Symbol letters of Feynman integrals from Gram determinants

Xiaofeng Xu

based on work in collaboration with

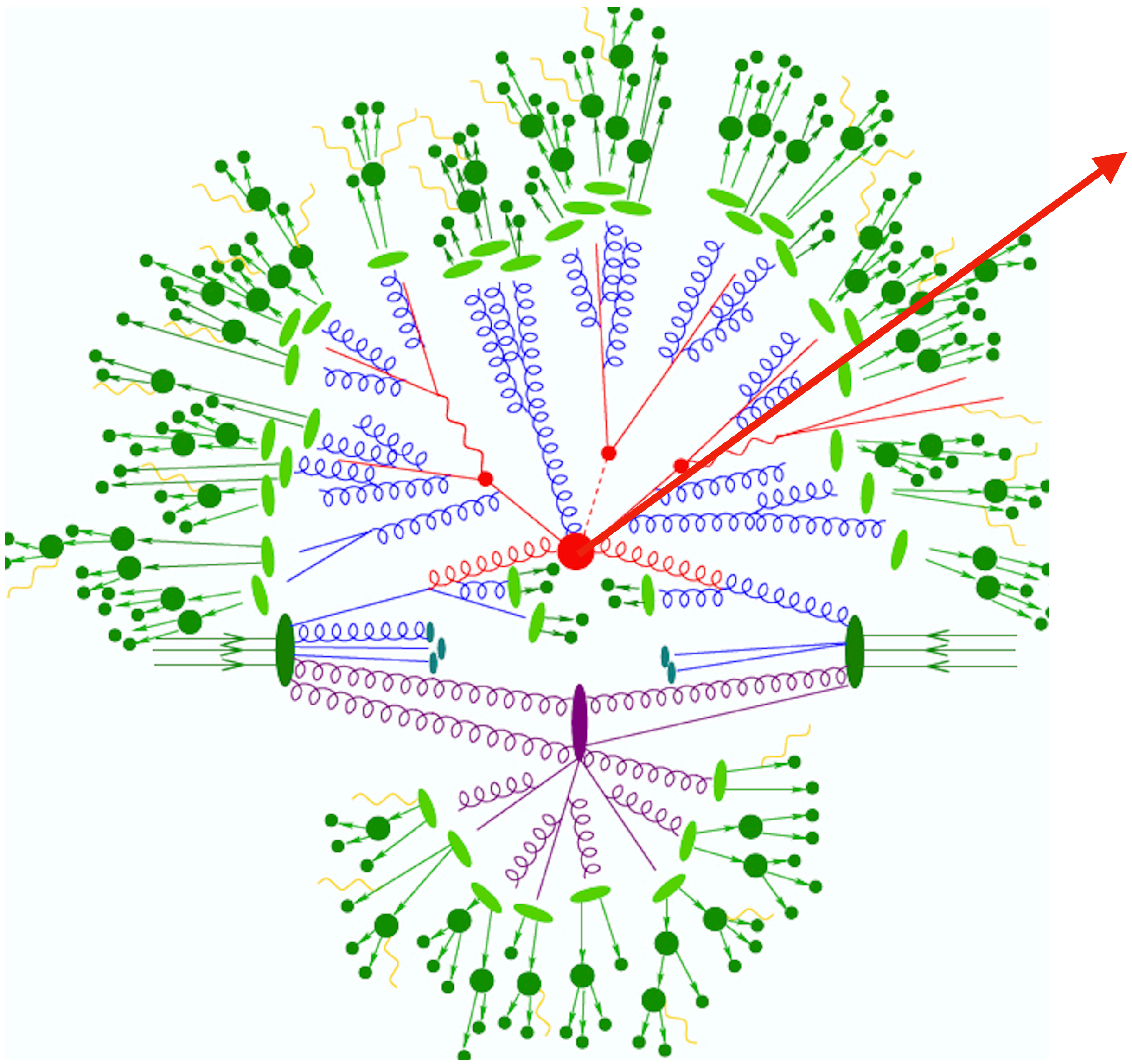
X. Jiang, J.Liu, L.Yang

Amplitudes 2024, IAS

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Introduction

Proton-proton collision at LHC

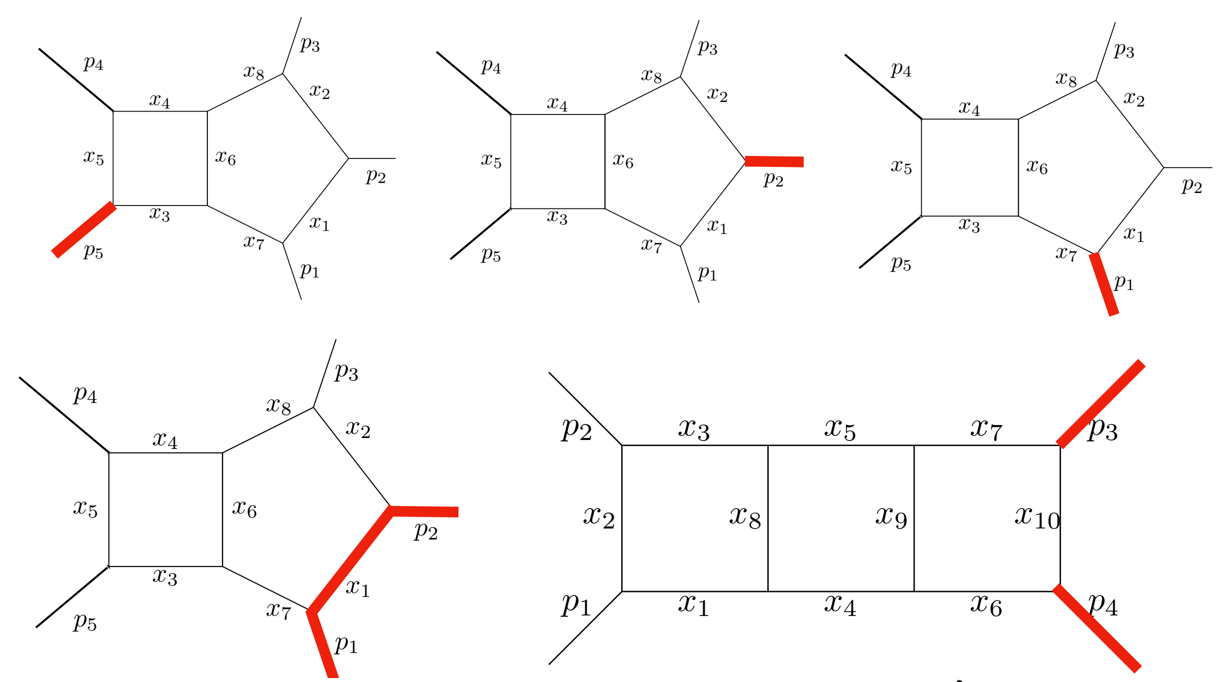


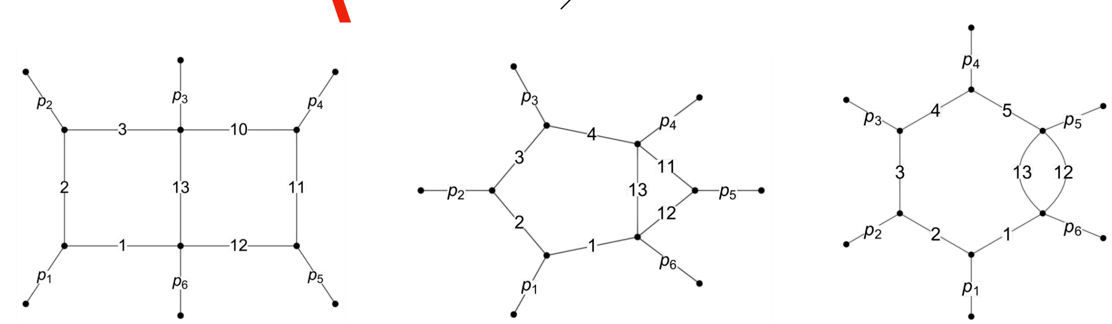
[S. Abreu and et al. ‘2020]

[S. Badger and et al.’2023]

[Johannes M. Henn and et al.’2024]

Precise predictions of hard scattering processes require higher order corrections





(a) Double-Box (db) (b) Pentagon-Triangle (pt) (c) Hexagon-Bubble (hb)

Introduction

- Differential equations method

[Henn '2013]

$$d\vec{f}_0(\vec{x},\epsilon) = dA_0(\vec{x},\epsilon)\vec{f}_0(\vec{x},\epsilon)$$

Differential equations for master integrals

↓

$$d\vec{f}(\vec{x},\epsilon) = \epsilon dA(\vec{x})\vec{f}(\vec{x},\epsilon)$$

Canonical differential equations(CDEs)

↓

$$d\vec{f}(\vec{x},\epsilon) = \epsilon \sum_k A_k d\log\alpha_k(\vec{x})\vec{f}(\vec{x},\epsilon)$$

α_k is called a letter

The solutions are expressed as iterated functions

$$F = \int d\log\alpha_n \cdots \int d\log\alpha_2 \int d\log\alpha_1$$

Introduction

- Challenges:

  1. Symbolic integration-by-parts (IBP) reduction
  2. Determine the letters from partial derivatives

- Benefits of letters:

  [C. Dlapa, J.Henn, and K.Yan ' 2020]

  1. Letters as input to find the canonical basis
  2. Bootstrap coefficient matrices for CDEs
