \sqrt{G_{N,11}}\delta(\sum p_-^i)\delta(\sum p_+^i)\delta^9(\sum \vec{p}^i)\delta^{16}(\lambda_\alpha^i \bar{\eta}_I^i)\frac{1}{(\langle 1,2\rangle\langle 2,3\rangle\langle 1,3\rangle)^2}$$

Two simple changes:

1) Express the amplitude starting from the polarization state created by the D3 branes.

2) Set the momenta equal to the N, n values.

After a bunch of redefinitions of the η variables we get the two and three point amplitudes

$$\hat{\mathcal{A}}_2^c \sim \delta^8(\vec{p}+\vec{p}')\delta^4(\bar{\eta}_J+\bar{\eta}'_J)\delta^4(\eta^K-\eta'^K)\,, \qquad N=N'\,, \qquad n=n'$$

$$\begin{aligned}\hat{\mathcal{A}}_3^c &\sim \delta^8(\sum\vec{p}^i)\delta^4(\bar{\eta}_J^{cm}+\bar{\eta}_J^3)\delta^4(\eta^{cmK}-\eta^{3K})\times \\ &\quad \times l_p^{9/2}\frac{R_-}{R_9^{5/2}}\frac{(n_1N_2-n_2N_1)^2N_3^2}{(N_1N_2)^2\sqrt{N_1N_2N_3}}\delta(\sum p_+^i)\delta^4(\bar{\eta}_J^r)\,.\end{aligned}$$

(normalization)

First line comes from the U(1) in $U(N_3)$.

The second line should come from $SU(N_3)$. $N_3=N_1+N_2$

We indeed reproduce this amplitude from the index computation.

The extra factors come from carefully relating the index to the amplitude.

We could now take the large N and n limits and recover the amplitudes in completely un-compactified eleven dimensions.

Conclusions so far

- We obtained precise agreement with the expected amplitude, even at finite N, n.
- It is a simple test of the BFSS conjecture.

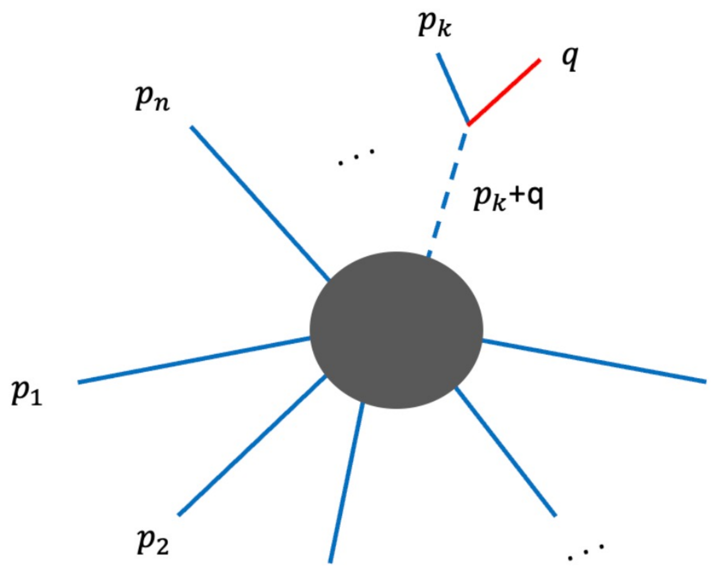
Soft factors and the three point amplitudes

Previous related discussions: Miller, Strominger, Tropper, Wang; Tropper, Wang

Soft factors in gravity amplitudes

$$\mathcal{A}_{n+1}(p_1,\cdots p_n,q) \sim (\underset{\uparrow \atop \frac{1}{q}}{S_0} + \underset{\uparrow \atop q^0}{S_1})\mathcal{A}_n(p_1,\cdots,p_n) + \cdots\ , \qquad \text{as} \quad q \to 0$$

They involve the three point amplitude



Why should be have such a simple behavior?

Look at the theory at long distances. The n point amplitude is like an n-point vertex.

The soft factors is putting this vertex in a curved background. → covariantizing the interaction.

