THE 5-MASS KITE INTEGRAL FAMILY ON TWO TORI

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Speakers include:

Samuel Abreu	Mikhail Ivanov
Zvi Bern	Diksha Jain
Freddy Cachazo	Renata Kallosh
Lucile Cangemi	Hayden Lee
Francois Charton	Juan Maldacena
Kevin Costello	Andrew McLeod
Stefano De Angelis	Ian Moult
Federica Devoto	Shruti Paranjape
Carolina Figueiredo	Franziska Porkert
Mathieu Giroux	Yael Shadmi
Tobias Hansen	Stephen Sharpe
Yifei He	Marcus Spradlin
Johannes Henn	Bernd Sturmfels
Aidan Herderschee	Lorenzo Tancredi
Mina Himwich	Xiaofeng Xu
	Zahra Zahraee

Organizers:
Nima Arkani-Hamed, Jacob Bourjaily,
Hofie Hannesdottir, Sebastian Mizera

Image credits: Gaia Fontana

Carl P. Feinberg Program in Cross-Disciplinary Innovation

IAS INSTITUTE FOR ADVANCED STUDY

Franziska Porkert

based on 2401.14307 with
Mathieu Giroux, Andrzej Pokraka and Yoann Sohnle

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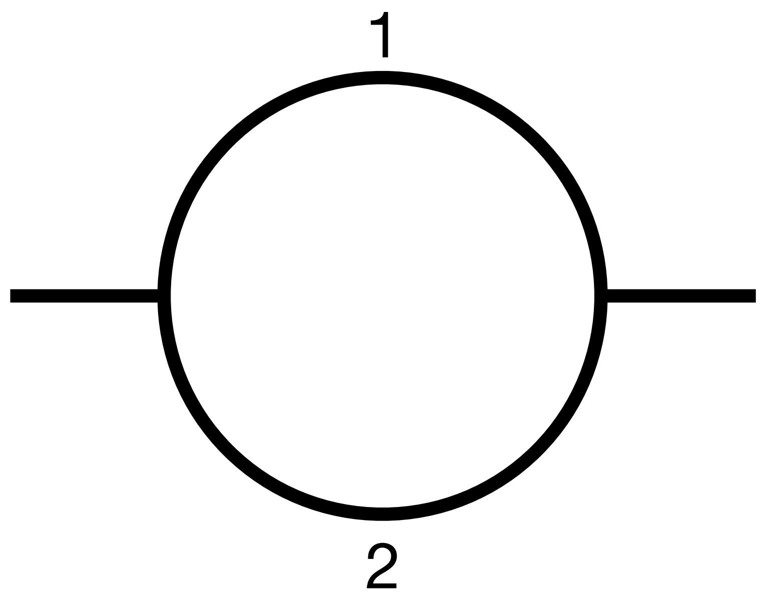
MOTIVATION

The kite integral family is the most general 2-loop 2-point Feynman integral family.

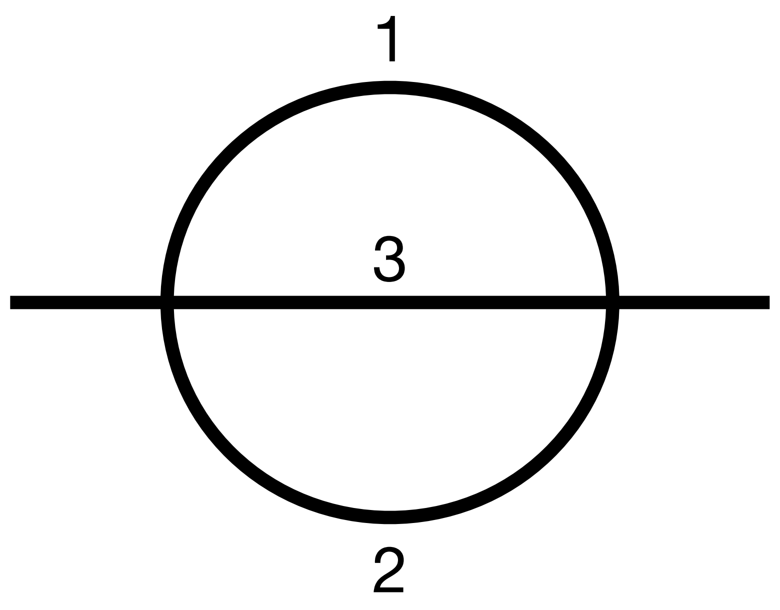
1 Loop: Bubble



elliptics

Sunrise: Solved

[Bogner, Müller-Stach, Weinzierl;19]

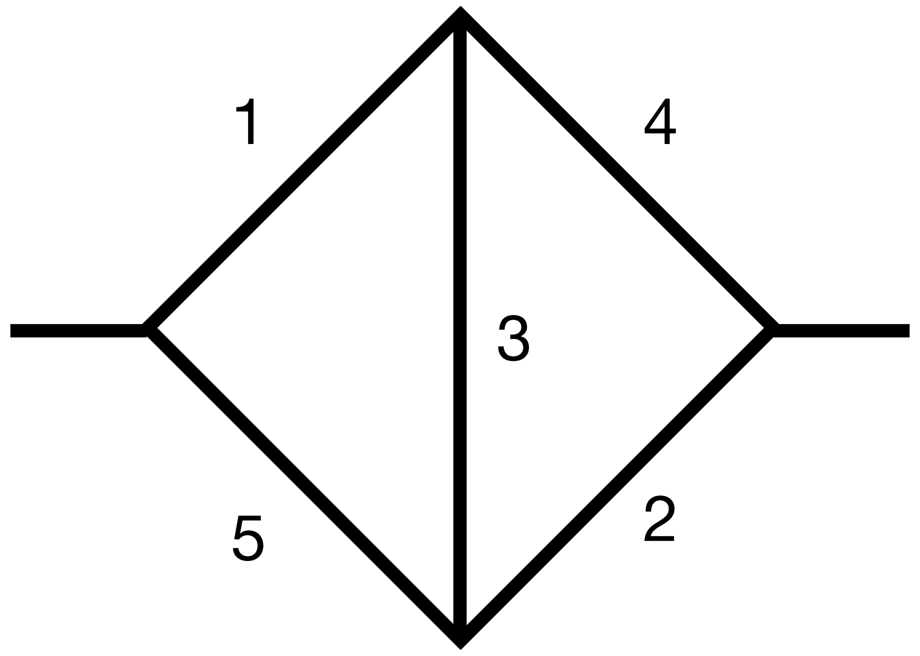


5 irreducible scalar products:
$\{\ell_1^2,\, \ell_2^2,\, \ell_1 \cdot \ell_2,\, \ell_1 \cdot p,\, \ell_2 \cdot p\}$

more
elliptics

2 Loop: Kite

[Giroux, Pokraka, FP, Sohnle; 24]



5 propagators in
the most generic case

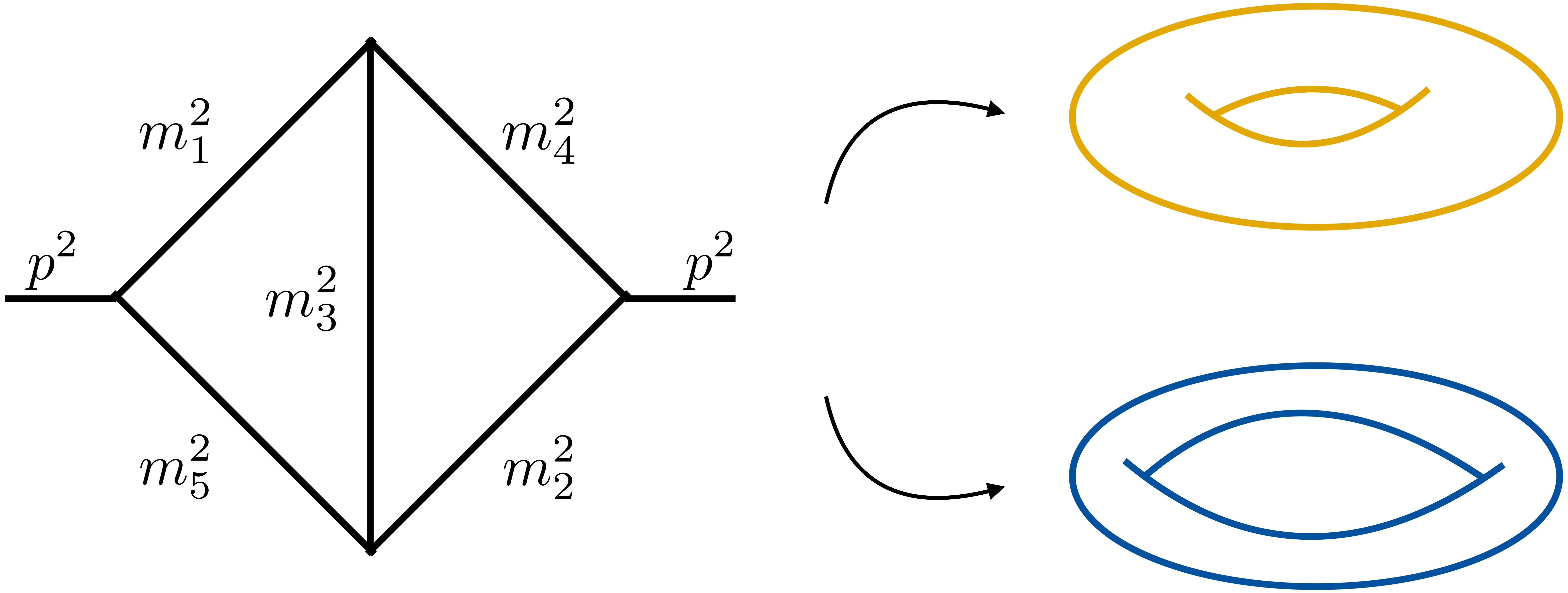
We analytically **solve all scalar 2-loop 2-point** Feynman integrals!

MOTIVATION

Tori & higher genus/dimensional geometries appear in multi-loop Feynman integrals. **Talk by Lorenzo**

But: Only few results with **multiple tori** or **many kinematic parameters on the torus** so far.

[Adams, Chaubey, Weinzierl;18 | Müller, Weinzierl;22 | Duhr, Klemm, Nega, Tancredi;22 | Görgen, Nega, Tancredi, Wagner; 23 **...**]

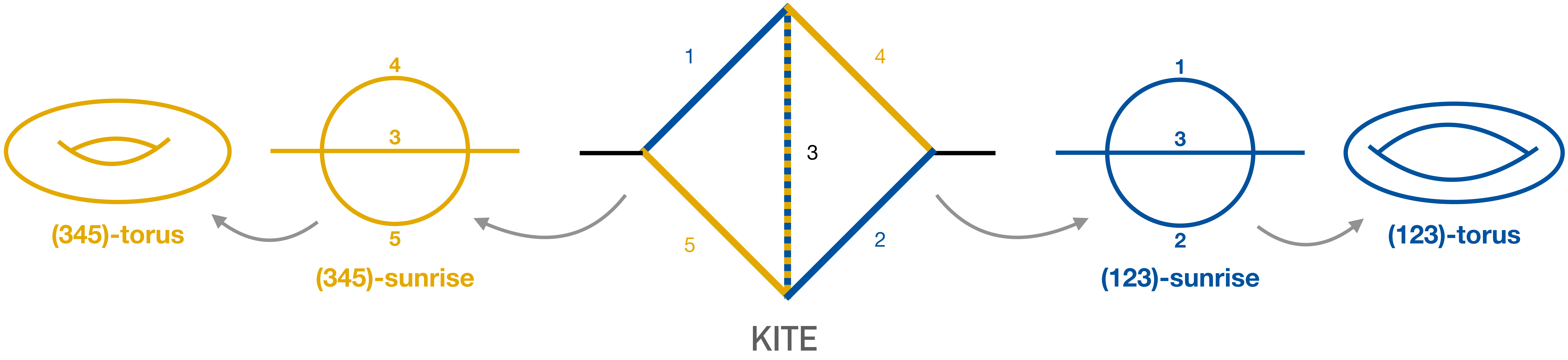


Kite integral: Two tori & five kinematic parameters

THE KITE INTEGRAL FAMILY

Goal: Solve the kite integral family analytically in $d = 2 - 2\varepsilon$.

$$I_\nu = \frac{e^{2\gamma_E\varepsilon}\mu^{2(|\nu|-d)}}{(i\pi)^d}\int\frac{\mathrm{d}^d\ell_1\mathrm{d}^d\ell_2}{D_1^{\nu_1}D_2^{\nu_2}D_3^{\nu_3}D_4^{\nu_4}D_5^{\nu_5}} \qquad \begin{array}{lll} D_1 = -\ell_1^2 + \boxed{m_1^2} & D_2 = -(\ell_2 - p)^2 + \boxed{m_2^2} & D_3 = -(\ell_1 - \ell_2)^2 + \boxed{m_3^2} \\ D_4 = -\ell_2^2 + \boxed{m_4^2} & D_5 = -(\ell_1 - p)^2 + \boxed{m_5^2} & \end{array}$$



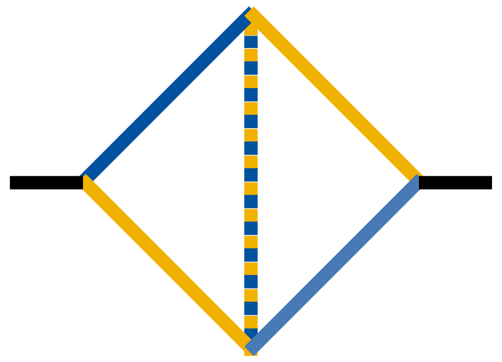
Solved with 3 masses/only one sunrise sub topology. [Adams, Bogner, Schweitzer, Weinzierl | Broedel, Duhr, Dulat, Penante, Tancredi]

THE BASIS OF MASTER INTEGRALS

The kite integral family has **30** master integrals (MIs).