  3. Bootstrap analytic expressions for Feynman integrals

  [L.J.Dixon and et al.'2011 , L.J.Dixon and et.al ,2013
  L.M. Drummond and et al. '2015 , D. Chicherin and et al.' 2018,
  S. Caron-Hout and et al. '2019]

Introduction

- Recent progress

1. Symbol letters for one-loop integrals are fully understood

   [M. Spradlin and A. Volovich ‘2011, N. Arkani-Hamed and E. Y. Yuan’ 2017
   S. Abreu and et al. ‘2017, S. Caron-Huot and A. Pokraka ‘2021,
   J. Chen and et al. ‘2022]

2. Attempts have been made from different perspectives for high-loops

   - Landau singularities [T.Dennen and et al. ’ 2016, I. Prlina and et al. ’ 2018
     H. S. Hannesdottir and et al. ’2022,
     L. Lippstreu and et al. ‘2022, C. Dlapa and et al. ‘2023]
   - Schubert problems [Q.Yang ’ 2022, S. He and et al. ’ 2022
     R. Morales and et al. ’2023,
     S. He and et al. ‘2023]
   - Intersection theory [J.Chen and et al. ‘2024]

Recursive structures of Baikov representation

[X.Jiang and L. Yang' 2023]

Standard Baikov representation

$$N = \frac{L(L+1)}{2} + LE$$

$$I_{a_1,a_2,\ldots,a_n} = \frac{C}{G(p_1,\ldots,p_E)^{\frac{d-E-1}{2}}} \int \prod_i^N dx_i \frac{G(l_1,\ldots,l_L,p_1,\ldots,p_E)^{\frac{d-L-E-1}{2}}}{x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N}}$$

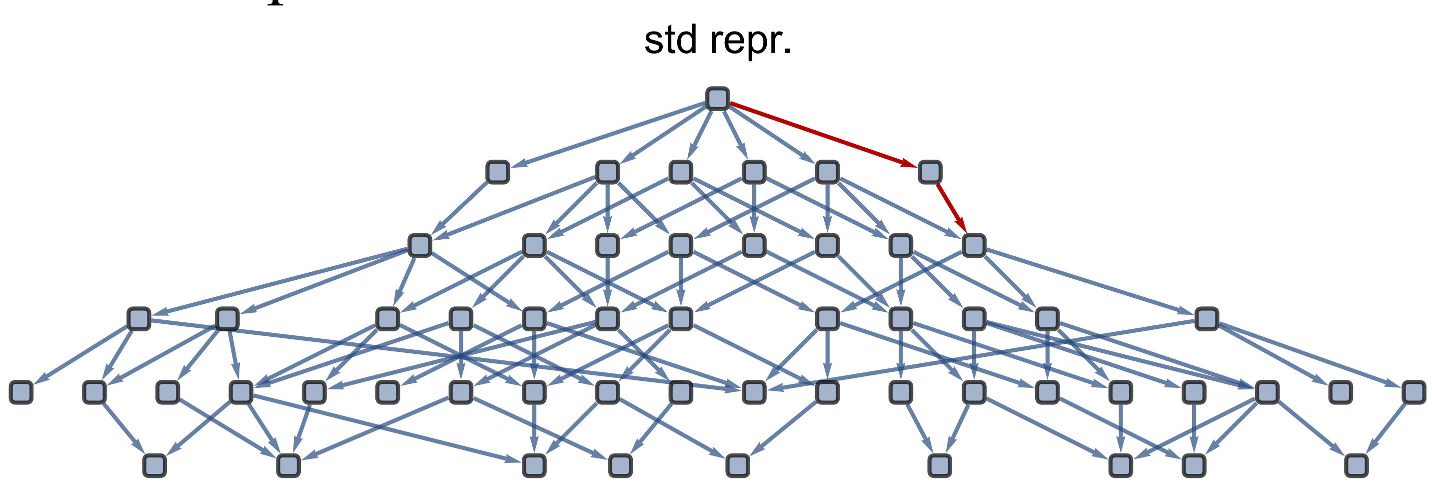
propagators and irreducible scalar product (ISP)

$$G(\{\mathbf{k}\},\{\mathbf{q}\}) \equiv \det(k_i \cdot q_j) \qquad G(\mathbf{k}) \equiv G(\{\mathbf{k}\},\{\mathbf{k}\})$$

