Recursive Landau Analysis

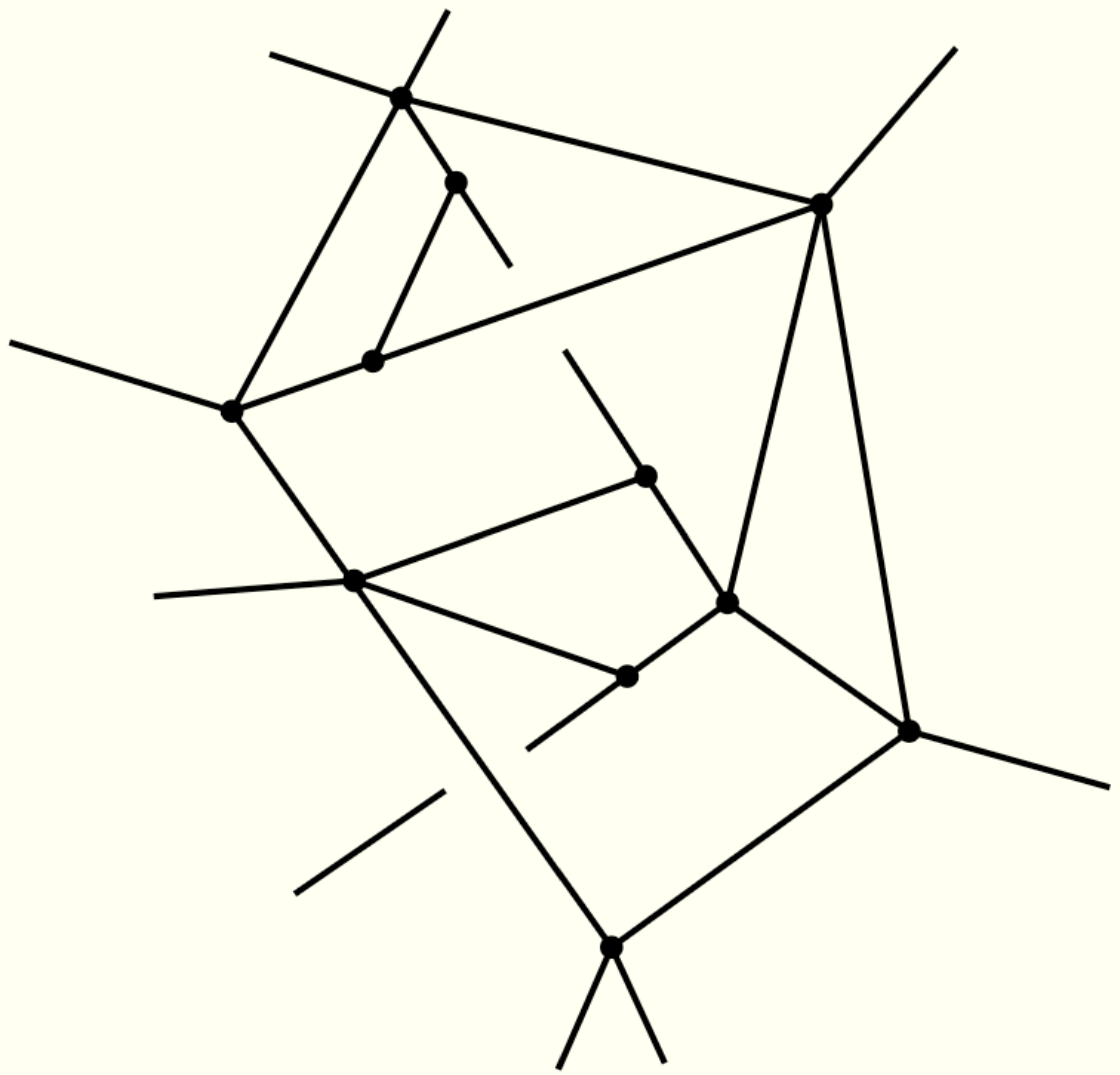
Mathieu Giroux (McGill)