We should be able to consistently couple to a slowly changing background gravitational field.

Consistency of the soft factors implies Poincare symmetry: Translations + Lorentz

This is particularly important when we think about the matrix model because the results from the matrix model are not obviously Lorentz invariant!

We will make a series of assumptions and derive the soft factors → Lorentz symmetry.

Assumptions

- 1) Amplitudes are suitably analytic.
- 2) When the soft q and one external line add up to zero the amplitude factorizes

$$\mathcal{A}_{n+1} \sim \mathcal{A}_3 \frac{1}{(p_i+q)^2} \mathcal{A}_n \ , \qquad \text{for} \quad (p_i+q)^2 \to 0$$

- 3) The three point function is the standard relativistic one.
- 4) All other singularities, branch cuts, etc, are subleading as a function of q. (they give higher powers of q as q → 0).

We do not assume that the amplitude is Lorentz invariant.

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We expect that these properties are true in the matrix model.

1) Is an assumption.
2) Follows from scattering properties in the matrix model and the existence of the on-shell intermediate state
3) Explicit computation of the 3-pt amplitude.
4) Follows from phase space volume considerations for multiparticle states.

We will now argue for the soft theorems for any theory obeying the above assumptions.

We use a contour deformation argument similar to the one used by BCFW to find the expression for the soft factor.

Britto, Cachazo, Feng, Witten

Arkani-Hamed, Cachazo, Kaplan

Difference: We will not pull the contour to infinity.

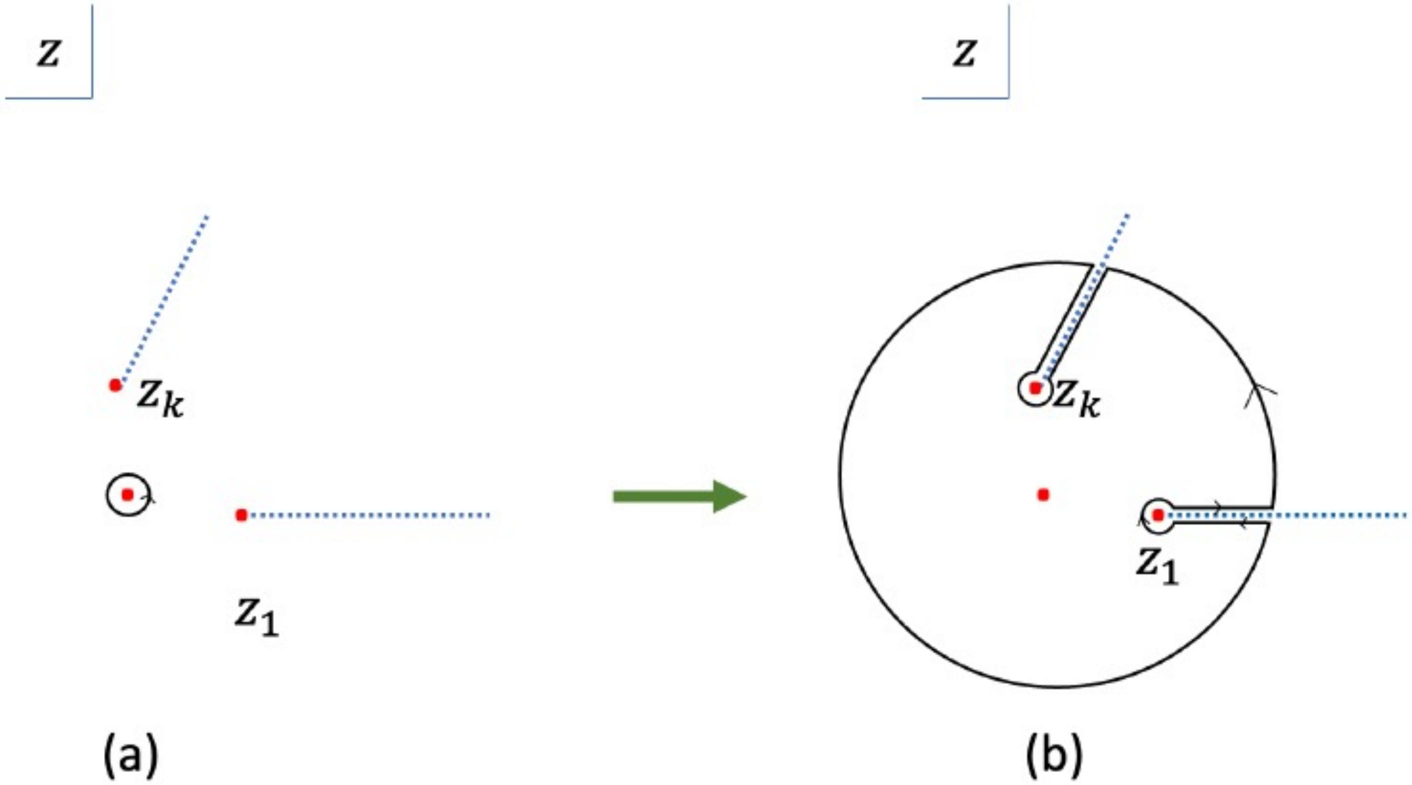
We introduce a complex deformation of the kinematics of the amplitude, z=0 is the original amplitude.