Loop-by-loop Baikov representation

$$\int u(\mathbf{x}) \prod_i^n \frac{dx_i}{x_i^{a_i}}, \quad u(\mathbf{x}) = \prod_j [P_j(\mathbf{x})]^{\alpha_j + \beta_j \epsilon}$$

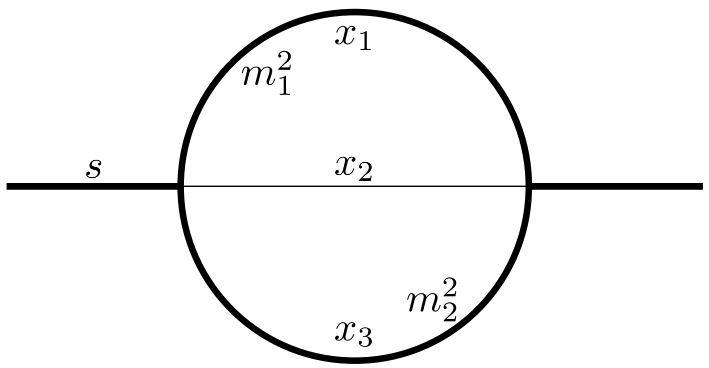
Relation graph of Baikov representation



Recursive structures of Baikov representation

Iterative structures of sunrise integrals

$$x_1 = l_1^2 - m_1^2\,,\ x_2 = (l_1 - l_2)^2\,,\ x_3 = (l_2 - p)^2 - m_2^2\,,$$
$$x_4 = l_2^2\,,\ x_5 = (l_1 - p)^2\,.$$



Baikov representation along the yellow path

$$G(p)^{\frac{2-d}{2}} G(l_1, l_2, p)^{\frac{d-4}{2}}$$

$$\downarrow$$

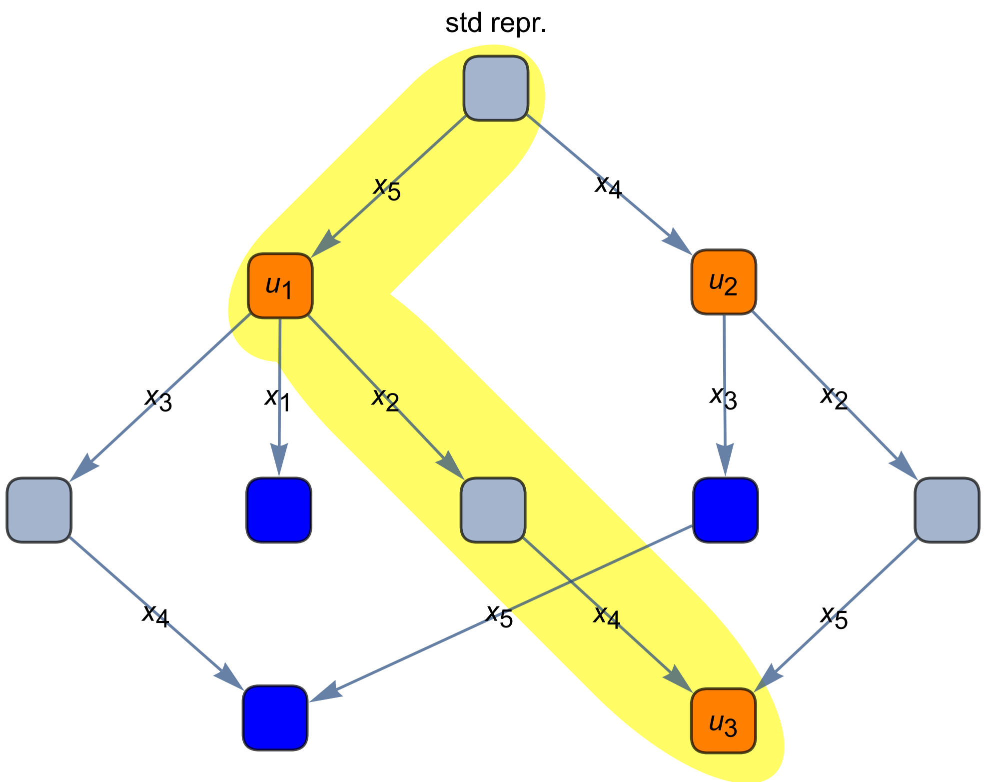
$$G(p)^{1-\frac{d}{2}} G(l_2, p)^{\frac{d-3}{2}} G(l_2)^{1-\frac{d}{2}} G(l_1, l_2)^{\frac{d-3}{2}}$$

$$\downarrow$$

$$G(p)^{1-\frac{d}{2}} G(l_2, p)^{\frac{d-3}{2}} G(l_1)^{\frac{d-2}{2}}$$

$$\downarrow$$

$$G(l_1)^{\frac{d-2}{2}} G(l_2 - p)^{\frac{d-2}{2}}$$



Rational symbol letters

- D-log form integrand and leading singularities of one-loop integral

[J. Chen and et al. '2021]

$$u(\mathbf{x})\varphi(\mathbf{x}) \sim \sqrt{G(\mathbf{0})} \bigwedge_{i=1}^{E+1} \frac{dx_i}{x_i} \left[G(\mathbf{x})\right]^{-1/2-\epsilon}$$

$$= \left[G(\mathbf{x})\right]^{-\epsilon} \bigwedge_{i=1}^{E+1} \frac{dx_i}{x_i} \sqrt{\frac{G(0_i, x_{i+1}, \ldots, x_{E+1})}{G(0_{i-1}, x_i, \ldots, x_{E+1})}}$$

E is odd

- Get a pinched pole at $\mathbf{x} = \mathbf{0}$ if $G(\mathbf{0}) = 0$.

- Rational letters as Gram determinant

$$G(\mathbf{0}) = G(\mathbf{x})\Big|_{\text{maximal cut}}$$

Rational symbol letters

- Analyze the pinched singularities of maximal cut integrals
- Example: one-fold integral

$$\int dz(z-a)^{\gamma_1}(z-b)^{\gamma_2}(Az^2+Bz+C)^{\gamma_3}\frac{1}{(z-a)(z-b)\sqrt{Az^2+Bz+C}},$$

simple poles of the integrand and maximal cut Gram determinants:

$$z=a \longrightarrow \{a-b, Aa^2+Ba+c\}$$

$$z=b \longrightarrow \{a-b, Ab^2+Bb+c\}$$

Rational letter as $G(\mathbf{x})\Big|_{\text{"maximal cut"}}$

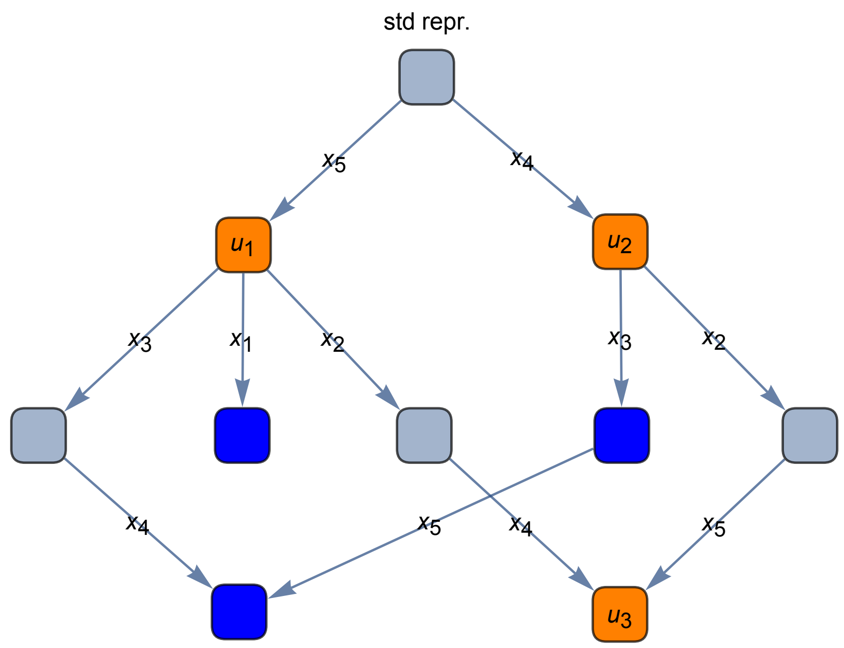
"maximal cut" means cut of the propagators and ISPs at some poles.

Rational symbol letters for sunrise integrals

- Baikov representation for top sector with $d = 2 - 2\epsilon$

$$\tilde{u}_1(x_4) = \left[\tilde{G}(p)\tilde{G}(l_2)\right]^{\epsilon}\left[\tilde{G}(l_1,l_2)\tilde{G}(l_2,p)\right]^{-1/2-\epsilon},$$
$$\tilde{G}(p) = s\,,\quad \tilde{G}(l_2,p) = -\lambda(x_4,s,m_2^2)/4\,,$$
$$\tilde{G}(l_2) = x_4\,,\quad \tilde{G}(l_1,l_2) = -(x_4-m_1^2)^2/4\,,$$

where $\lambda(x,y,z) \equiv x^2+y^2+z^2-2xy-2yz-2zx$



- Simple poles of x_4

$$x_4 = m_1^2 \to \tilde{G}(l_2,p)\Big|_{x_4=0} \sim \lambda(m_1^2,m_2^2,s)$$

$$x_4 = 0 \to \tilde{G}(l_1,l_2)\Big|_{x_4=0} \sim m_1^2$$

$$x_4 = 0 \to \tilde{G}(l_2,p)\Big|_{x_4=0} \sim s - m_2^2$$

→ spurious pole

Ansatz for irrational symbol letters

- Sylvester identity for Gram determinant:

$$G(\mathbf{q}, q_i)G(\mathbf{q}, q_j) = G(\{\mathbf{q}, q_i\}, \{\mathbf{q}, q_j\})^2 + G(\mathbf{q})G(\mathbf{q}, q_i, q_j)$$

- Similar to the construction of d-log form integrand

$$\partial_B \log \frac{(B+\sqrt{-DE})}{(B-\sqrt{-DE})}\bigg|_{DE} = \frac{2\sqrt{-DE}}{-AC},$$

$$\partial_B \log \frac{(B+\sqrt{AC})}{(B-\sqrt{AC})}\bigg|_{DE} = \frac{2}{\sqrt{AC}},$$

with B^2+DE=AC.

- Algebraic letters:

$$d\log \frac{G(\{\mathbf{q}, q_i\}, \{\mathbf{q}, q_j\}) + \sqrt{-G(\mathbf{q})G(\mathbf{q}, q_i, q_j)}}{G(\{\mathbf{q}, q_i\}, \{\mathbf{q}, q_j\}) - \sqrt{-G(\mathbf{q})G(\mathbf{q}, q_i, q_j)}}\Bigg|_{\text{"maximal cut"}}$$

$$d\log \frac{G(\{\mathbf{q}, q_i\}, \{\mathbf{q}, q_j\}) + \sqrt{G(\mathbf{q}, q_i)G(\mathbf{q}, q_j)}}{G(\{\mathbf{q}, q_i\}, \{\mathbf{q}, q_j\}) - \sqrt{G(\mathbf{q}, q_i)G(\mathbf{q}, q_j)}}\Bigg|_{\text{"maximal cut"}}$$

Selection rules for symbol letters

https://github.com/windfolgen/Baikovletter

$$d\log \frac{G(\{\mathbf{q}, q_i\}, \{\mathbf{q}, q_j\}) + \sqrt{-G(\mathbf{q})G(\mathbf{q}, q_i, q_j)}}{G(\{\mathbf{q}, q_i\}, \{\mathbf{q}, q_j\}) - \sqrt{-G(\mathbf{q})G(\mathbf{q}, q_i, q_j)}}\Bigg|_{\text{"maximal cut"}}$$

$$d\log \frac{G(\{\mathbf{q}, q_i\}, \{\mathbf{q}, q_j\}) + \sqrt{G(\mathbf{q}, q_i)G(\mathbf{q}, q_j)}}{G(\{\mathbf{q}, q_i\}, \{\mathbf{q}, q_j\}) - \sqrt{G(\mathbf{q}, q_i)G(\mathbf{q}, q_j)}}\Bigg|_{\text{"maximal cut"}}$$

1. The power of Gram determinants in Baikov representation should be half integer, for those appearing in square root

2. The Gram determinants in square root are linked by one-direction arrow in the graph of the recursive structure

3. The expression in the square root is not a perfect square

Symbol letters for sunrise integrals

- Gram determinant appearing in square roots

$$\{G(), G(l_1, p), G(l_1 - p, p), G(l_2, p), G(l_2 - p, p), G(l_1, l_2), G(l_1 - p, l_2 - p)\}\,.$$