Two elliptic curves (1 MI)

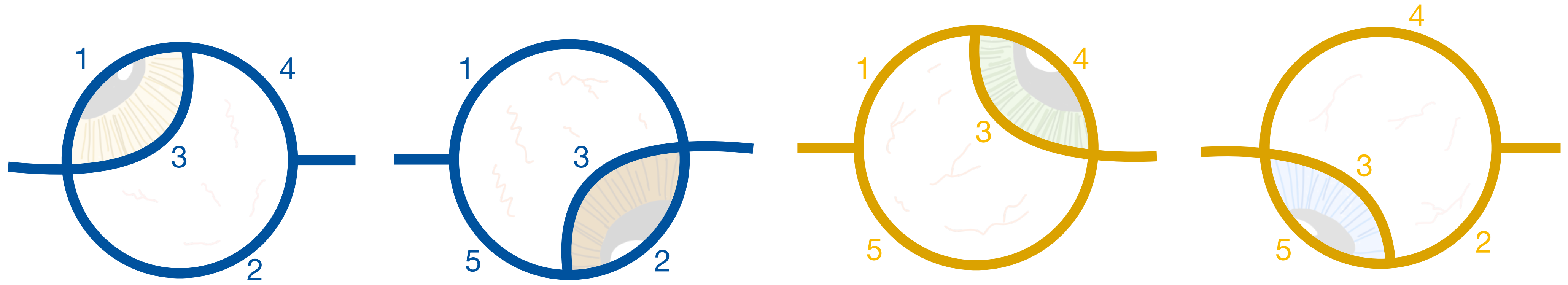
The top sector:



I_{11111}

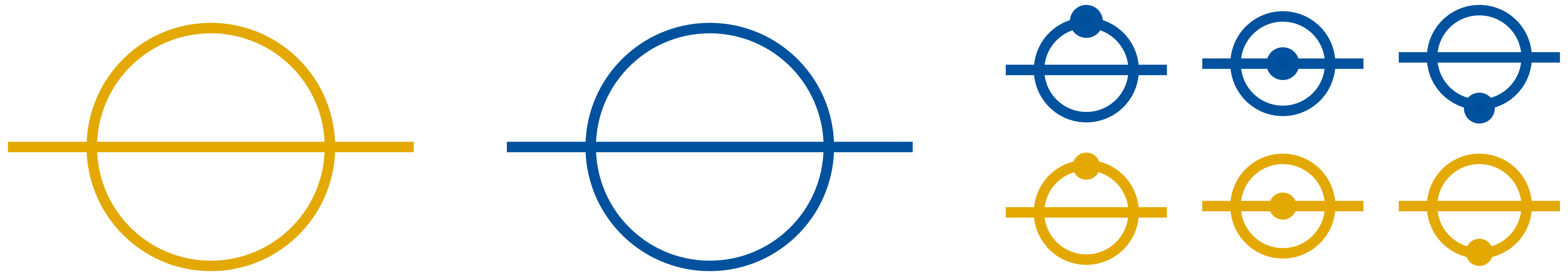
One elliptic curve (12 MI)

Four *eyeball* sub-topologies:

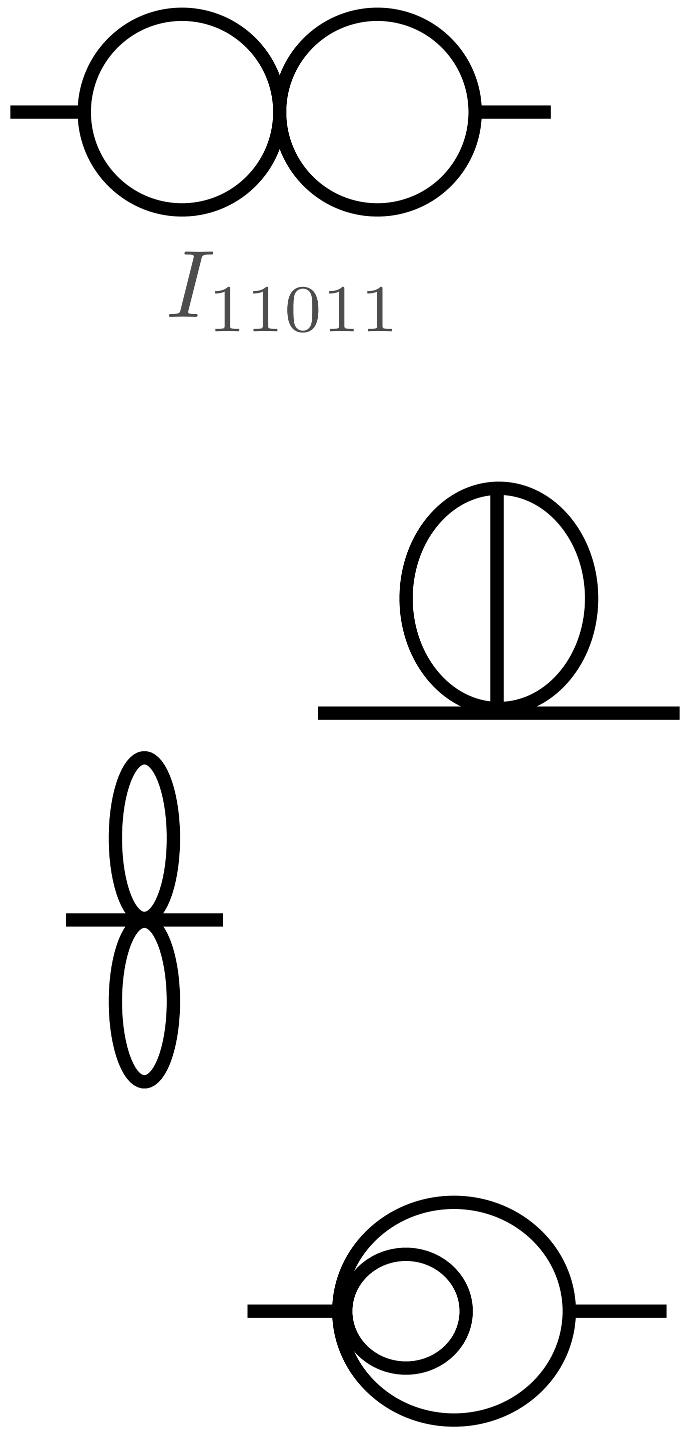


I_{11110} I_{11101} I_{10111} I_{01111}

The two sunrises:



Not elliptic (17 MI)



I_{11011}

METHOD OF DIFFERENTIAL EQUATIONS

We compute the kite integral family analytically with ***differential equations***.

- Set up a **differential equation** w.r.t the external (kinematic) parameters

$$\mathrm{d}\mathbf{I}(\mathbf{X}) = A(\mathbf{X},\varepsilon)\mathbf{I}(\mathbf{X}) \quad \text{with} \quad \mathrm{d} = \sum \mathrm{d}X_i \partial_{X_i} \quad \text{where} \quad X_0 = \frac{p^2}{m_3^2} \quad \text{and} \quad X_i = \frac{m_i^2}{m_3^2} \text{ for } i \in \{1,2,4,5\}$$

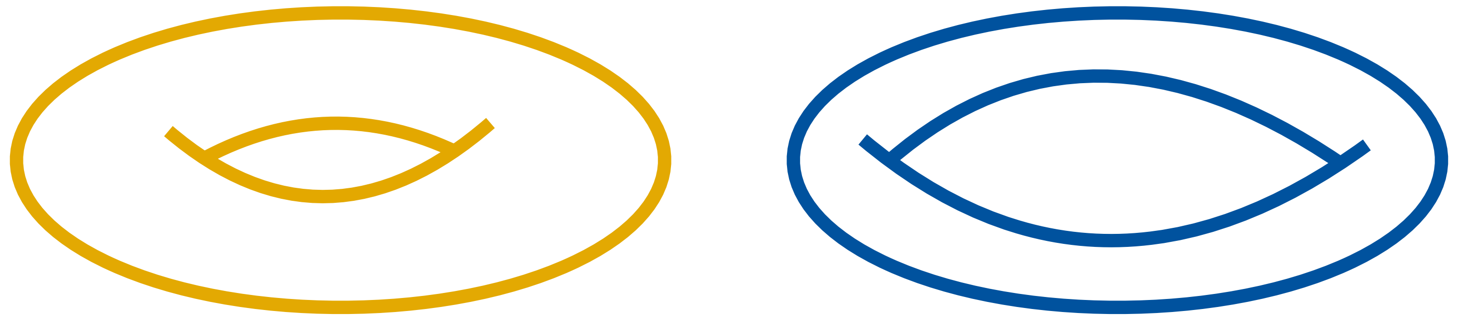
- Find an ε**-form differential equation** & solve in terms of **iterated integrals.** [Henn]

$$\mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X})$$

$$\mathbf{J}(\mathbf{X}) = \mathbb{P}\mathrm{exp}\left(\varepsilon \int_\gamma B\right) \cdot \mathbf{J}\left(\text{some point } \mathbf{X}^0\right) = \left(1 + \varepsilon \int_\gamma B + \varepsilon^2 \int_\gamma B \int_\gamma B + \dots\right) \cdot \mathbf{J}(\mathbf{X}^0)$$

See talks by Lorenzo Tancredi, Samuel Abreu

1. Setup: Parametrization of the kinematic space on the two tori



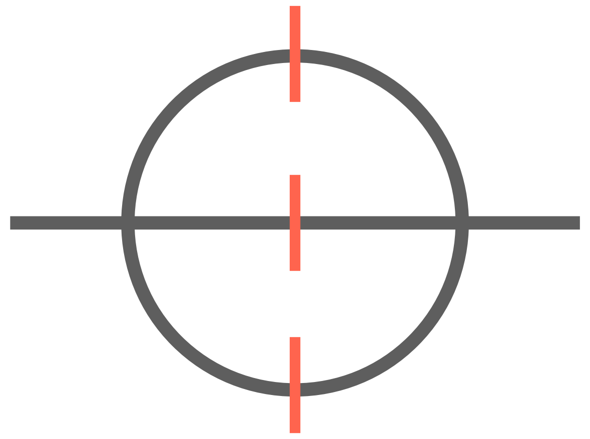
2. Transformation: Finding an $\mathcal{E}$-form differential equation (on the tori)

$$\mathbf{dJ}(\underline{X}) = \varepsilon B(\underline{X})\mathbf{J}(\underline{X})$$

3. Solution: The singularity structure and iterated integrals on the tori

$$I_\gamma(\omega_1, \ldots, \omega_k; \lambda) = \int_0^\lambda \mathrm{d}\lambda_1 f_1(\lambda)_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 f_2(\lambda_2) \cdots \int_0^{\lambda_{k-1}} \mathrm{d}\lambda_k f_k(\lambda_k) \text{ with } f_j(\lambda)\mathrm{d}\lambda = \gamma^* \omega_j$$

THE SUNRISE ELLIPTIC CURVE

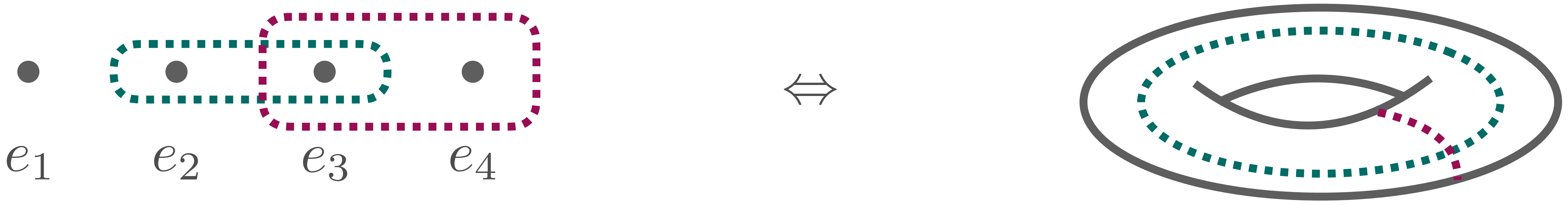


$$\sim \int_{\mathcal{C}} \frac{\mathrm{d}x}{\sqrt{(x-e_1)(x-e_2)(x-e_3)(x-e_4)}}$$