with Simon Caron-Huot and Miguel Correia

ArXiv: 2406.05241

Let's first set up the stage

$G =$



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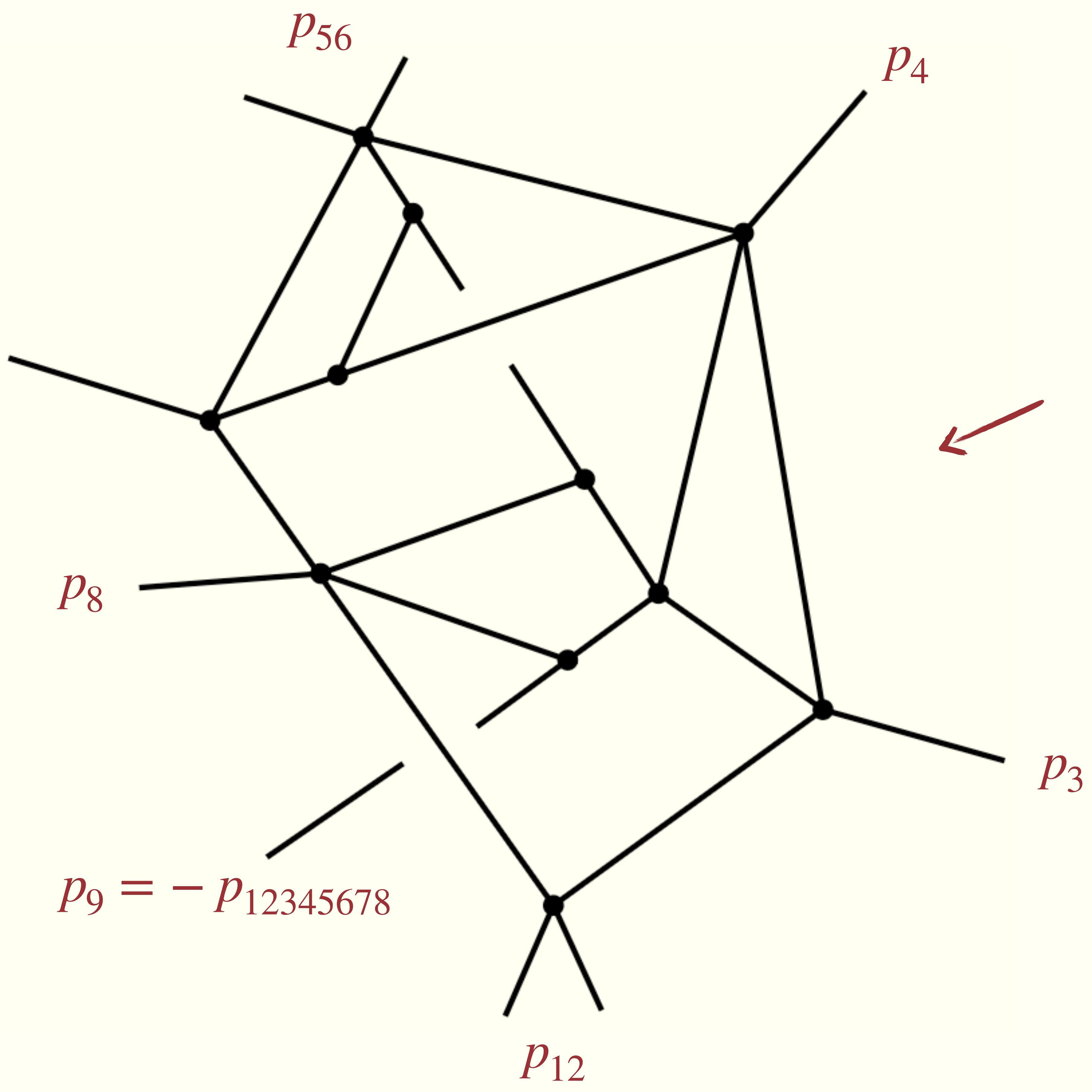

A function of $X_G = \{p_i \cdot p_j\}_{i,j=1}^{n-1}$ and internal masses on the kinematic space

$$p_I \equiv \sum_{i \in I} p_i$$
$$s_I \equiv p_I^2$$

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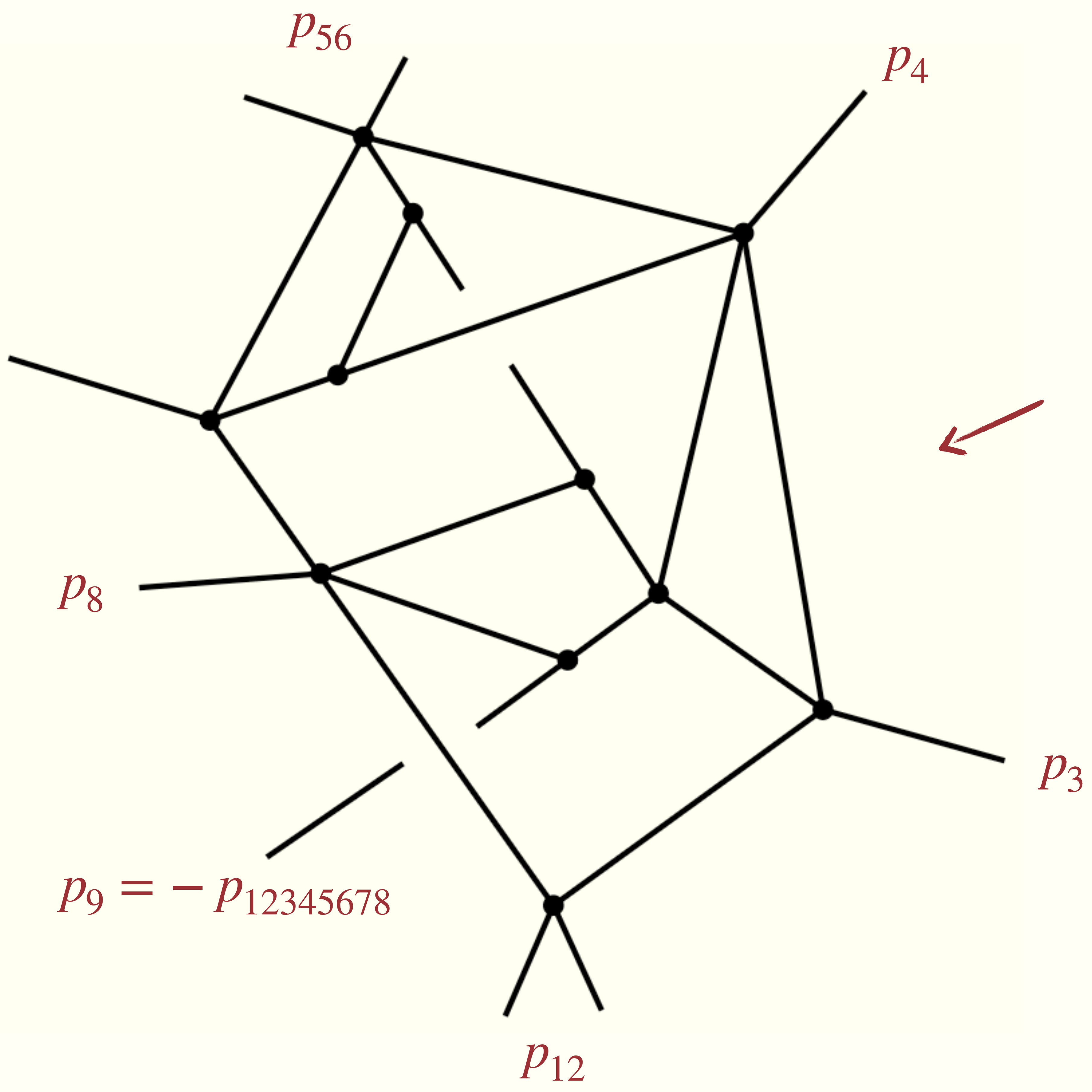

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What's the analytic structure of G ?