- Algebraic letters

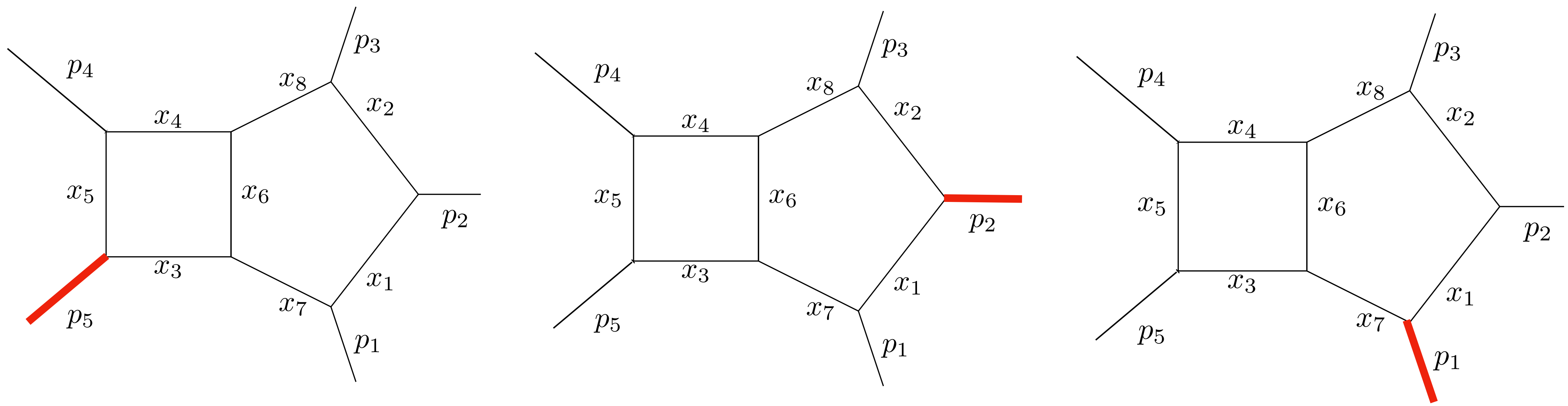
$$d\log \frac{G(\{p\},\{l_2\}) + \sqrt{-G()G(l_2,p)}}{G(\{p\},\{l_2\}) - \sqrt{-G()G(l_2,p)}}\Bigg|_{x_1=x_2=x_3=0, x_4=m_1^2}$$

$$d\log \frac{G\left(\{l_2 - p\},\{l_2\}\right) + \sqrt{-G()G\left(l_2, l_2 - p\right)}}{G\left(\{l_2 - p\},\{l_2\}\right) - \sqrt{-G()G\left(l_2, l_2 - p\right)}}\Bigg|_{x_1=x_2=x_3=0, x_4=m_1^2}$$

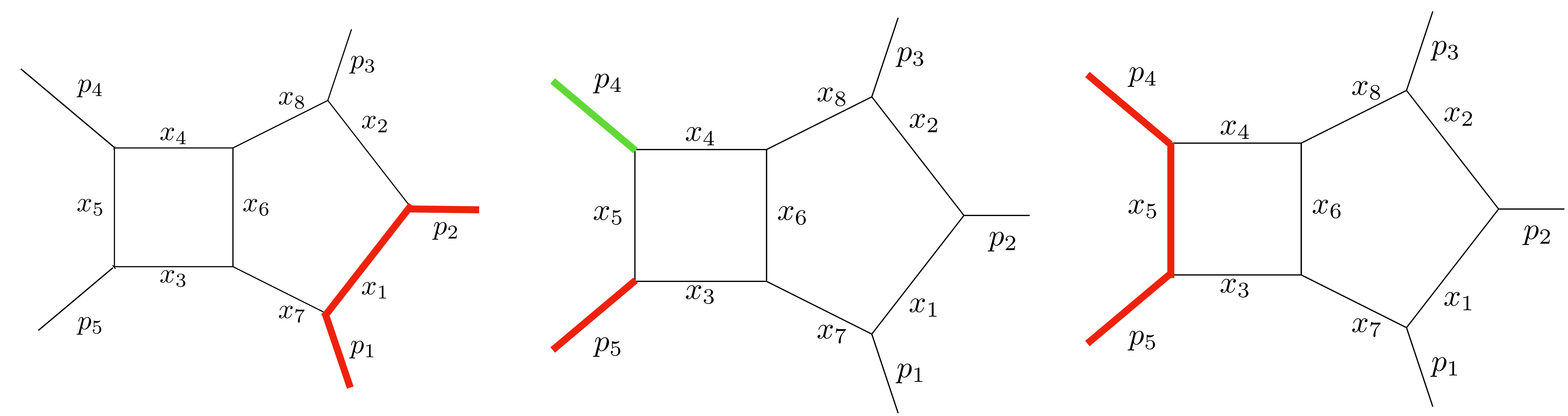
- Two independent algebraic letters

$$\left\{\log\left(-\frac{m_1^2 + m_2^2 - s + \lambda\left(m_1, m_2, s\right)}{m_1^2 + m_2^2 - s - \lambda\left(m_1, m_2, s\right)}\right), \log\left(-\frac{m_1^2 - m_2^2 + s + \lambda\left(m_1, m_2, s\right)}{m_1^2 - m_2^2 + s - \lambda\left(m_1, m_2, s\right)}\right)\right\}$$

Examples



[S. Abreu and et al. ‘2020]



[S. Badger and et al.’2023]

127 master integrals

Checked with CDEs

Rational letters

- 43 rational letters

$$\{m_4^2,\ m_5^2,\ s_{12},\ s_{15},\ s_{23},\ s_{34},\ s_{45},\ s_{12}+s_{23},\ m_5^2-s_{15},\ m_4^2-s_{34},\ s_{15}-s_{34},\ s_{12}-s_{45},\ s_{23}-s_{45},\ s_{12}+s_{15}-s_{34},$$

$$s_{12}+s_{23}-s_{45},\ s_{15}-s_{23}-s_{34},\ m_5^2 s_{23}-s_{15} s_{45},\ m_4^2 s_{12}-s_{34} s_{45},\ m_5^2-s_{15}+s_{23}-s_{45},\ m_4^2+s_{12}-s_{34}-s_{45},$$

$$m_4^2 s_{12} s_{15}+m_5^2 s_{23} s_{34}-s_{15} s_{34} s_{45},\ m_4^2 s_{23}-m_5^2 s_{23}-m_4^2 s_{45}+s_{15} s_{45},\ m_4^2 s_{12}-m_5^2 s_{12}+m_5^2 s_{45}-s_{34} s_{45},$$

$$m_4^2 s_{15}-m_4^2 s_{34}-s_{15} s_{34}+s_{23} s_{34}+s_{34}^2,\ m_5^2 s_{15}-s_{12} s_{15}-s_{15}^2-m_5^2 s_{34}+s_{15} s_{34},\ m_4^4-2 m_4^2 s_{15}+s_{15}^2-2 m_4^2 s_{23}-2 s_{15} s_{23}+s_{23}^2,$$

$$m_5^4-2 m_5^2 s_{12}+s_{12}^2-2 m_5^2 s_{34}-2 s_{12} s_{34}+s_{34}^2,\ m_4^4-2 m_4^2 m_5^2+m_5^4-2 m_4^2 s_{45}-2 m_5^2 s_{45}+s_{45}^2,$$

$$m_4^4-2 m_4^2 s_{15}+s_{15}^2-4 m_5^2 s_{23}-2 m_4^2 s_{45}+2 s_{15} s_{45}+s_{45}^2,\ m_5^4-4 m_4^2 s_{12}-2 m_5^2 s_{34}+s_{34}^2-2 m_5^2 s_{45}+2 s_{34} s_{45}+s_{45}^2,$$

$$m_4^2 s_{12}-m_5^2 s_{15}+s_{15}^2+m_5^2 s_{23}-2 s_{15} s_{23}+s_{23}^2+m_5^2 s_{34}-s_{15} s_{34}+s_{23} s_{34}+s_{15} s_{45}-s_{23} s_{45}-s_{34} s_{45},$$