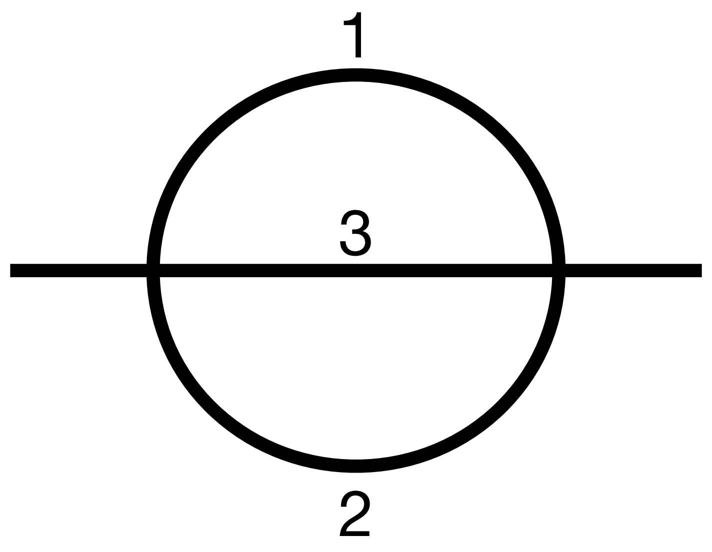
Rational functions in the kinematics

The **massive sunrise** defines an **elliptic curve** $y^2=(x-e_1)(x-e_2)(x-e_3)(x-e_4)$ isomorphic to a **torus**.

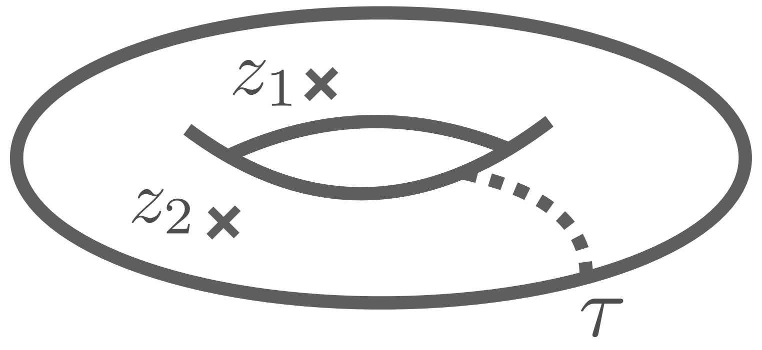
[Laporta, Remiddi;04] [Bloch, Vanhove,13] [Bogner, Weinzierl, Müller-Stach;19] …



SUNRISE PARAMETERS ON THE TORUS



$$\xrightarrow{\qquad\qquad\qquad} \quad y^2 = (x-e_1)(x-e_2)(x-e_3)(x-e_4)$$



PERIODS: $\psi_1 = 2\int_{e_2}^{e_3} \frac{\mathrm{d}x}{y} \sim K(k^2)$ and $\psi_2 = 2\int_{e_4}^{e_3} \frac{\mathrm{d}x}{y} \sim K(1-k^2)$ with $K(k^2) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}$

Ratios of polynomials in the kinematic parameters

$$\longrightarrow \boxed{\tau = \frac{\psi_2}{\psi_1}}$$

PUNCTURES VIA ABEL'S MAP: Point on the elliptic curve $\longmapsto$ point on the torus: $(x,y) \mapsto z^\pm = \pm\frac{1}{\psi_1}\int_{e_1}^{x} \frac{\mathrm{d}t}{y(t)}$

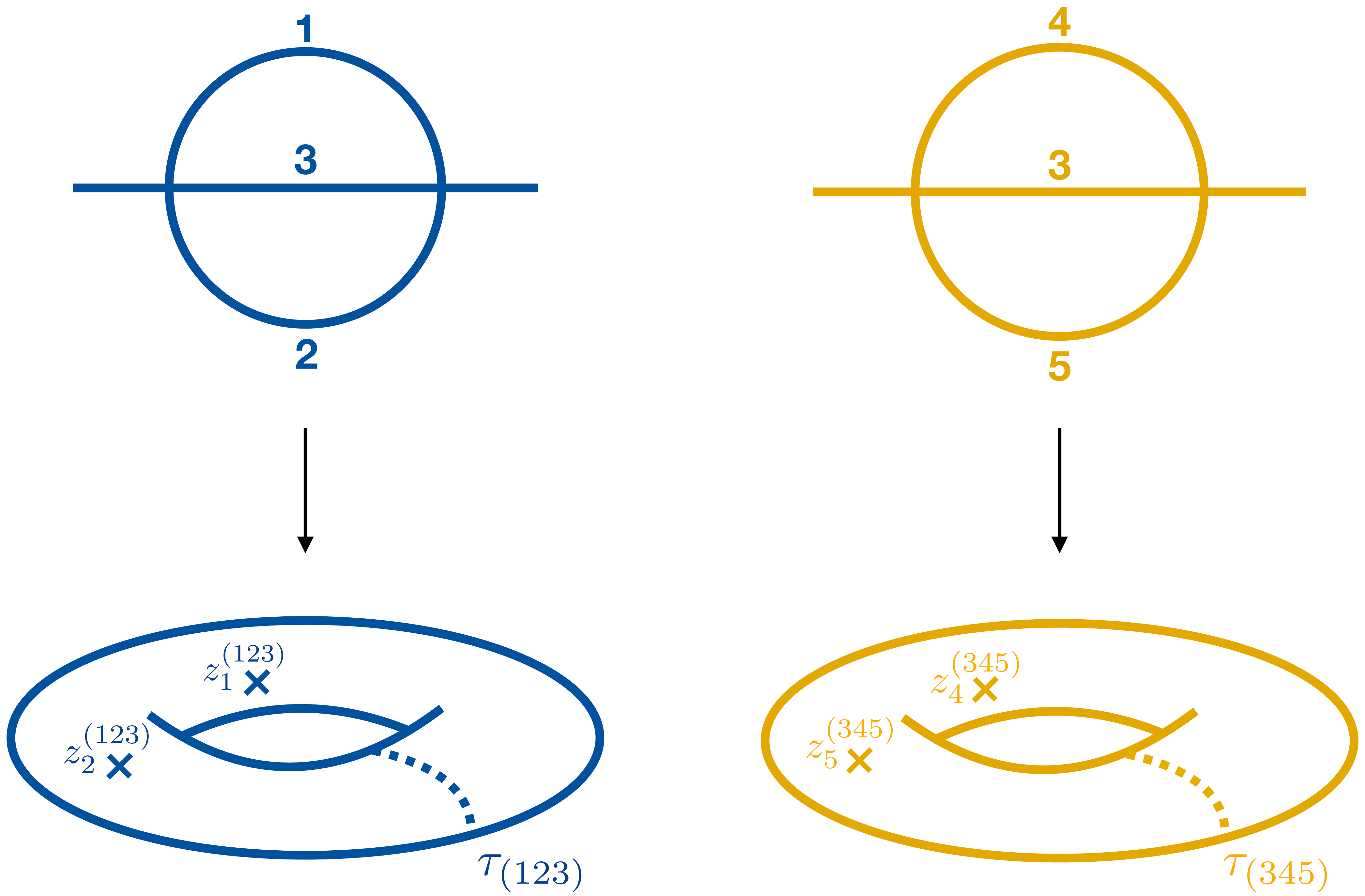
Ratios of polynomials in the kinematic parameters

$$\longrightarrow \boxed{z_i \sim \frac{F(\sqrt{u_i}, k^2)}{2K(k^2)}}$$

[Bogner, Weinzierl, Müller-Stach;19]

with $F(\sqrt{u}, k^2) = \int_0^{\sqrt{u}} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}$

SUNRISE PARAMETERS ON THE TORI

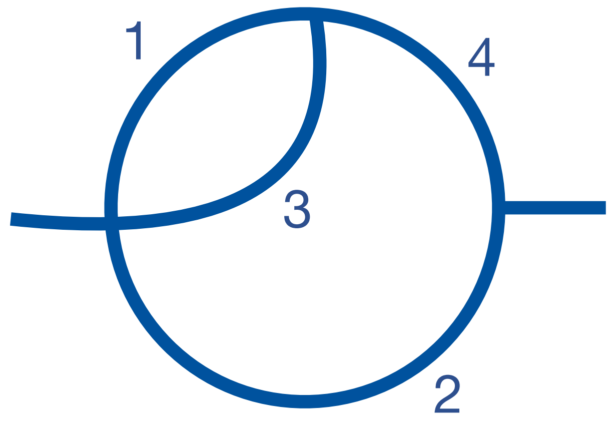


The sunrises can be parametrized on the two tori by

$$\{\tau_{(123)}, z_1^{(123)}, z_2^{(123)}\} \text{ and } \{\tau_{(345)}, z_4^{(345)}, z_5^{(345)}\}$$

Next: Full kinematic space of the kite on **both** tori!

THE 1234 EYEBALL ON THE 123 TORUS



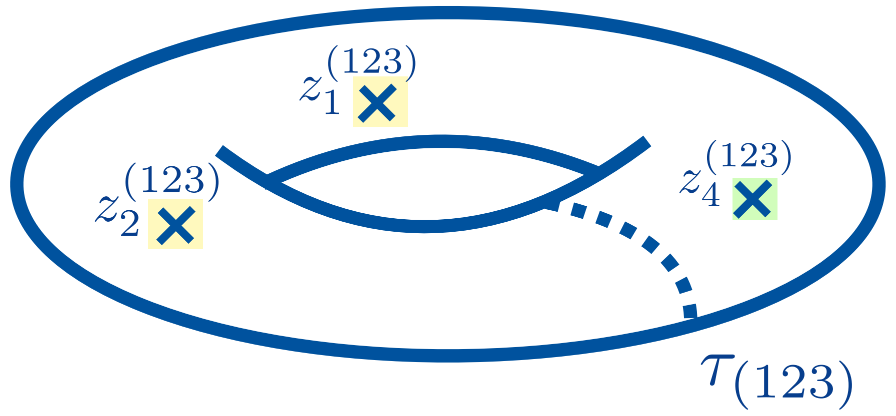
1234-eyeball

- 4 masses + p : 4 parameters
- **(123)** - torus

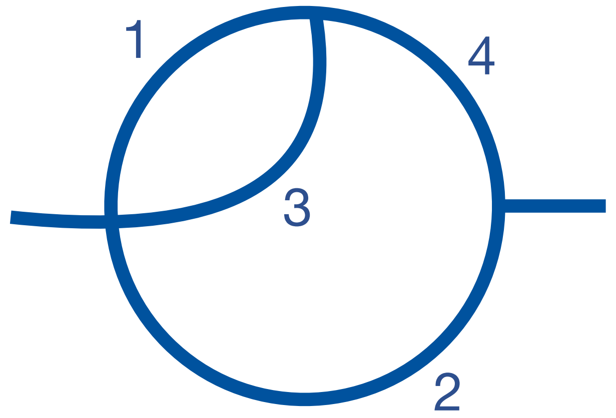
→

Period $\tau_{(123)}$

+ 3 punctures $z_1^{(123)}$, $z_2^{(123)}$ & $z_4^{(123)}$



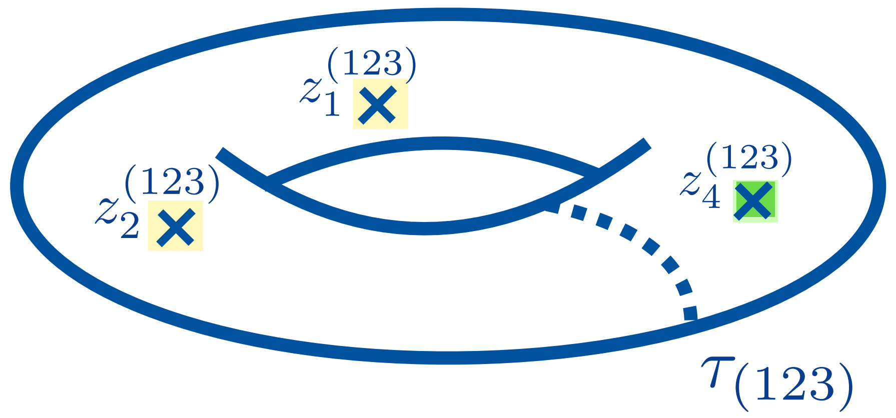
THE 1234 EYEBALL ON THE 123 TORUS



- 4 masses + p : 4 parameters
- **(123)** - torus

→ Period $\tau_{(123)}$

+ 3 punctures $z_1^{(123)}$, $z_2^{(123)}$ & $z_4^{(123)}$

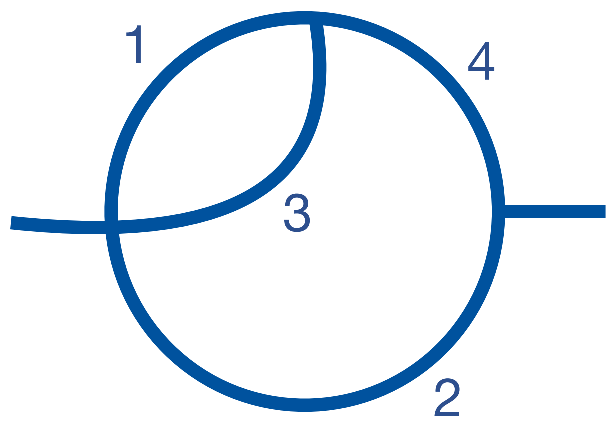


We can find this **new puncture** by integrating over the maximal cut in two dimensions:

$$\int \frac{\mathrm{d}X_4}{\underbrace{\sqrt{\lambda(1,X_1,X_4)\lambda(X_0,X_2,X_4)}}_{\text{(1234) - Eyeball MC @ 2d}}} = \frac{1}{2}\psi_1^{(123)}\left(\underbrace{\frac{F(u_4\,,k_2)}{K(\,k_2\,)}}_{z_4^{(123)}} - 1\right) \quad \text{with} \quad u_4^{(123)} = \frac{(\sqrt{X_0}+\sqrt{X_2})^2-(\sqrt{X_1}-1)^2}{4\sqrt{X_1}}\,\frac{(1+\sqrt{X_1})^2-X_4}{(\sqrt{X_0}+\sqrt{X_2})^2-X_4}$$

Källen function $\lambda(a,b,c) = a^3+b^3+c^3+2ab+2ac+2bc$

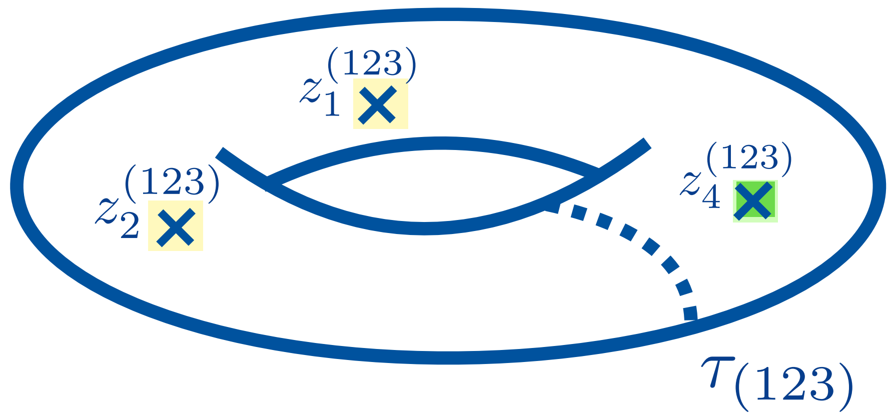
THE 1234 EYEBALL ON THE 123 TORUS



- 4 masses + p : 4 parameters
- **(123) -** torus

→ Period $\tau_{(123)}$

+ 3 punctures $z_1^{(123)}$, $z_2^{(123)}$ & $z_4^{(123)}$



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Källen function $\lambda(a,b,c)=a^3+b^3+c^3+2ab+2ac+2bc$

We can also find the **sunrise punctures** from the maximal cut via limits:

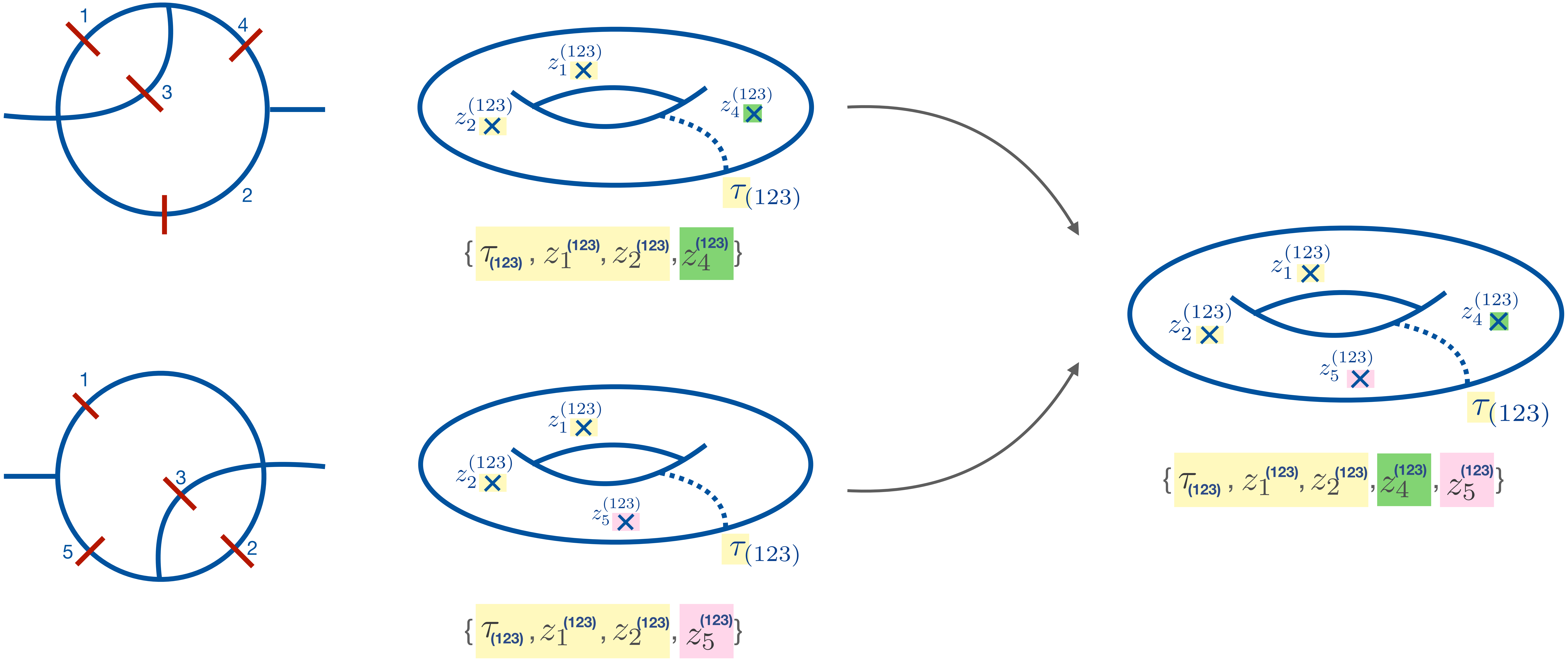
$z_2^{(123)}$ from $z_4^{(123)}$ by $u_2^{(123)} = u_4^{(123)}|_{m_4^2\to\infty}$

$z_1^{(123)}$ from $z_2^{(123)}$ by $u_1^{(123)} = u_2^{(123)}|_{m_1^2\leftrightarrow m_2^2}$

All parameters for the 1234 eyeball from a simple integral over its maximal cut in 2D.

THE KITE ON THE 123 TORUS

We can find *all kite punctures* on the **123-torus** from the *eyeball maximal cuts*.



$$\{\tau_{(123)}, z_1^{(123)}, z_2^{(123)}, z_4^{(123)}\}$$

$$\{\tau_{(123)}, z_1^{(123)}, z_2^{(123)}, z_5^{(123)}\}$$

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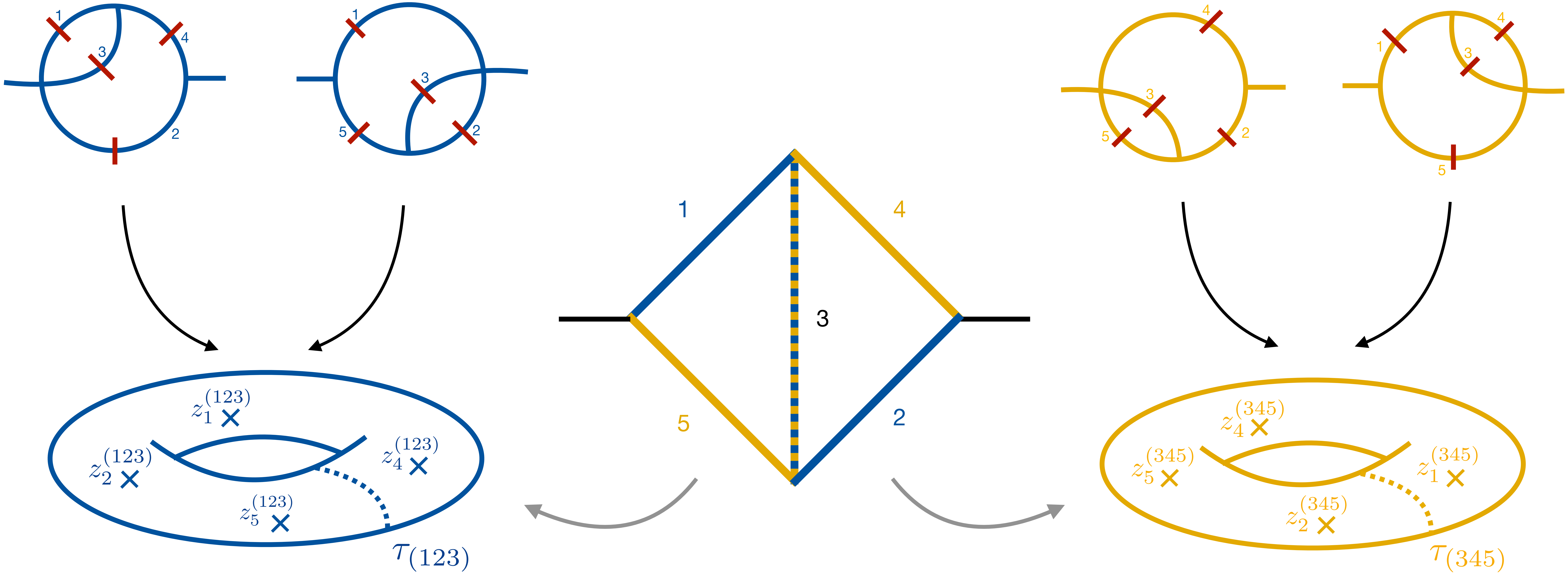
THE KITE ON THE TWO TORI