LET'S FIRST SET UP THE STAGE

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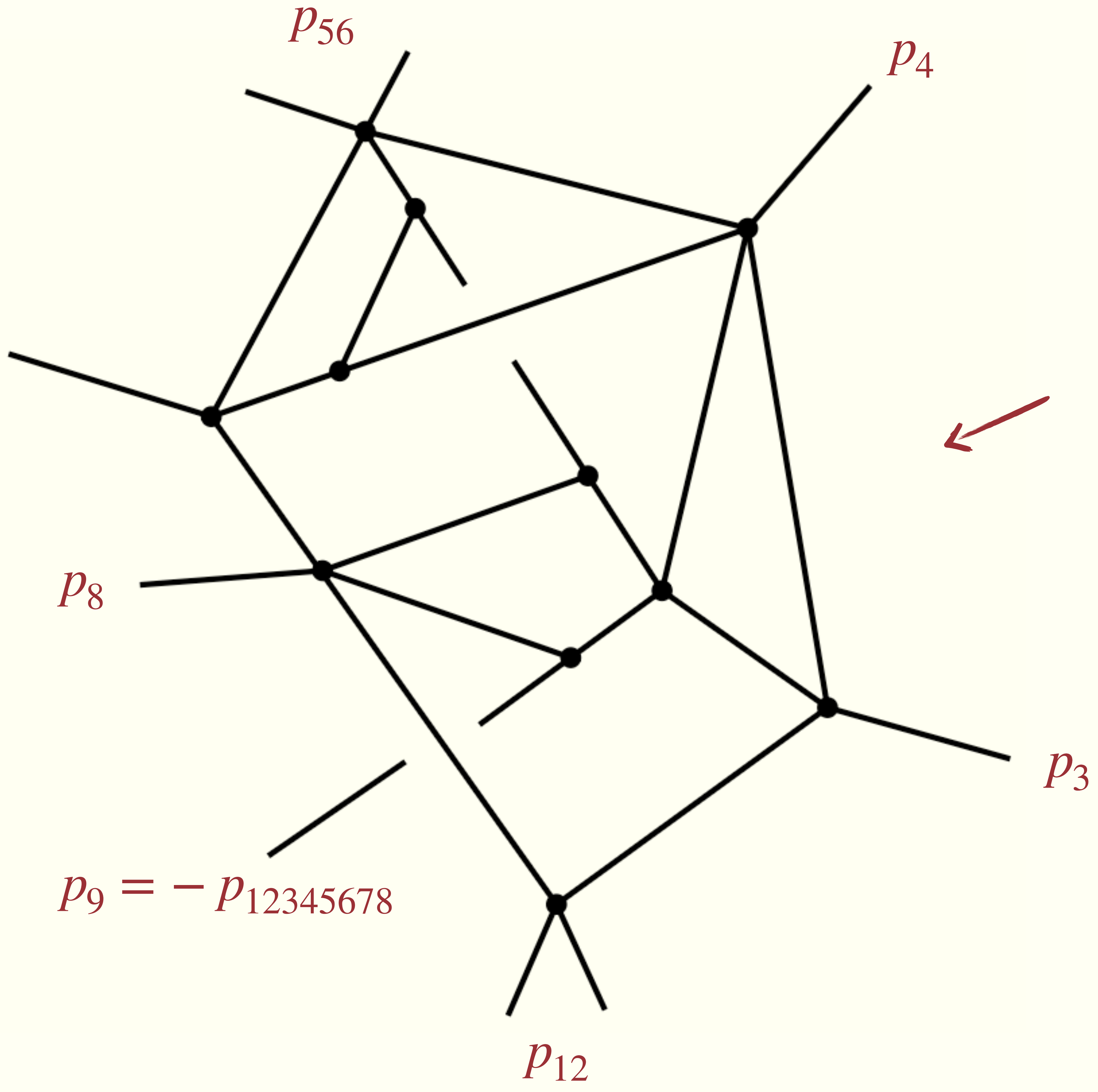

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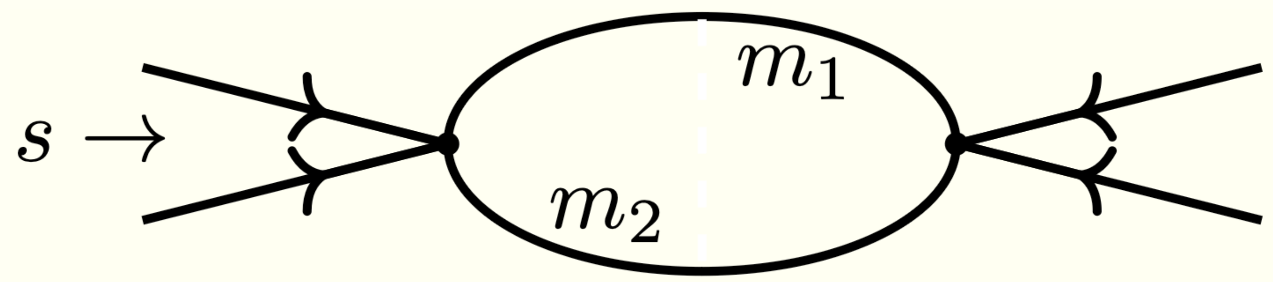
$$p_I \equiv \sum_{i \in I} p_i$$
$$s_I \equiv p_I^2$$

In other words, where are its *kinematic* singularities ?