$$m_4^2 s_{12}+s_{12}^2+m_4^2 s_{15}+s_{12} s_{15}+m_5^2 s_{23}-m_4^2 s_{34}-2 s_{12} s_{34}-s_{15} s_{34}+s_{34}^2-s_{12} s_{45}-s_{15} s_{45}+s_{34} s_{45},$$

$$m_4^4 s_{23}-m_4^2 m_5^2 s_{23}+m_4^2 s_{12} s_{23}-m_4^2 s_{23} s_{34}+m_5^2 s_{23} s_{34}-m_4^4 s_{45}+m_4^2 s_{15} s_{45}+m_4^2 s_{34} s_{45}-s_{15} s_{34} s_{45},$$

$$m_4^2 m_5^2 s_{12}-m_5^4 s_{12}-m_4^2 s_{12} s_{15}+m_5^2 s_{12} s_{15}-m_5^2 s_{12} s_{23}+m_5^4 s_{45}-m_5^2 s_{15} s_{45}-m_5^2 s_{34} s_{45}+s_{15} s_{34} s_{45},$$

$$m_4^4 s_{23}^2-4 m_4^2 m_5^2 s_{23} s_{45}+2 m_4^2 s_{15} s_{23} s_{45}-2 m_4^2 s_{23}^2 s_{45}+s_{15}^2 s_{45}^2-2 s_{15} s_{23} s_{45}^2+s_{23}^2 s_{45}^2,$$

$$m_5^4 s_{12}^2-4 m_4^2 m_5^2 s_{12} s_{45}-2 m_5^2 s_{12}^2 s_{45}+2 m_5^2 s_{12} s_{34} s_{45}+s_{12}^2 s_{45}^2-2 s_{12} s_{34} s_{45}^2+s_{34}^2 s_{45}^2,$$

$$m_4^2 m_5^2 s_{23}-m_5^4 s_{23}-m_4^2 s_{15} s_{23}+m_5^2 s_{15} s_{23}-m_5^2 s_{23}^2-m_4^2 m_5^2 s_{45}+m_4^2 s_{15} s_{45}+m_5^2 s_{15} s_{45}-s_{15}^2 s_{45}+m_5^2 s_{23} s_{45}+s_{15} s_{23} s_{45}-s_{15} s_{45}^2,$$

$$m_4^4 s_{12}-m_4^2 m_5^2 s_{12}+m_4^2 s_{12}^2-m_4^2 s_{12} s_{34}+m_5^2 s_{12} s_{34}+m_4^2 m_5^2 s_{45}-m_4^2 s_{12} s_{45}-m_4^2 s_{34} s_{45}-m_5^2 s_{34} s_{45}-s_{12} s_{34} s_{45}+s_{34}^2 s_{45}+s_{34} s_{45}^2,$$

$$m_5^4 s_{12} s_{23}+m_5^4 s_{23}^2+m_4^2 m_5^2 s_{12} s_{45}-m_5^2 s_{12} s_{15} s_{45}-m_5^2 s_{12} s_{23} s_{45}-2 m_5^2 s_{15} s_{23} s_{45}+m_5^2 s_{23} s_{34} s_{45}+s_{12} s_{15} s_{45}^2+s_{15}^2 s_{45}^2-s_{15} s_{34} s_{45}^2,$$

$$m_4^4 s_{12}^2+m_4^4 s_{12} s_{23}+m_4^2 s_{12} s_{15} s_{45}+m_4^2 m_5^2 s_{23} s_{45}-m_4^2 s_{12} s_{23} s_{45}-2 m_4^2 s_{12} s_{34} s_{45}-m_4^2 s_{23} s_{34} s_{45}-s_{15} s_{34} s_{45}^2+s_{23} s_{34} s_{45}^2+s_{34}^2 s_{45}^2,$$

$$m_4^4 s_{12}^2+2 m_4^2 m_5^2 s_{12} s_{23}+m_5^4 s_{23}^2+2 m_4^2 s_{12} s_{15} s_{45}-2 m_5^2 s_{15} s_{23} s_{45}-2 m_4^2 s_{12} s_{34} s_{45}+2 m_5^2 s_{23} s_{34} s_{45}+s_{15}^2 s_{45}^2-2 s_{15} s_{34} s_{45}^2+s_{34}^2 s_{45}^2,$$

$$m_4^2 s_{12}^2+m_4^2 s_{12} s_{15}-m_5^2 s_{12} s_{15}+m_4^2 s_{12} s_{23}+m_5^2 s_{12} s_{23}+m_4^2 s_{15} s_{23}-m_5^2 s_{15} s_{23}+m_5^2 s_{23}^2-m_4^2 s_{12} s_{34}+m_5^2 s_{12} s_{34}-$$

$$m_4^2 s_{23} s_{34}+m_5^2 s_{23} s_{34}+s_{12} s_{15} s_{45}+s_{15}^2 s_{45}-s_{12} s_{23} s_{45}-s_{15} s_{23} s_{45}-s_{12} s_{34} s_{45}-2 s_{15} s_{34} s_{45}+s_{23} s_{34} s_{45}+s_{34}^2 s_{45},$$

$$m_4^4 s_{12}^2-2 m_4^2 s_{12}^2 s_{15}+s_{12}^2 s_{15}^2+2 m_4^2 m_5^2 s_{12} s_{23}-2 m_4^2 s_{12}^2 s_{23}-4 m_4^2 s_{12} s_{15} s_{23}+2 m_5^2 s_{12} s_{15} s_{23}-2 s_{12}^2 s_{15} s_{23}+$$

$$m_5^4 s_{23}^2-2 m_5^2 s_{12} s_{23}^2+s_{12}^2 s_{23}^2+2 m_4^2 s_{12} s_{23} s_{34}-4 m_5^2 s_{12} s_{23} s_{34}+2 s_{12} s_{15} s_{23} s_{34}-2 m_5^2 s_{23}^2 s_{34}-2 s_{12} s_{23}^2 s_{34}+$$

$$s_{23}^2 s_{34}^2+2 m_4^2 s_{12} s_{15} s_{45}-2 s_{12} s_{15}^2 s_{45}-2 m_5^2 s_{15} s_{23} s_{45}+2 s_{12} s_{15} s_{23} s_{45}-2 m_4^2 s_{12} s_{34} s_{45}+2 s_{12} s_{15} s_{34} s_{45}+$$

$$2 m_5^2 s_{23} s_{34} s_{45}+2 s_{12} s_{23} s_{34} s_{45}+2 s_{15} s_{23} s_{34} s_{45}-2 s_{23} s_{34}^2 s_{45}+s_{15}^2 s_{45}^2-2 s_{15} s_{34} s_{45}^2+s_{34}^2 s_{45}^2\}$$