$\{\tau_{(123)}, z_1^{(123)}, z_2^{(123)}, z_4^{(123)}, z_5^{(123)}\}$

$\{\tau_{(345)}, z_1^{(345)}, z_2^{(345)}, z_4^{(345)}, z_5^{(345)}\}$

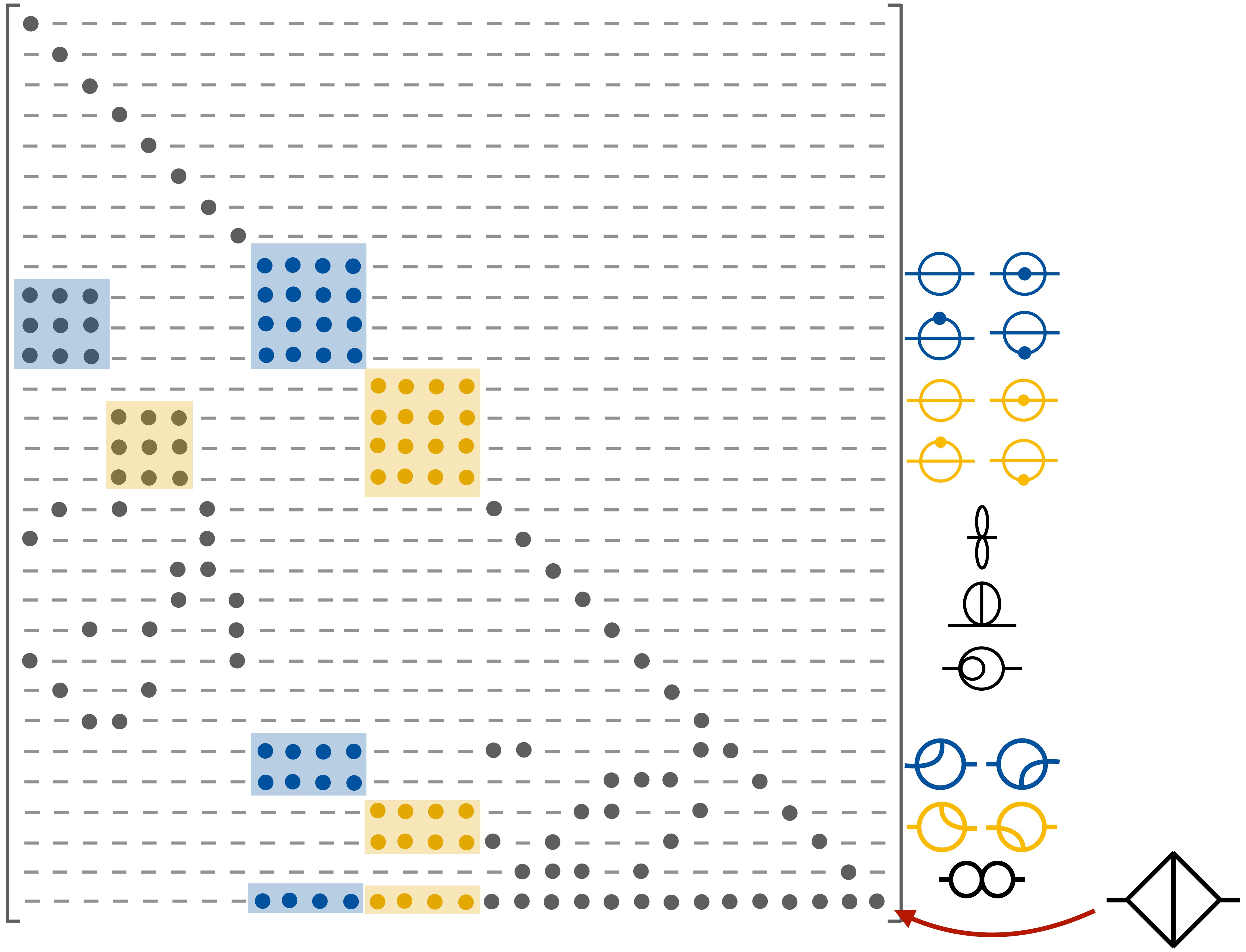
All parameters for the full kite on both tori from integrals over eyeball maximal cuts in 2D

1. Setup: Parametrization of the kinematic space on the two tori

2. Transformation: Finding an ε-form differential equation (on the tori)

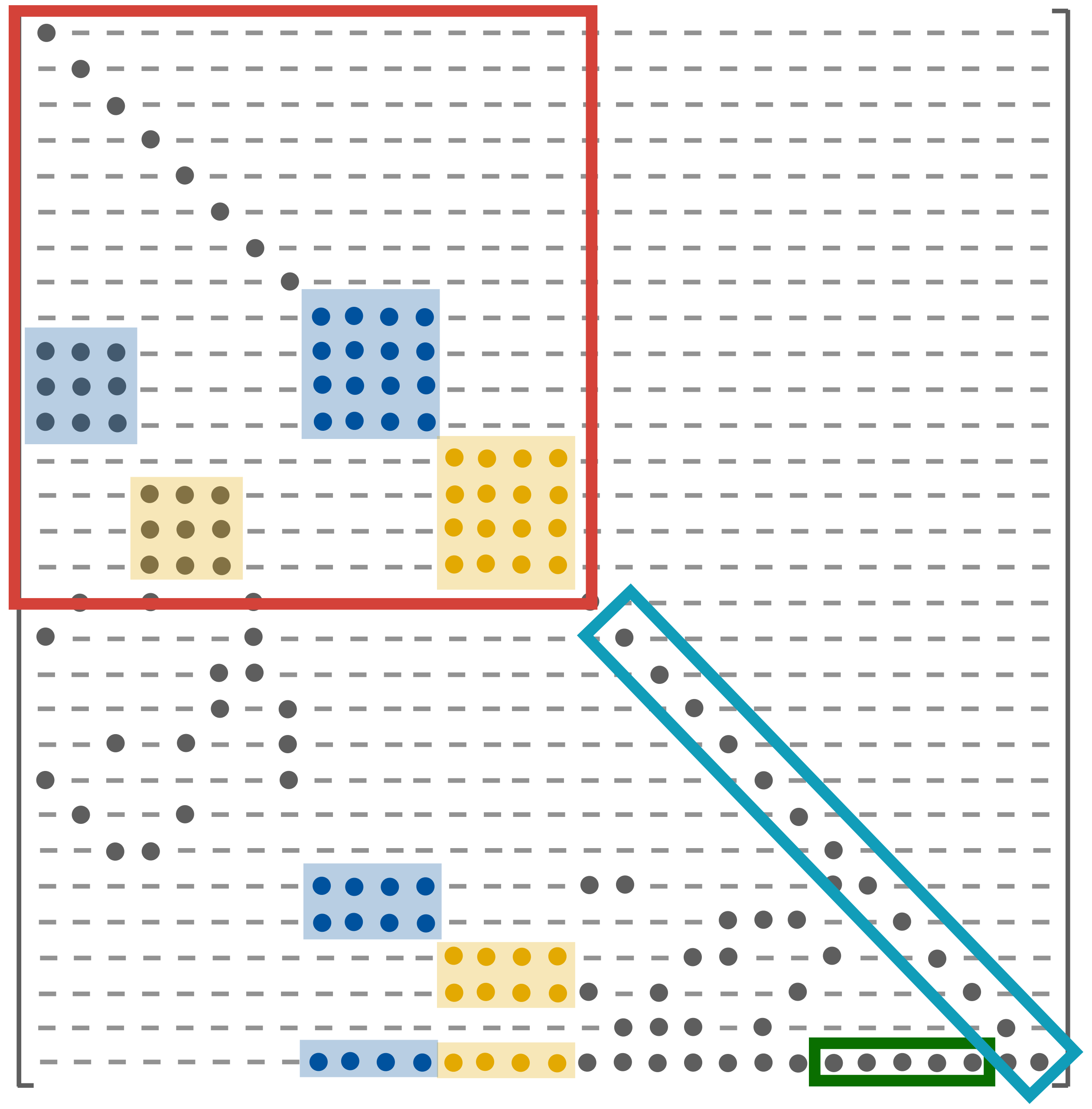
Initial: $\mathrm{d}\mathbf{I}(\mathbf{X}) = \mathbf{A}(\mathbf{X},\varepsilon)\mathbf{I}(\mathbf{X})$ $\longrightarrow$ Goal: $\mathbf{J} = \mathbf{U}\mathbf{I}$ with $\mathrm{d}\mathbf{J} = \varepsilon\mathbf{B}(\mathbf{X})\mathbf{J}$ & $\varepsilon\mathbf{B} = \mathrm{d}\mathbf{U}\mathbf{U}^{-1} + \mathbf{U}\mathbf{A}\mathbf{U}^{-1}$

$$A(\mathbf{X},\varepsilon) =$$



3. Solution: The singularity structure and iterated integrals on the tori

TOWARDS ε-FORM: STEP 0-2



Step 0: Sunrise sectors in ε-form [Bogner, Müller-Stach,Weinzierl;19]

Introduces $\partial_0\psi_1^{(ijk)}$ and $\psi_1^{(ijk)}$ with (ijk) $\in$ { **(123)** , **(345)** }

Step 1: Diagonale in ε-form
(Leading singularities / Division by maximal cuts)

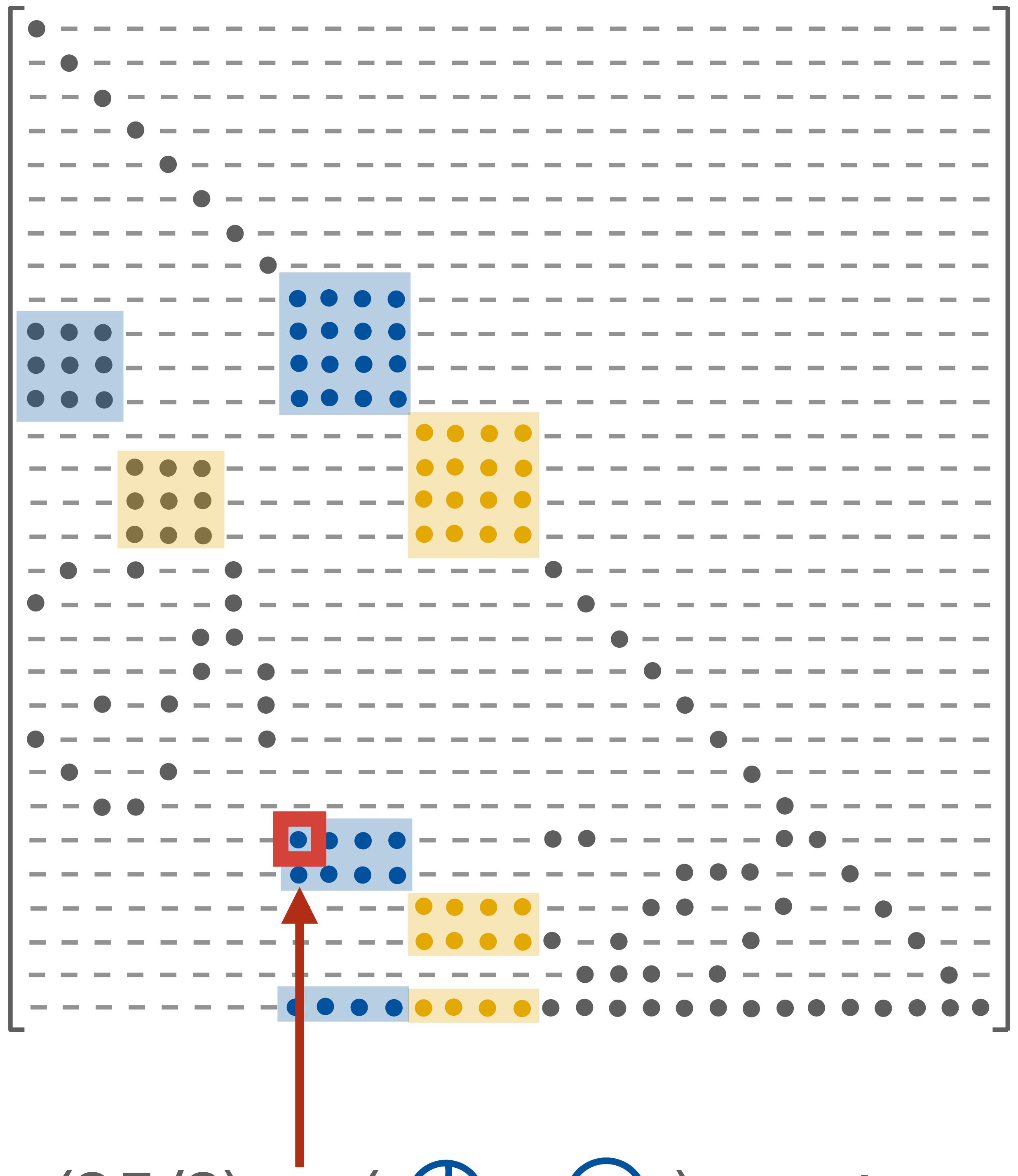
Step 2: Remove the remaining non-elliptic ε^0- terms.

New differential equation:

$$\mathrm{d}\mathbf{J}_2(\mathbf{X}) = B_2(\varepsilon, \mathbf{X})\mathbf{J}_2(\mathbf{X})$$

TOWARDS ε -FORM: STEP 3

Remove the ε^0- terms in the (,) sector via $\mathbf{U}_3$.