Two special momenta: q_{soft} , p_n

$$\mathcal{A}_{n+1} = \frac{1}{2\pi i}\oint \frac{dz}{z}\mathcal{A}_{n+1}(z)$$

We used four dimensional kinematics. But it should be possible to extend it to the 11 dimensional one.

Deform the contour



We only pick up poles near the origin, which appear when q → 0.

The contour is shifted to a finite position, not to infinity.

Except for the poles, the rest of the contour has a subleading behavior in the limit q → 0.

Picking up the residues and expanding for small q, we find

$$\mathcal{A}_{n+1} = S_0 \mathcal{A}_n + S_1 \mathcal{A}_n + \text{subleading}$$

With

$$S_0 \mathcal{A}_n \propto \sum_k \frac{\langle n,k\rangle^2}{\langle n,s\rangle^2} \frac{[s,k]}{\langle s,k\rangle} A_n(0)$$

$$S_1 \mathcal{A}_n = \frac{1}{2} \sum_k \frac{\langle n,k\rangle}{\langle n,s\rangle} \frac{[s,k]}{\langle s,k\rangle} \bar{\lambda}^s_{\dot\beta} \frac{\partial \mathcal{A}_n}{\partial \bar{\lambda}^k_{\dot\beta}}$$

There is only one aspect of this that is important for us:

These expressions are selecting one of the particles, the particle n in this case. We could repeat this argument with other particles, say m, or m'.

$$S_0 \mathcal{A}_n \propto \sum_k \frac{\langle n, k \rangle^2}{\langle n, s \rangle^2} \frac{[s, k]}{\langle s, k \rangle} \mathcal{A}_n(0)$$

$$S_1 \mathcal{A}_n = \frac{1}{2} \sum_k \frac{\langle n, k \rangle}{\langle n, s \rangle} \frac{[s, k]}{\langle s, k \rangle} \bar{\lambda}^s_{\dot{\beta}} \frac{\partial \mathcal{A}_n}{\partial \bar{\lambda}^k_{\dot{\beta}}}$$

Demanding that the answer is independent of the ``special'' particle we get a constraint

Leading soft factor → energy momentum conservation for the n-point amplitude.

Subleading soft factor → Lorentz invariance of the n-point amplitude.

Tropper, Wang

Conclusion

- We have argued for the soft limits using some assumptions that we think are true in the matrix model.
- We found that consistency implies Lorentz invariance.

Conclusions

- We reviewed the BFSS conjecture for the non-perturbative S-matrix.
- We discussed the simplest amplitude: the 3pt amplitude.
- We discussed an application to soft factors and Lorentz invariance (which depended on extra assumptions).
- Hopefully, in the future one could compute more interesting amplitudes…

Thank you