Algebraic letters

- 62 algebraic letters

- An example

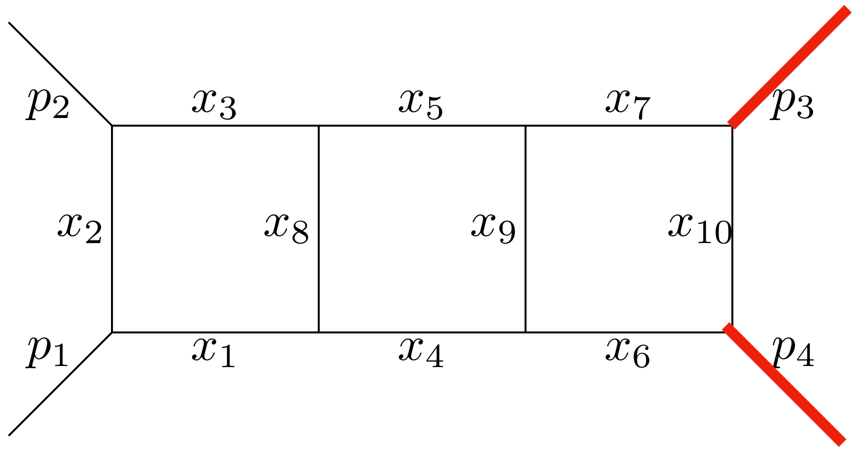
$$\mathrm{dlog}\Big[-\Big(\Big(2\, m_4^2 m_5^2 s_{12} - m_5^4 s_{12} + m_5^2 s_{12}^2 - m_5^2 s_{12} s_{34} + m_5^2 s_{12} s_{45} - s_{12}^2 s_{45} + m_5^2 s_{34} s_{45} + 2\, s_{12} s_{34} s_{45} - s_{34}^2 s_{45} + \sqrt{(m_5^4 - 2\, m_5^2 s_{12} + s_{12}^2 - 2\, m_5^2 s_{34} - 2\, s_{12} s_{34} + s_{34}^2)\,(m_5^4 s_{12}^2 - 4\, m_4^2 m_5^2 s_{12} s_{45} - 2\, m_5^2 s_{12}^2 s_{45} + 2\, m_5^2 s_{12} s_{34} s_{45} + s_{12}^2 s_{45}^2 - 2\, s_{12} s_{34} s_{45}^2 + s_{34}^2 s_{45}^2)}\Big)\Big/ \Big(-2\, m_4^2 m_5^2 s_{12} + m_5^4 s_{12} - m_5^2 s_{12}^2 + m_5^2 s_{12} s_{34} - m_5^2 s_{12} s_{45} + s_{12}^2 s_{45} - m_5^2 s_{34} s_{45} - 2\, s_{12} s_{34} s_{45} + s_{34}^2 s_{45} + \sqrt{(m_5^4 - 2\, m_5^2 s_{12} + s_{12}^2 - 2\, m_5^2 s_{34} - 2\, s_{12} s_{34} + s_{34}^2)\,(m_5^4 s_{12}^2 - 4\, m_4^2 m_5^2 s_{12} s_{45} - 2\, m_5^2 s_{12}^2 s_{45} + 2\, m_5^2 s_{12} s_{34} s_{45} + s_{12}^2 s_{45}^2 - 2\, s_{12} s_{34} s_{45}^2 + s_{34}^2 s_{45}^2)}\Big)\Big)\Big]$$

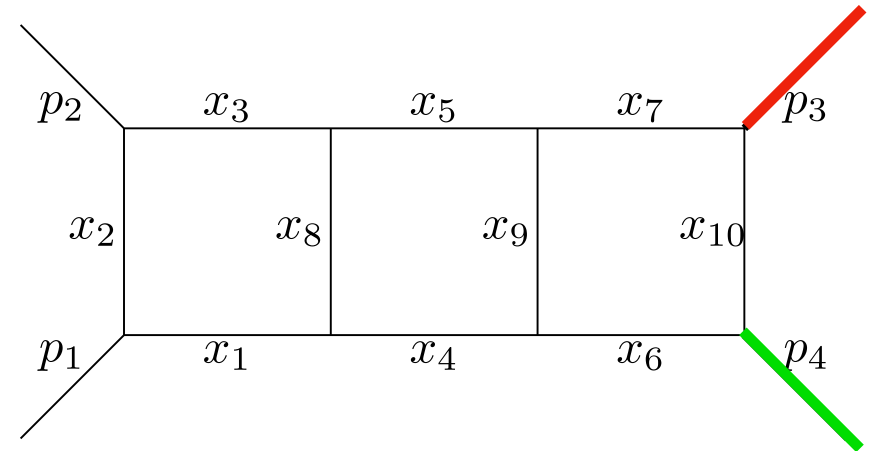
Constructed from

$$d\log\left(\frac{G\left(\{p_1+p_2,p_3+p_4,l_2\},\{p_1+p_2,p_3+p_4,p_4\}\right)+\sqrt{-G\left(p_1+p_2,p_3+p_4\right)G\left(l_2,p_1+p_2,p_4,p_3+p_4\right)}}{G\left(\{p_1+p_2,p_3+p_4,l_2\},\{p_1+p_2,p_3+p_4,p_4\}\right)-\sqrt{-G\left(p_1+p_2,p_3+p_4\right)G\left(l_2,p_1+p_2,p_4,p_3+p_4\right)}}\right)$$

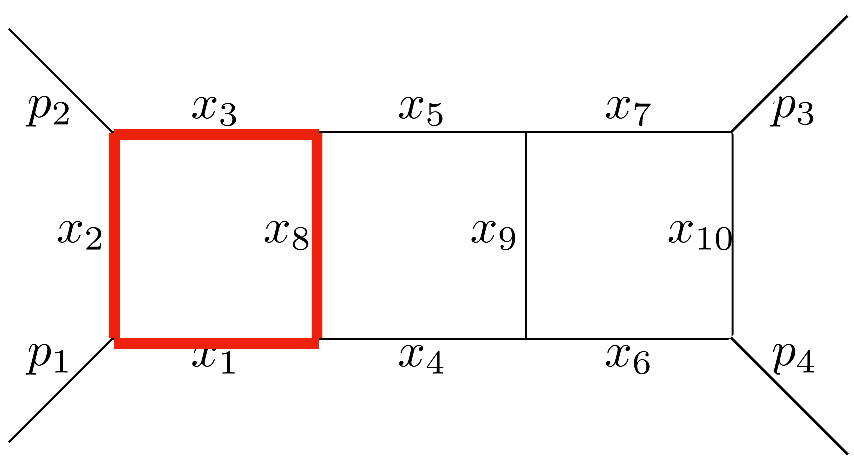
with $\{x_1\to 0, x_2\to 0, x_3\to 0, x_4\to 0, x_5\to 0, x_6\to 0, x_7\to 0, x_8\to 0, x_9\to 0, x_{10}\to 0, x_{11}\to s_{12}\}$