(25/9) or (,) - entry

Example: (,) - Entry

Make an ansatz for $\mathbf{U}_3$ & require: $\left(\mathrm{d}\mathbf{U}_3\mathbf{U}_3^{-1} + \mathbf{U}_3\mathbf{B}_2\mathbf{U}_3^{-1}\right)_{25,9} \sim \varepsilon$

$$\big\downarrow$$

$$(U_3)_{25,9} = -\int_0^{X_4} \mathrm{d}X_4' \,(B_2)_{25,9}\,|_{\varepsilon\to 0}$$

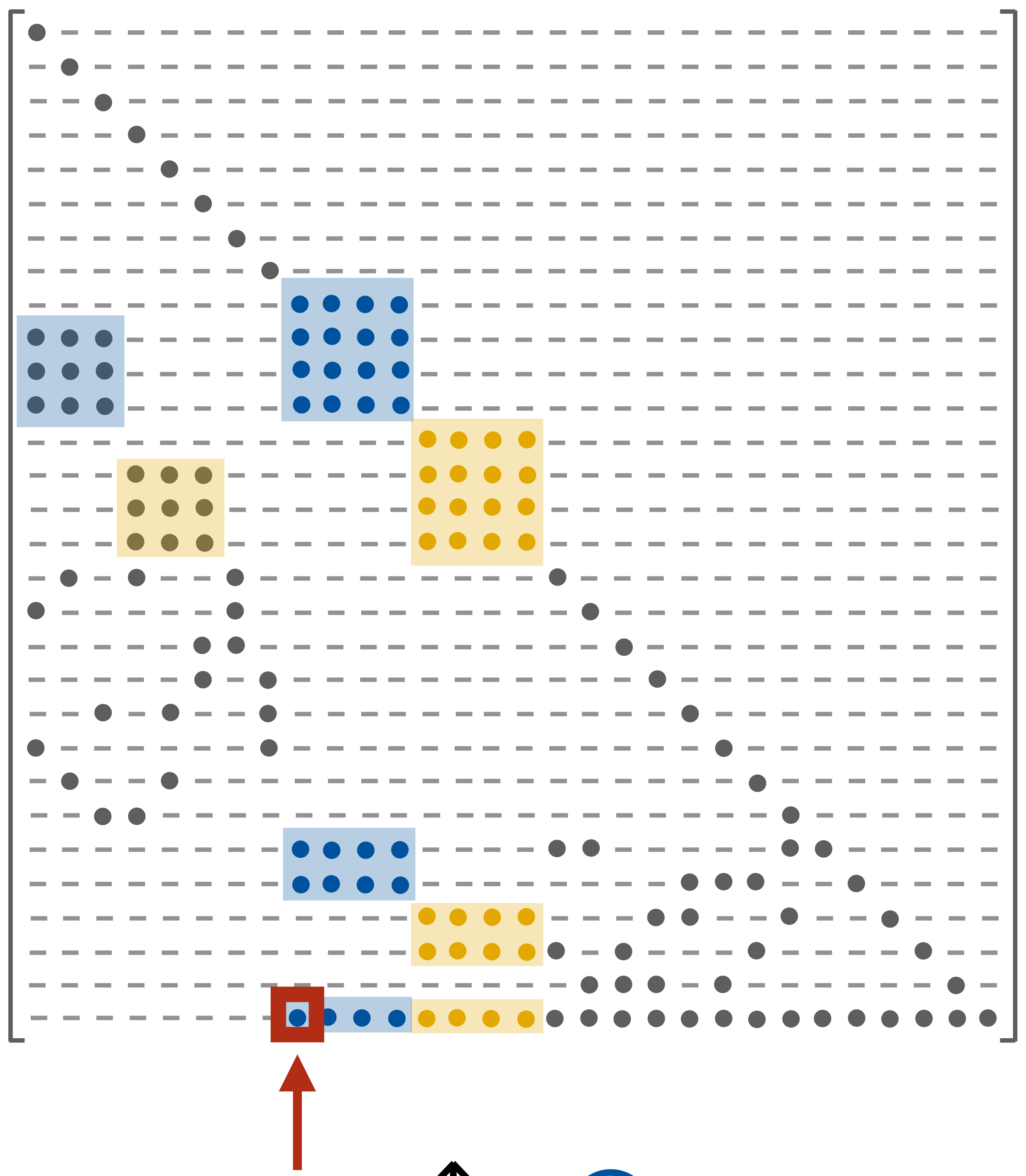
contains puncture:

$$\boxed{\frac{F(u_4\,,k_2)}{K(k_2)}}$$

All punctures for the kite on the two tori also appear naturally in the ε-form diff. equation.

TOWARDS ε -FORM: STEP 4

Remove the ε^0- terms in the (◇, ⊖) sector via $\mathbf{U}_4$.



(30/9) or (◇, ⊖) - entry

Example: (◇, ⊖) - entry

Ansatz for $\mathbf{U}_4$ with non-trivial entry υ & require ε-form after transformation

$$\Rightarrow \quad -\tilde{B} = \mathrm{d}\upsilon \qquad \text{with} \quad \tilde{B} = \sigma(X_i, \mathrm{d}X_i)\,\psi_1 + \rho(X_i, \mathrm{d}X_i)\,\partial_0\psi_1$$

Entry of the deq B_2

Entry of $\mathbf{U}_4$

Modular transformation:

$$(c\,\tau + d)\underbrace{\left(\mathrm{d}\upsilon + \tilde{B}\right)}_{0} + c\boxed{(\upsilon\,\mathrm{d}\tau + \rho\,\psi_1\,\partial_0\tau\,)} = 0$$

$$\Rightarrow \upsilon = -\psi_1\partial_0\tau \sum \rho|_{\mathrm{d}X_j} \frac{\partial X_j}{\partial \tau}$$

Compute from the inverse of the Jacobian

→ Need **all** parameters (i.e. punctures) on the torus

Work on the tori to find the transformation but this requires full parametrization/all punctures.

1. Setup: Parametrization of the kinematic space on the two tori

2. Transformation: Finding an ε-form differential equation (on the tori)

$$\mathrm{d}\mathbf{J} = \varepsilon B(\underline{X})\mathbf{J}$$

$$\mathbf{J}(\underline{X}) = \mathbb{P}\exp\left(\varepsilon\int_\gamma B\right)\cdot\mathbf{J}\left(\text{ some point } \underline{X}^0\right) = \left(1+\epsilon\int_\gamma B+\varepsilon^2\int_\gamma B\int_\gamma B+\dots\right)\mathbf{J}\left(\underline{X}^0\right)$$

3. Solution: The singularity structure and iterated integrals on the tori

$$I_\gamma(\omega_1,\dots,\omega_k;\lambda) = \int_0^\lambda \mathrm{d}\lambda_1 f_1(\lambda)_1 \int_0^{\lambda_1}\mathrm{d}\lambda_2 f_2(\lambda_2)\cdots\int_0^{\lambda_{k-1}}\mathrm{d}\lambda_k f_k(\lambda_k) \text{ with } f_j(\lambda)\mathrm{d}\lambda = \gamma^*\omega_j$$

g-KERNELS AND EMPLS

Formal solution for the kite: $\mathbf{J}(\underline{z},\underline{\tau}) = \left(1 + \varepsilon \int_\gamma B + \varepsilon^2 \int_\gamma B \int_\gamma B + \dots\right) \mathbf{J}\left(\underline{z}^0, \underline{\tau}^0\right)$

The natural integration kernels for iterated integrals on a torus are the **g-kernels** and we use their combinations to **Kronecker forms**

$$\omega_k^{\text{Kronecker}}(z,\tau) = (2\pi)^{2-k} \left(g^{(k-1)}(z,\tau)\mathrm{d}z + (k-1) g^{(k)}(z,\tau) \frac{\mathrm{d}\tau}{2\pi i} \right)$$

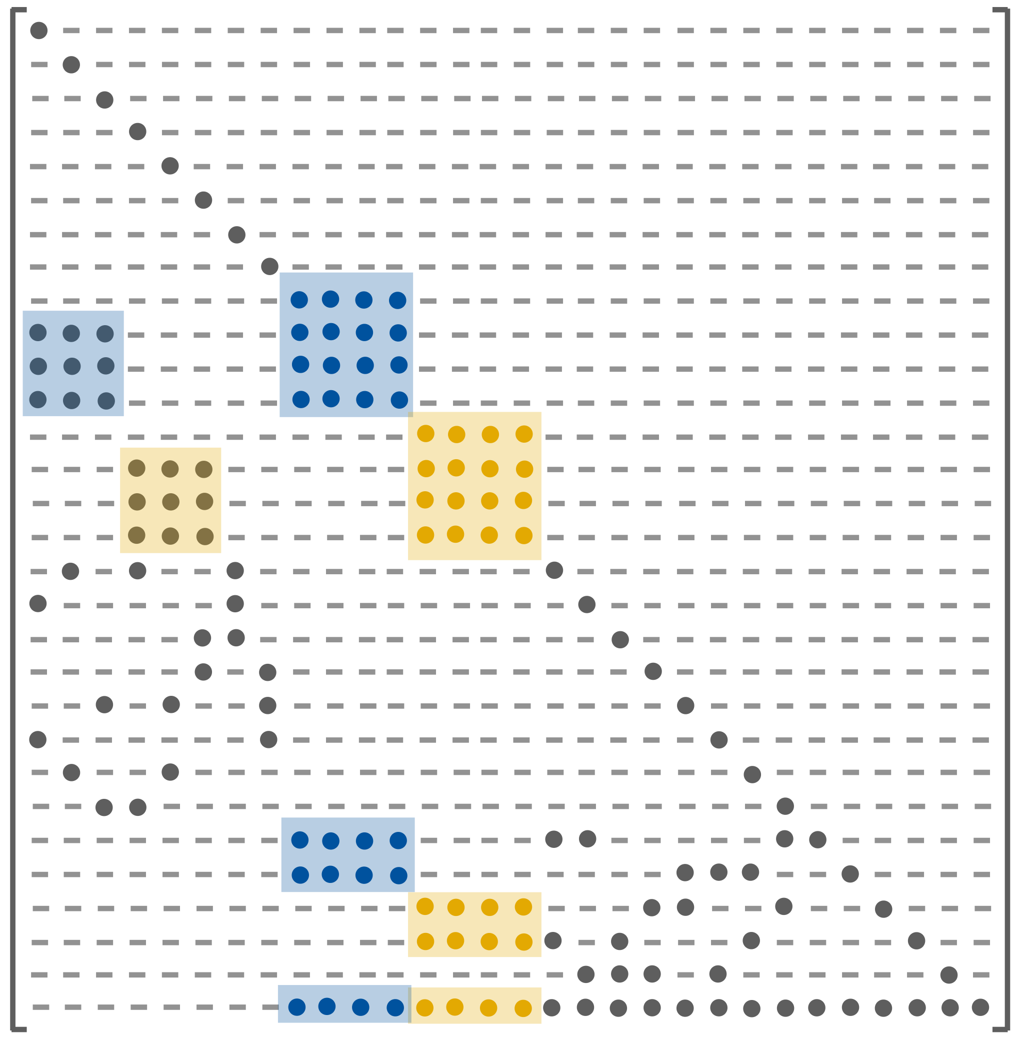
g-kernels: coefficients of the Kronecker-Eisenstein series

Iterated integrals over Kronecker forms along z are **elliptic multiple polylogarithms**:
[Brown, Levin | Brödel, Duhr, Dulat, Penante, Tancredi | Brödel, Matthes, Schlotterer]

$$\tilde{\Gamma}\left(\begin{matrix} n_1 & \dots & n_k \\ w_1 & \dots & w_k \end{matrix} ; z \right) = \int_0^z \mathrm{d}z_1 \, g^{(n_1)}(z_1 - w_1) \tilde{\Gamma}\left(\begin{matrix} n_2 & \dots & n_k \\ w_2 & \dots & w_k \end{matrix} ; z \right)$$

Reorganize the differential equation B in terms of $\omega_k^{\text{Kronecker}}$ & modular forms $\eta_k(\tau)$.

THE DEQ IN KRONECKER-FORMS



Entry by entry in the differential equation:

1. Determine, which Kronecker forms appear.

Decide on torus (123) or (345)

$$\omega_k\left(\underbrace{\sum_{n=1}^{5} c_n z_n^{(ijk)}}_{\mathcal{L}_i(\underline{z})}, c\cdot\tau^{(ijk)}\right)$$

Integer (only c=1,2 for the kite)

Modular behavior:

ω_k is a quasi-modular form of weight k-2

Find from q-expansion $(q=e^{i\pi\tau})$

→ Singularities!

2. Numerically determine the linear combination of Kronecker forms to fix the $\mathrm{d}z$ part

3. Numerically fix the remaining $\mathrm{d}\tau$ part with forms $\eta_2(\tau)$ & $\eta_4(\tau)$

THE DEQ IN KRONECKER-FORMS

Any diagonal entry B_{ii} is a linear combination of dlogs:

$$B_{ii} = \mathrm{d}\log A_1 + \cdots + \mathrm{d}\log A_m \xrightarrow{\text{q-expansion}} \mathrm{d}\log A_j = \mathrm{d}\log\left(A_j^{(0)} + \mathcal{O}(q)\right)$$

Any diagonal entry B_{ii} is a expressible in ω_2:

$$B_{ii} = \sum_j c_j \omega_2(\mathcal{L}_j(z), \tau) \xrightarrow{\text{q-expansion}} \omega_2(\mathcal{L}_j(z), \tau) = \mathrm{d}\log\left(\sin(\pi\mathcal{L}_j(z))\right) + \mathcal{O}(q^2)$$

By comparing the leading order in q, we can find the appearing arguments $\mathcal{L}_i(\underline{z})$!

The zero-loci of these $\mathcal{L}_i(\underline{z})$ are singularities on the tori.

Find 17 $\mathcal{L}_i(\underline{z})$ on each of the tori.