Let us make sure we are on the same page

Normal threshold

(± branches)



$$\overset{D=3}{=} \frac{\sqrt{\pi}}{\sqrt{s}} \log\left[\frac{m_1 + m_2 + \sqrt{s}}{m_1 + m_2 - \sqrt{s}}\right]$$

Second type

singularity

Well understood at one-loop; can be much *harder* beyond!

Having good control over this question would be enormously useful for

Differential equations and numerical integration of Feynman integrals
(boundary conditions, analytic continuation and contour deformations)

[See Franziska's, Hayden's and Samuel's talks]

Symbol calculus and bootstrap of Feynman integrals
(singularities constrain the letters)

[See Andrew's, Francois's and Xiaofeng's talks]

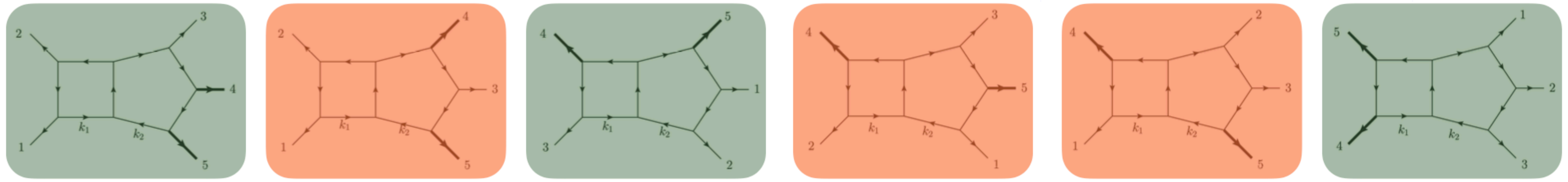
Knowing singularities *beforehand* has proven central for state-of-the-art phenomenological applications — e.g.,

[Samuel Abreu's slide]

Computing Feynman Integrals: Alphabets and Letters

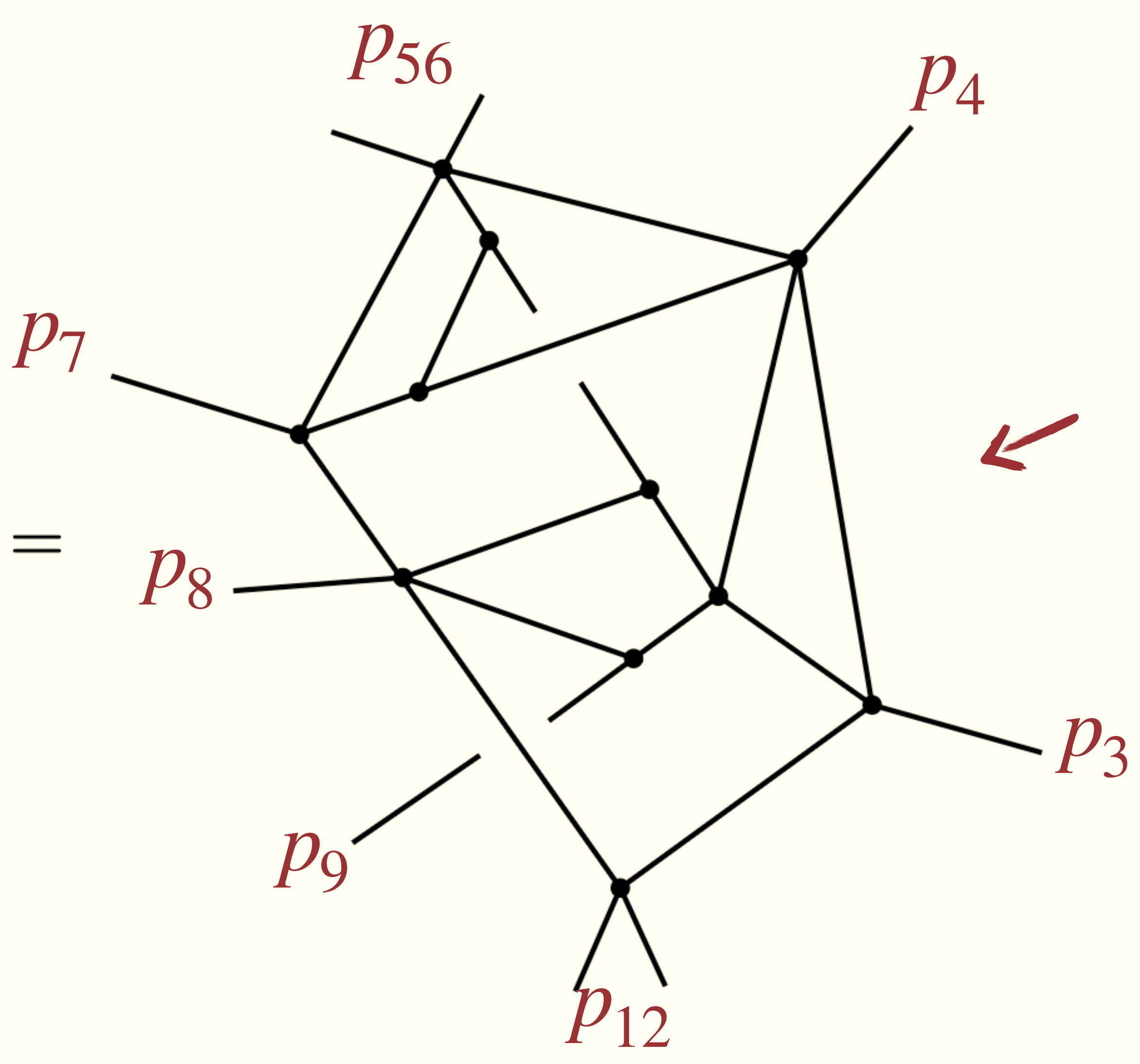
$$d\vec{\mathcal{J}}(x,\epsilon) = \epsilon\left(\sum_i A_i\, d\log W_i(x)\right)\vec{\mathcal{J}}(x,\epsilon)$$