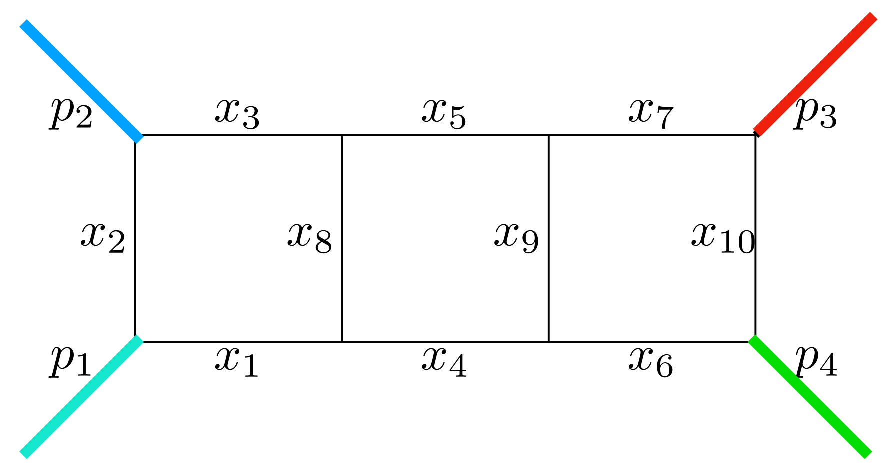
Three-loop trip-box with masses





Checked with CDEs





The letters for first case:

$$\left\{m^2, s, t, m^2 - t, 4m^2 - s, m^2 - s - t, 2m^2 - s - t, m^4 - m^2 t + st, m^4 - 2m^2 t + t(s+t)\right\}$$

$$\{d\log\left(-\frac{\sqrt{s\left(s-4m^2\right)} - 2m^2 + s}{\sqrt{s\left(s-4m^2\right)} + 2m^2 - s}\right), d\log\left(-\frac{\sqrt{s\left(s-4m^2\right)} - 2m^2 + s + 2t}{\sqrt{s\left(s-4m^2\right)} + 2m^2 - s - 2t}\right)\}$$

Summary and outlook

Summary

1. Recursive structure of Baikov representation
2. Present an algorithmic approach for constructing symbol letters
3. Bootstrap two loop five-points and three loop four points integrals

Outlook

1. Explore the relationship between our method and Schubert problems
2. Investigate Landau singularities in Baikov representation
3. Bootstrap Feynman integrals with letters

Feynman integrals in embedding space

- Embed a vector in d dimension to d+2

[P.A.M.Dirac' 1936,
S.Weinberg' 2010,
D. Simmons-Duffin' 2014]

$$X^I \equiv (x^\mu, X^-, X^+)\,,$$

with $(XY) = 2x \cdot y + X^-Y^+ + X^+Y^-\,.$

- Consider a propagator $(l - p_i)^2 - m_i^2$

$$Y_i \equiv (l^\mu, l^2, 1)\,, \qquad X_i = \{p_i, -p_i^2 + m_i^2, 1\}$$

- Integration measure

$$\int d^d l \to \int \frac{\langle Y d^{d+1} Y\rangle}{(Y X_I)^d}\delta(Y^2)$$

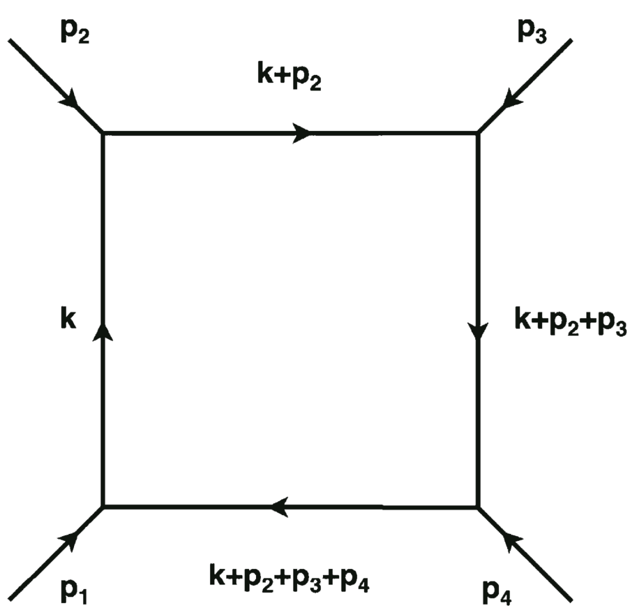
with $\langle Y d^{d+1} Y\rangle = \epsilon_{I_1 I_2 \cdots I_{d+2}} Y^{I_1} dY^{I_2} \cdots dY^{I_{d+2}}$ and $X_I \equiv (0,1,0)$

Schubert problems

[S. He, X. Jiang, J. Liu, and Q. Yang' 2023]

- Introduce 4 embedding vectors $\{X_0, X_1, X_2, X_3\}$

- Schubert problems for box integrals



$$Y X_0 = Y X_1 = Y X_2 = Y X_3 = Y^2 = 0$$

- Schubert problems for triangle integrals

$$Y X_0 = Y X_1 = Y X_2 = Y X_I = Y^2 = 0$$

- Letters as cross ratio

$$\frac{(Y_1^+ Y_2^+)(Y_1^- Y_2^-)}{(Y_1^+ Y_2^-)(Y_1^- Y_2^+)}$$

Schubert problems

- Dual vectors

$$\tilde{X}_0^{M_0} = \epsilon^{M_0 M_1 M_2 M_3 M_4 M_5} X_1^{M_1} \ldots X_5^{M_5} ,$$
$$\tilde{X}_1^{M_0} = \epsilon^{M_0 M_1 M_2 M_3 M_4 M_5} X_0^{M_1} X_2^{M_2} \ldots X_5^{M_5} ,$$
$$\vdots$$
$$\tilde{X}_5^{M_0} = \epsilon^{M_0 M_1 M_2 M_3 M_4 M_5} X_0^{M_1} X_1^{M_2} \ldots X_4^{M_5} ,$$

$$X_4 \equiv X_I$$

with following relations

$$X_i \tilde{X}_j = (-1)^i X_0 \wedge X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \delta_{i,j} ,$$
$$\tilde{X}_i \tilde{X}_j = G\left(\{X_0, \ldots, X_{i-1}, X_{i+1}, X_5\}, \{X_0, \ldots, X_{j-1}, X_{j+1}, X_5\}\right) .$$

- Solutions and letters

$$Y_{\text{box}} = \tilde{X}_4 + c_{\text{box}} \tilde{X}_5$$
$$Y_{\text{tri}} = \tilde{X}_3 + c_{\text{tri}} \tilde{X}_5$$

$$\frac{\left(G\left(\{X_0, X_1, X_2, X_3\}, \{X_0, X_2, X_3, X_4\}\right) + \sqrt{G\left(X_0, X_1, X_2, X_3\right) G\left(X_0, X_2, X_3, X_4\right)}\right)^2}{\left(G\left(\{X_0, X_1, X_2, X_3\}, \{X_0, X_2, X_3, X_4\}\right) - \sqrt{G\left(X_0, X_1, X_2, X_3\right) G\left(X_0, X_2, X_3, X_4\right)}\right)^2}$$