Examples: $\frac{1}{2}(z_2 - z_4)$, $\frac{1}{2}(z_1 + z_2 + z_4 - z_5)$

THE RESULT IN ITERATED INTEGRALS

Boundary point: We choose for the initial point: $m_i = m > 0$ and $p^2 \to 0$

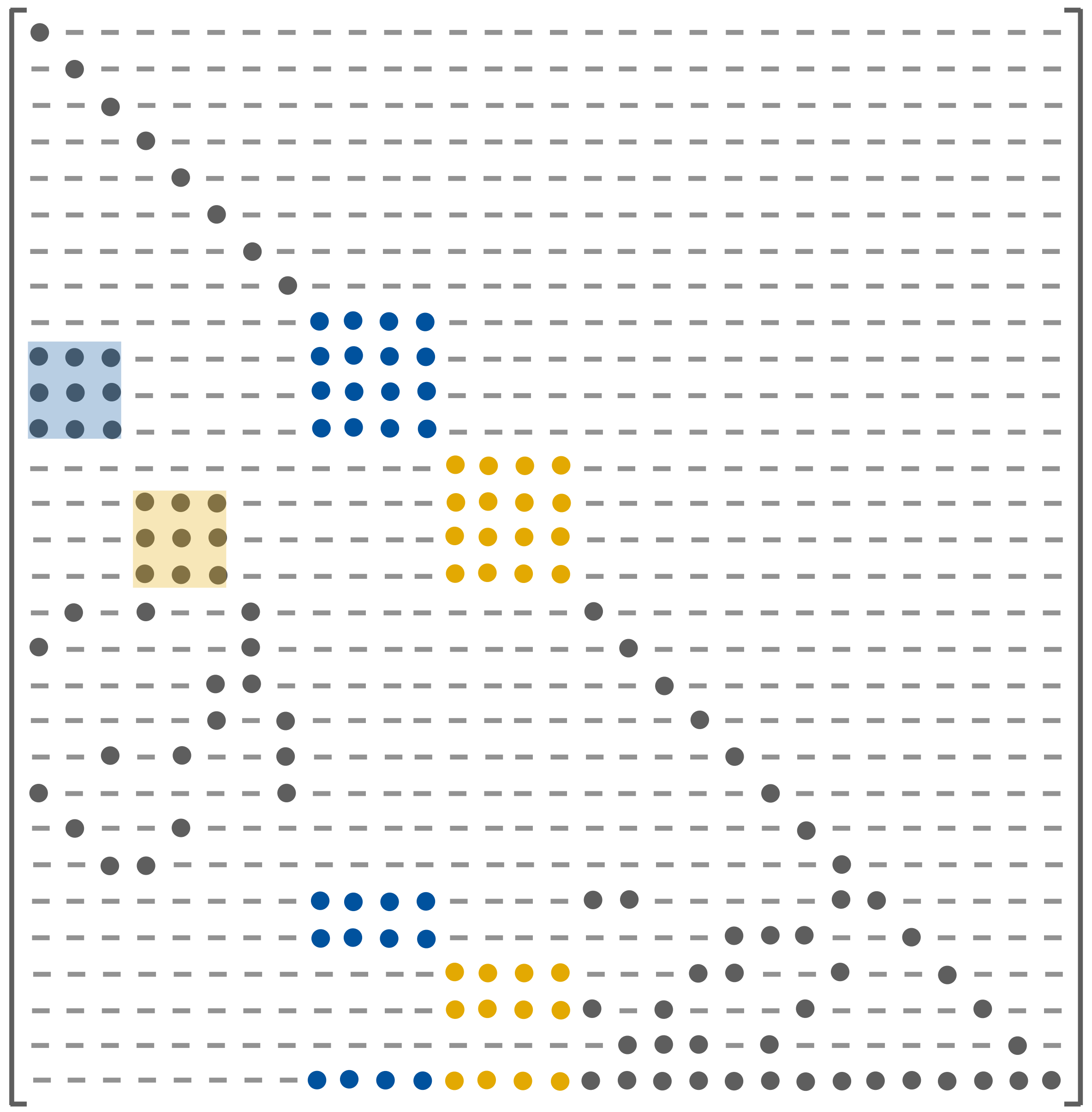
$$\mathbf{J}_0 = \varepsilon^4 \left(I_{11000} \times \vec{1}_8, \frac{\sqrt{3} I_{11100}}{2}, \vec{0}_2, -\frac{\sqrt{3} I_{11100}}{4}, \frac{\sqrt{3} I_{11100}}{2}, \vec{0}_2, -\frac{\sqrt{3} I_{11100}}{4}, i\sqrt{3} I_{11100}, \vec{0}_3, i\sqrt{3} I_{11100}, \vec{0}_9 \right)$$

Result in iterated integrals:

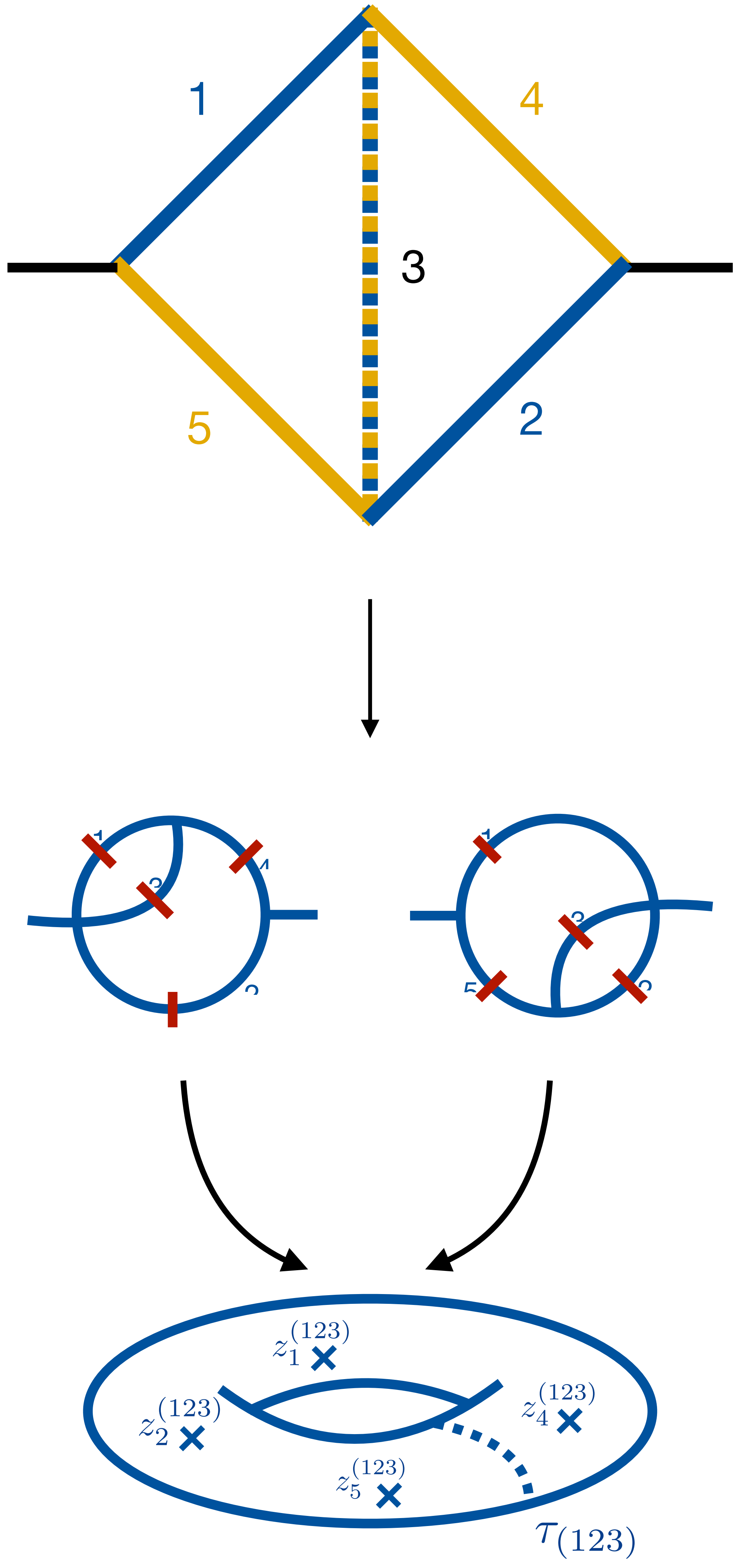
A good choice of parametrization ensures that the elliptic curves don't mix in the integrals.

$$J_i = \mathbf{J}_0 + \mathbf{J}_0 \varepsilon \left(\sum \int \omega_k^{(123)} + \sum \int \omega_k^{(345)} \right)$$

$$+ \mathbf{J}_0 \varepsilon^2 \left(\sum \int \omega_k^{(123)} \int \omega_k^{(123)} + \sum \int \omega_k^{(345)} \int \omega_k^{(345)} \right) + \ldots$$



NO terms of the form $\int \omega_k^{(123)} \int \omega_k^{(345)}$



SUMMARY

1. Parametrization of the kinematic space on the two tori:
   Punctures from integrals of 2D maximal cuts of all eyeball sub topologies.

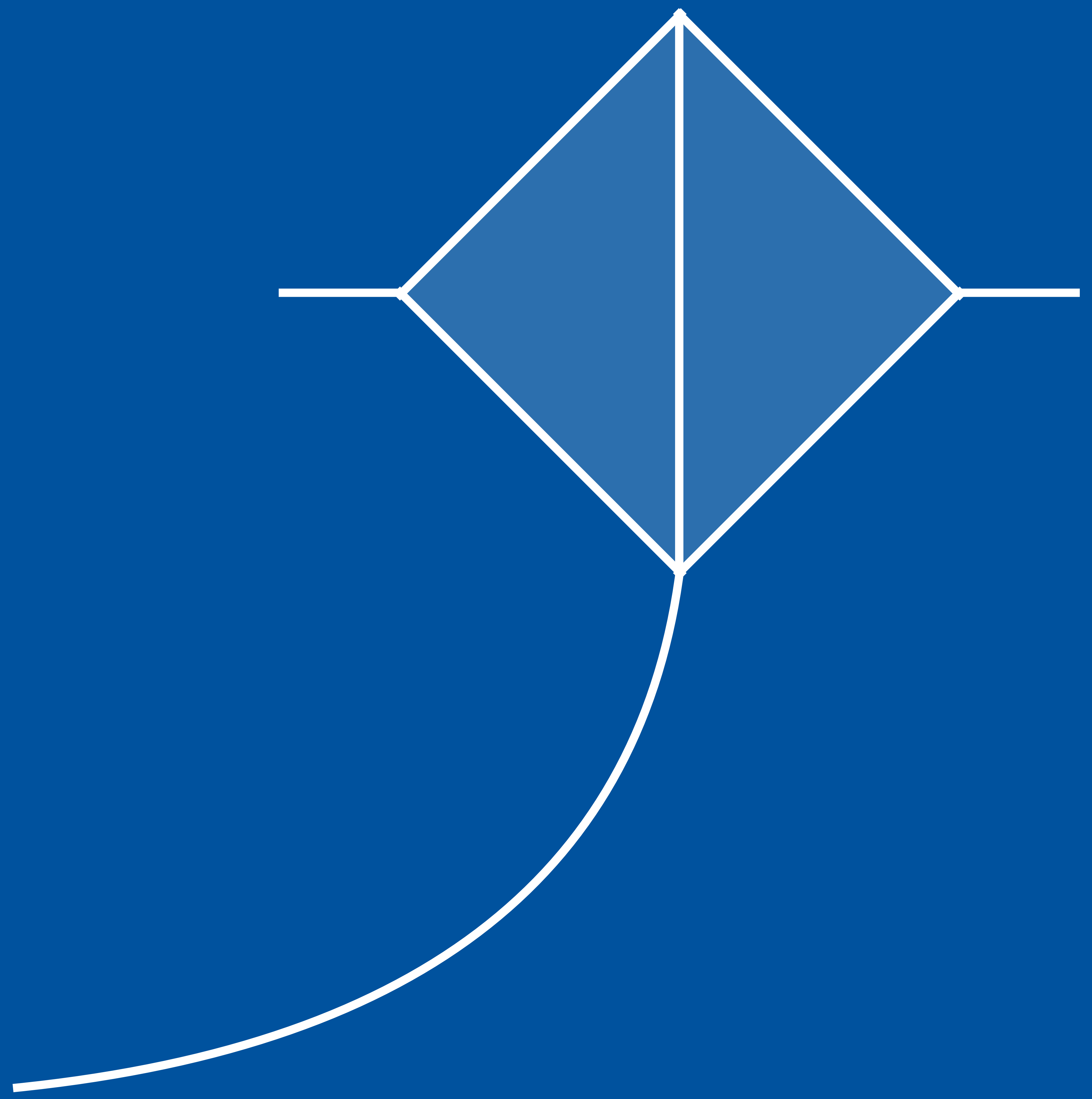
2. Finding an ε-form differential equation on the tori
   Use the parametrization on both tori + modular transformation

3. The singularity structure and iterated integrals on the tori
   Use the q-expansion to find singularites/arguments of Kronecker forms

OUTLOOK

- Other multi-scale Feynman integral families with several elliptic curves: Can we parametrize & solve them in a similar way?
- More efficient numerics ?
- Multi-scale problems related to Calabi-Yau & hyperelliptic curves

Thank you!