- Getting diff. eq. relies on IBPs: difficult to do analytically...
- If the W_i are known, determine the A_i from numerical IBPs!
  - ✓ removes the IBP bottleneck, allows to attack multi-scale problems
- The W_i give singularities of Feynman integrals ⇒ Landau conditions
  - ✓ Factorisation of work: determine W_i without computing the differential equation!
  - ✓ Active area of research in Amplitudes area: coactions, solving Landau conditions, principal A-determinants, Gram determinants, Schubert problem, ...
  - ✓ Two highlights: **[2311.14669, Fevola, Mizera, Telen], [2401.07632, Jiang, Liu, Xu, Yang, 24]**
- Baikovletter [2401.07632] misses one of the new five-point roots
  - ✓ Not really an issue, we know it's there



+ related work by [Abreu, Caron-Huot, Chicherin, Dixon, Gehrmann, Henn, Ita, McLeod, Mitev, Moriello, Page, Presti, Sotnikov, Tschernow, von Hippel, Wasser, Wilhelm, Zhang, Zoia, …]

What's our goal ?

$G =$ 

A function of $X_G = \{p_i \cdot p_j\}_{i,j=1}^{n-1}$ and internal masses on the kinematic space

Singularities are written as a list $\mathcal{L}(G)$ of polynomials in X_G

$$\mathcal{L}(G)_i = 0$$

The product over i is called the *Landau discriminant*

[Fevola, Mizera, Telen (2023)]

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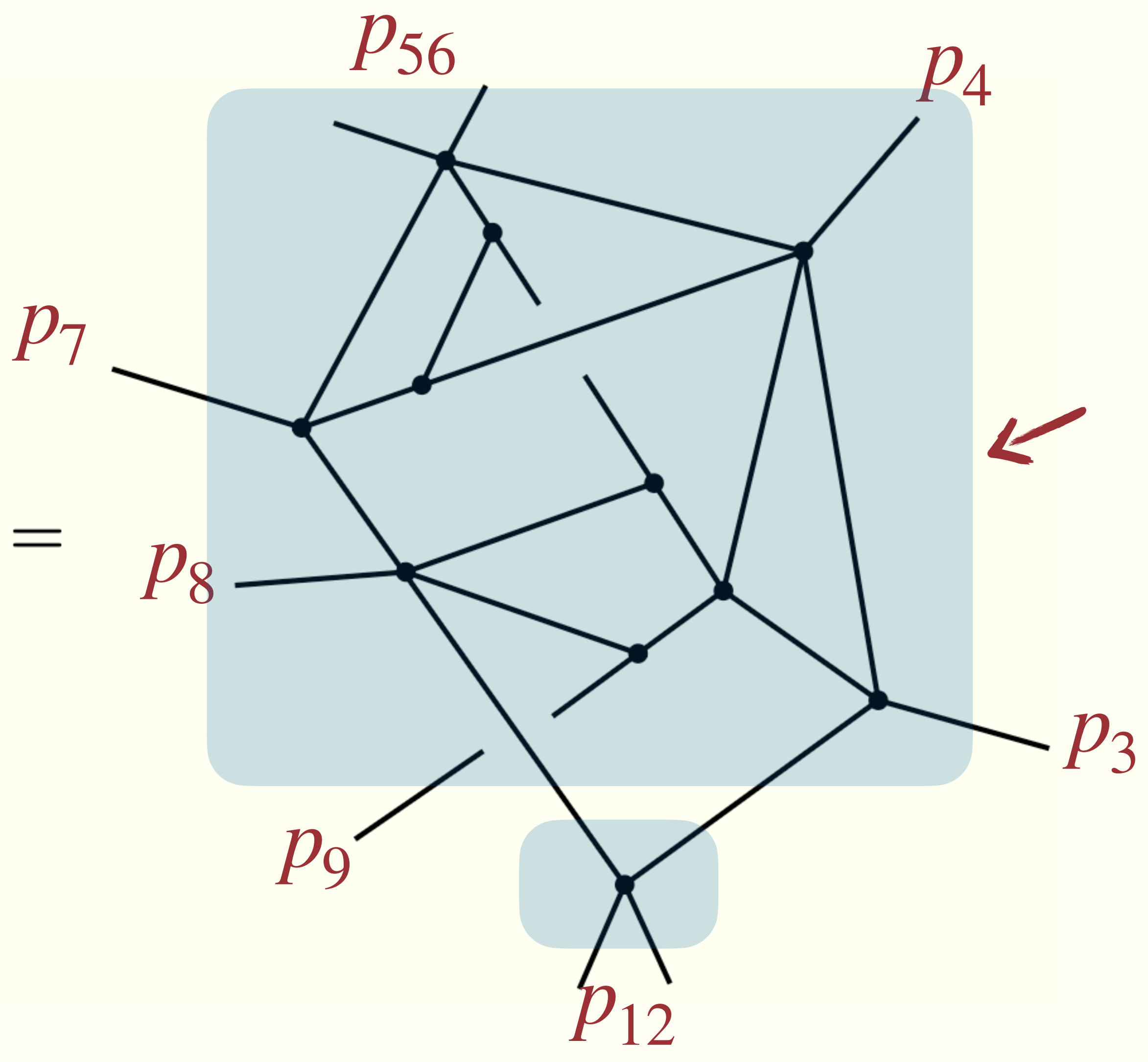
$G =$ 

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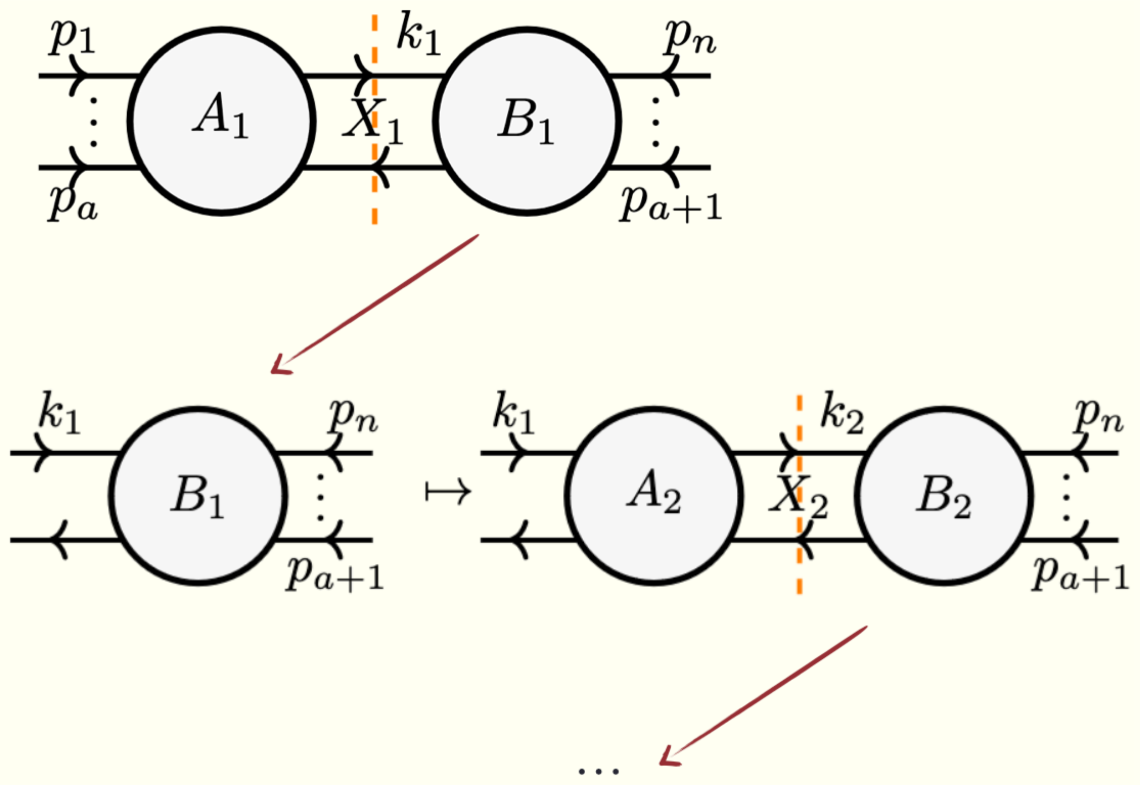
The goal of this talk is to learn how to compute these polynomials *recursively* in terms of those of subgraphs (we'll see that this is *surprisingly* efficient!)