Backup: Two Tori from Two Sunrises

The maximal cut of a sunrise integral with propagators α defines a quartic elliptic curve:

$$y_\alpha^2 = (x_\alpha - e_1^\alpha)(x_\alpha - e_2^\alpha)(x_\alpha - e_3^\alpha)(x_\alpha - e_4^\alpha)$$

Roots of the sunrises' elliptic curves:

$$\{e_i^{(123)}\} = \{-(\sqrt{X_1}+\sqrt{X_2})^2, -(1+\sqrt{X_0})^2, -(1-\sqrt{X_0})^2, -(\sqrt{X_1}-\sqrt{X_2})^2\}$$
$$\{e_i^{(345)}\} = \{-(\sqrt{X_4}+\sqrt{X_5})^2, -(1+\sqrt{X_0})^2, -(1-\sqrt{X_0})^2, -(\sqrt{X_4}-\sqrt{X_5})^2\}$$

Periods of the sunrises' elliptic curves:

$$\psi_1^\alpha = 2\int_{e_2^\alpha}^{e_3^\alpha} \frac{\mathrm{d}x_\alpha}{y_\alpha} = 2\frac{K(k_\alpha^2)}{c_4^\alpha} \quad \text{and} \quad \psi_2^\alpha = 2\int_{e_4^\alpha}^{e_3^\alpha} \frac{\mathrm{d}x_\alpha}{y_\alpha} = 2i\frac{K(1-k_\alpha^2)}{c_4^\alpha}$$

with $c_4^\alpha = \frac{1}{2}\sqrt{(e_3^\alpha - e_1^\alpha)(e_4^\alpha - e_2^\alpha)}$ & $k_\alpha^2 = \frac{(e_3^\alpha - e_2^\alpha)(e_4^\alpha - e_1^\alpha)}{(e_3^\alpha - e_1^\alpha)(e_4^\alpha - e_1^\alpha)}$

Backup: Two Tori from Two Sunrises

A point on the elliptic curve is mapped to a point on the torus via Abel's map:

$$(x, \pm y) \mapsto z^{\pm} = \pm \frac{1}{\psi_1} \int_{e_1}^{x} \frac{dx}{y} \mod \Lambda_{1,\tau} \quad = \pm \left[e^{i \arg(x-e_1) - \arg(x-e_2)} \frac{F(\sqrt{u_x}, k^2)}{2K(k^2)} + \frac{\tau}{2} \right] \quad \text{with} \quad u_x = \frac{x-e_2}{x-e_1} \frac{e_1-e_3}{e_2-e_3}$$

Punctures for the two sunrises in the kite:

$$u_1^{(123)} = \frac{(\sqrt{X_0} + \sqrt{X_1})^2 - (\sqrt{X_2} - 1)^2}{4\sqrt{X_2}} \qquad u_4^{(345)} = \frac{(\sqrt{X_0} + \sqrt{X_4})^2 - (\sqrt{X_5} - 1)^2}{4\sqrt{X_5}}$$

$$u_2^{(123)} = \frac{(\sqrt{X_0} + \sqrt{X_2})^2 - (\sqrt{X_1} - 1)^2}{4\sqrt{X_1}} \qquad u_5^{(345)} = \frac{(\sqrt{X_0} + \sqrt{X_5})^2 - (\sqrt{X_4} - 1)^2}{4\sqrt{X_4}}$$

Additional punctures from the eye-balls

$$u_4^{(123)} = u_2^{(123)} \frac{(1 + \sqrt{X_1})^2 - X_4}{(\sqrt{X_0} + \sqrt{X_2})^2 - X_4} \qquad u_1^{(345)} = u_5^{(345)} \frac{(1 + \sqrt{X_4})^2 - X_1}{(\sqrt{X_0} + \sqrt{X_5})^2 - X_1}$$

$$u_5^{(123)} = u_1^{(123)} \frac{(1 + \sqrt{X_2})^2 - X_5}{(\sqrt{X_0} + \sqrt{X_1})^2 - X_5} \qquad u_2^{(345)} = u_4^{(345)} \frac{(1 + \sqrt{X_5})^2 - X_2}{(\sqrt{X_0} + \sqrt{X_4})^2 - X_2}$$

Backup: Modular transformations

Under a modular transformation, the periods and punctures transform in the following way:

$$z \mapsto \frac{z}{c\tau + d} \quad , \quad \tau \mapsto \frac{a\tau + b}{c\tau + d}$$

$$\psi_1 \mapsto (c\tau + d)\psi_1 \quad , \quad \psi_2 \mapsto (a\tau + b)\psi_2$$

$$\partial_0 \psi_1 \mapsto (c\tau + d)\partial_t \psi_1 + c\psi_1 \partial_0 \tau$$

A quasi-modular form of weight k and depth p transforms in the following way:

$$f(z, \tau) \mapsto \sum_{i=0}^{p} (c\tau + d)^{k+2} \left(\frac{cz}{cz + d} \right)^{i} f_i(z, \tau)$$

Modular behavior of B

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\end{array}\right) \tag{C.1}
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Backup: Forms on the Torus

The g-kernels

$$F(z,\eta,q) = \pi\frac{\theta_1'(0,\tau)\theta_1(\pi(z+\eta),\tau)}{\theta_1(\pi z,\tau)\theta_1(\pi\eta,\tau)} = \sum_{\alpha=0}^{\infty}\eta^{\alpha-1}g^{(\alpha)}(z,\tau)$$

$$\begin{aligned}
g^{(0)}(z,\tau) &= 1\\
g^{(1)}(z,\tau) &= \pi\cot(\pi z) + 4\pi\sin(2\pi z)q^2 + \mathcal{O}(q^4)\,,\\
g^{(2)}(z,\tau) &= -\frac{\pi^2}{3} + 8\pi^2\cos(2\pi z)q^2 + \mathcal{O}(q^4)\,,\\
g^{(3)}(z,\tau) &= -8\pi^3\sin(2\pi z)q^2 + \mathcal{O}(q^4)\,,\\
g^{(4)}(z,\tau) &= -\frac{\pi^4}{45} - \frac{16\pi^4}{3}\cos(2\pi z)q^2 + \mathcal{O}(q^4)\,.
\end{aligned}$$

$$\omega_2(z,\tau) = \mathrm{d}\log\frac{\theta_1(\pi z,\tau)}{\eta(\tau)} = \mathrm{d}\log\sin(\pi z) + \mathcal{O}(q^2)\,,$$

The η - function

$$\eta_2(\tau) = [e_2(\tau) - 2e_2(2\tau)]\,\frac{\mathrm{d}\tau}{2\pi i} \quad \text{with} \quad e_2(\tau) = \frac{2\pi^2}{6}\frac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)}$$

$$\eta_4(\tau) = e_4(\tau)\frac{\mathrm{d}\tau}{(2\pi i)^3} \quad \text{with} \quad e_4(\tau) = \frac{\pi^4}{90}\left(\theta_2^8(0,\tau) + \theta_3^8(0,\tau) + \theta_4^8(0,\tau)\right)$$

Backup: More q-expansion behavior

$$
\begin{aligned}
\mathrm{d}\log\left(X_0|_{q^2}\right) &= \mathrm{d}\log\left(\sin^2\left(\pi z_1\right)\sin^2\left(\pi z_2\right)\right)\,,\\
\mathrm{d}\log\left(X_4|_{q^0}\right) &= \mathrm{d}\log\frac{\sin\left(\frac{\pi}{2}(2z_1+z_2+z_4)\right)\sin\left(\frac{\pi}{2}(2z_1+z_2-z_4)\right)\sin^2\left(\pi z_2\right)}{\sin^2\left(\pi(z_1+z_2)\right)\sin\left(\frac{\pi}{2}(z_2+z_4)\right)\sin\left(\frac{\pi}{2}(z_2-z_4)\right)}\,,\\
\mathrm{d}\log\left(\lambda_{024}|_{q^0}\right) &= \mathrm{d}\log\frac{\sin^2\left(\pi z_1\right)\sin^4\left(\pi z_2\right)}{\sin^2\left(\pi(z_1+z_2)\right)\sin^2\left(\frac{\pi}{2}(z_2+z_4)\right)\sin^2\left(\frac{\pi}{2}(z_2-z_4)\right)}\,,\\
\mathrm{d}\log\left(\lambda_{134}|_{q^0}\right) &= \mathrm{d}\log\frac{\sin^2\left(\pi z_1\right)\sin^2\left(\pi z_2\right)\sin^2\left(\pi z_4\right)}{\sin^2\left(\pi(z_1+z_2)\right)\sin^2\left(\frac{\pi}{2}(z_2+z_4)\right)\sin^2\left(\frac{\pi}{2}(z_2-z_4)\right)}\,.
\end{aligned}
$$

Backup: Initial condition detailed

$$m_i = m > 0 \text{ and } p^2 \to 0 \quad \Leftrightarrow \quad \begin{array}{l} \tau_0^{(123)} = \tau_0^{(345)} = i\infty \\ z_1^{(123)} = z_2^{(123)} = z_4^{(345)} = z_5^{(345)} = \dfrac{1}{3} \\ z_4^{(123)} = z_5^{(123)} = z_1^{(345)} = z_2^{(345)} = \dfrac{1}{2} - i\infty \end{array}$$

$$\downarrow$$

$$\mathbf{J}_0 = \varepsilon^4 \left(I_{11000} \times \vec{1}_8, \frac{\sqrt{3} I_{11100}}{2}, \vec{0}_2, -\frac{\sqrt{3} I_{11100}}{4}, \frac{\sqrt{3} I_{11100}}{2}, \vec{0}_2, -\frac{\sqrt{3} I_{11100}}{4}, i\sqrt{3} I_{11100}, \vec{0}_3, i\sqrt{3} I_{11100}, \vec{0}_9 \right)$$

- $I_{11000}(\underline{X}_0) = e^{2\gamma_{\mathrm{EM}}\varepsilon} \Gamma^2(\varepsilon)$
- $I_{11100}(\underline{X}_0) = \dfrac{e^{2\gamma_{\mathrm{EM}}\varepsilon} \Gamma(1+2\varepsilon)}{(-1)^{1+2\varepsilon} 3^{1/2+\varepsilon}} \left[\left(-e^{\frac{2i\pi}{3}}\right)^{-\varepsilon} F_\varepsilon\left(\frac{2i\pi}{3}\right) - \left(-e^{-\frac{2i\pi}{3}}\right)^{-\varepsilon} F_\varepsilon\left(-\frac{2i\pi}{3}\right) + \frac{\pi}{\varepsilon} \right]$

$$\text{with } F_\varepsilon(z) = \frac{3i\Gamma^2(\varepsilon+1)}{2\varepsilon^2 \Gamma(2\varepsilon+1)} \, {}_2F_1(-2\varepsilon, -\varepsilon, 1-\varepsilon, e^z)$$

Integrating along τ with the common base point $\tau_0^{(123)} = \tau_0^{(345)} = i\infty$ and fixed z_i we obtain:

$$J_{30}^{(2)} = \sum_{j=6}^{9} (-1)^{\delta_{j,6}+\delta_{j,7}} \left([\Omega^{(123)}_{2,j,1}, \Omega^{(123)}_{2,1,1}] - 2[\Omega^{(123)}_{2,j,1}, \Omega^{(123)}_{2,1,2}] - [\Omega^{(123)}_{2,j,1}, \Omega^{(123)}_{2,5,1}] + 2[\Omega^{(123)}_{2,j,1}, \Omega^{(123)}_{2,5,2}]\right)$$

$$+ \sum_{j=6}^{9} (-1)^{\delta_{j,10}+\delta_{j,11}} \left([\Omega^{(123)}_{2,j,1}, \Omega^{(123)}_{2,2,1}] - 2[\Omega^{(123)}_{2,j,1}, \Omega^{(123)}_{2,2,2}] - [\Omega^{(123)}_{2,j,1}, \Omega^{(123)}_{2,5,1}] + 2[\Omega^{(123)}_{2,j,1}, \Omega^{(123)}_{2,5,2}]\right)$$

$+((123) \leftrightarrow (345))$ with $\Omega^{d}_{a,b,c} = \omega_a(\mathcal{L}^{d}_{b}, c\tau_d)$ and $[\omega_1, \omega_2] = \int_{i\infty} \omega_1(t_1) \int_{i\infty}^{t_1} \omega_2(t_2)$