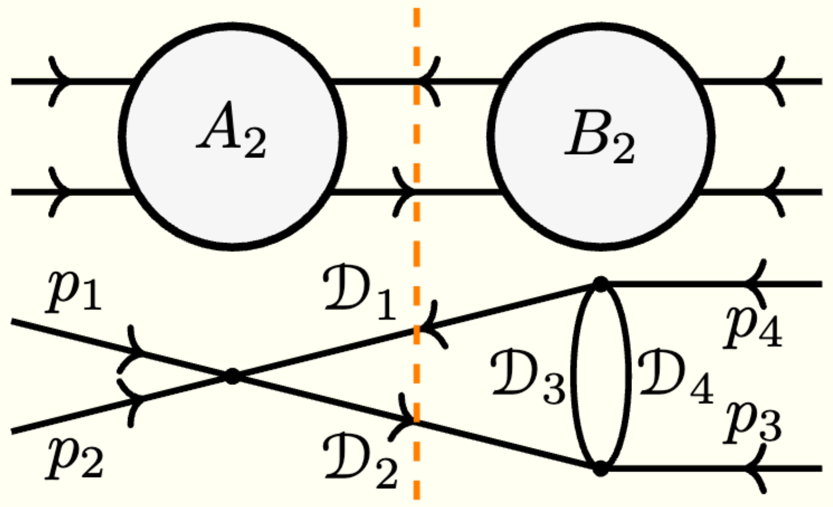
Outline

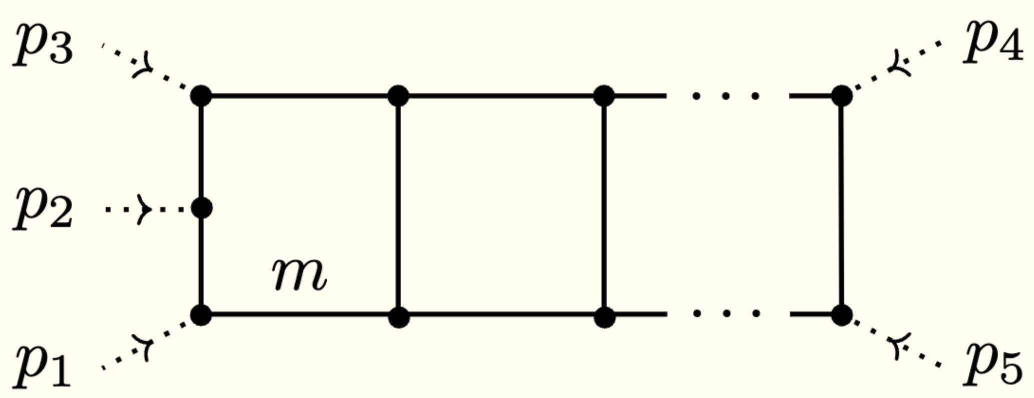
Recursion via unitarity



Proof of principle examples:

Recursively finding singularities

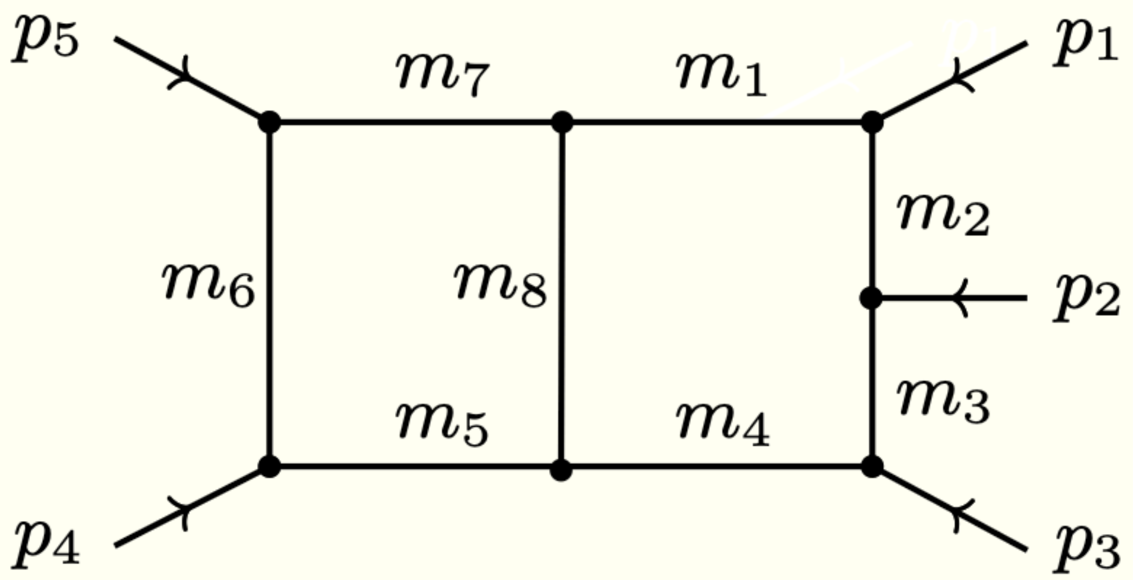




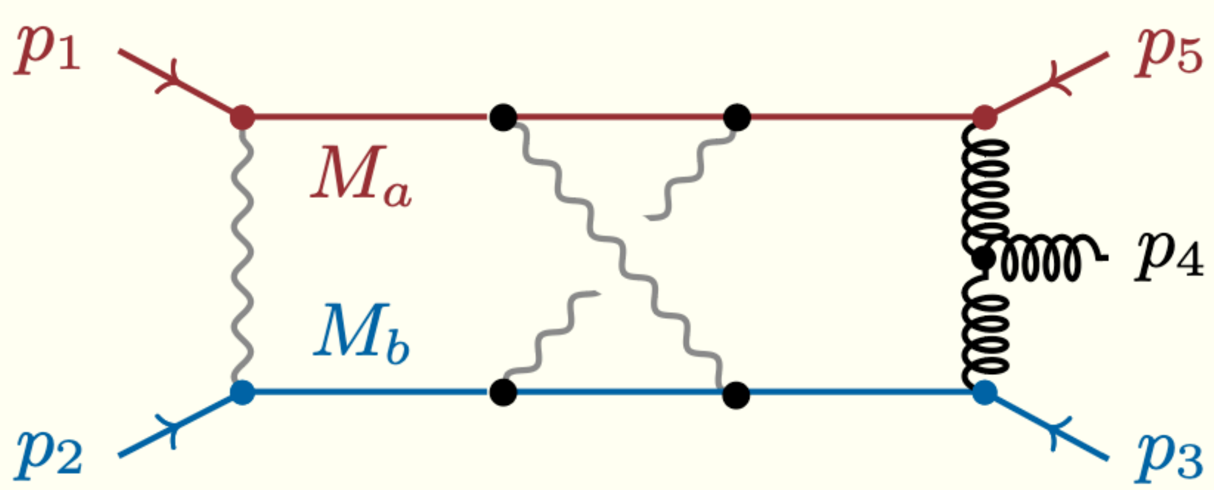
Checks and new analytic predictions:

Leading singularities

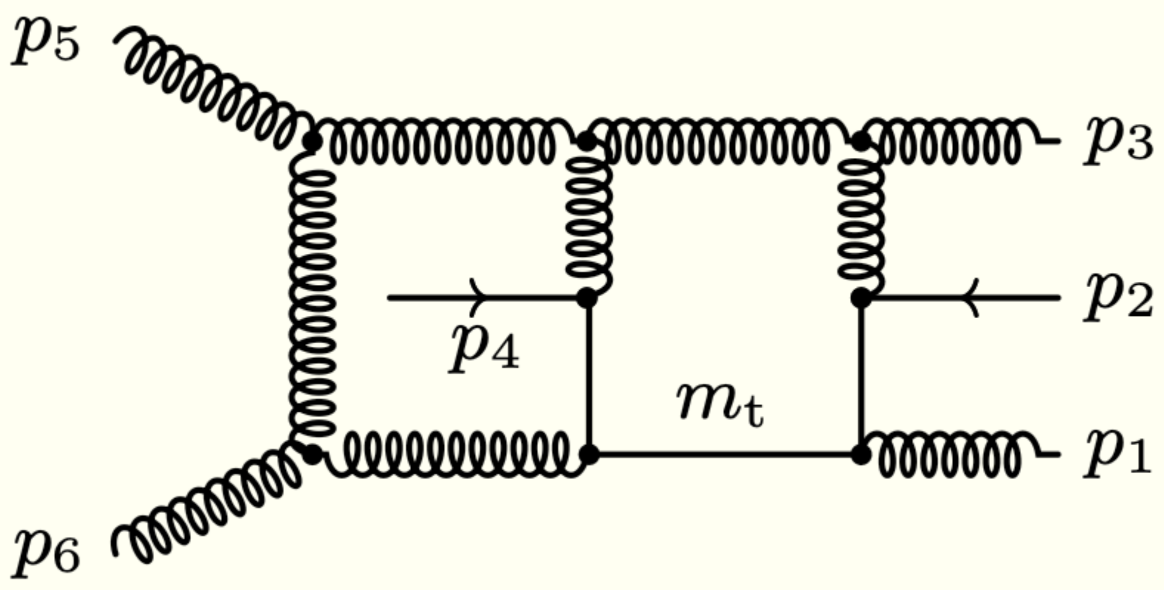
(Generic kinematic pentabox)



(Three-loop QED+QCD boX)



(Non-planar massive hexabox)



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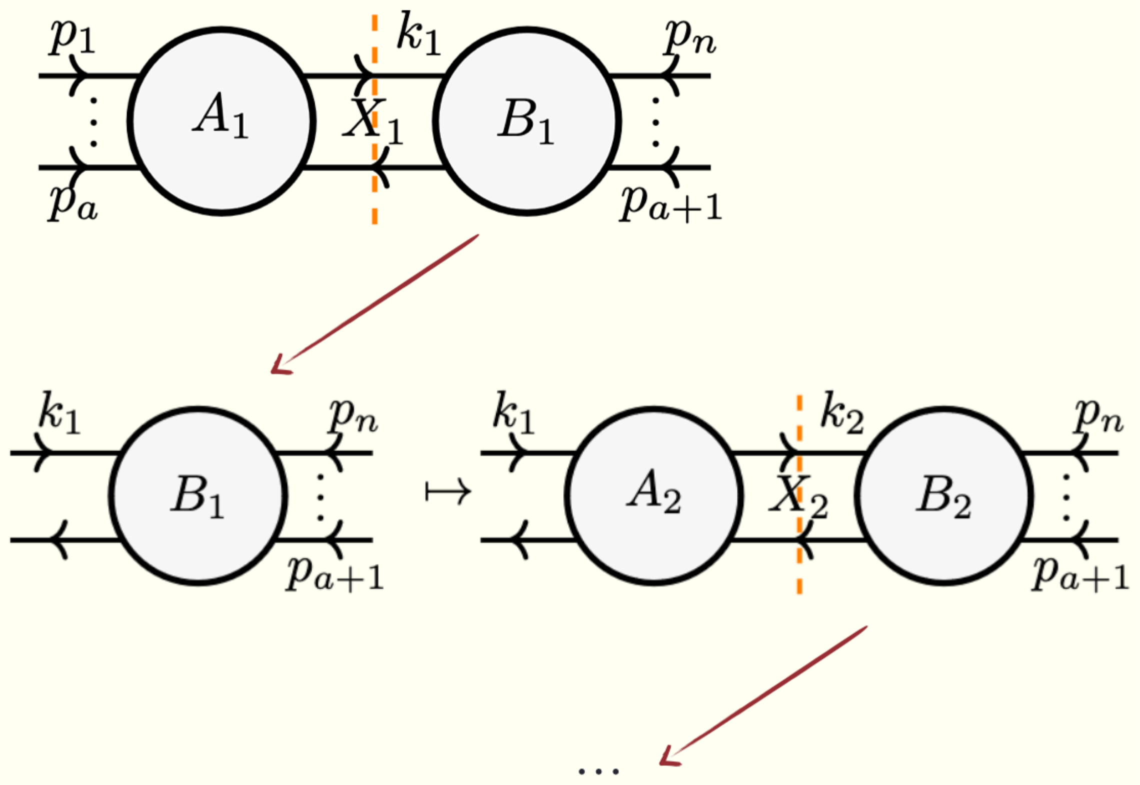

Proof of principle examples:

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Unitarity and thresholds

Unitarity of the S-matrix implies that

$$SS^\dagger = \mathbb{1}$$

$$S = \mathbb{1} + iT \implies \frac{1}{2i}(T - T^\dagger) = \frac{1}{2}TT^\dagger$$

↖ Separation between free and interacting parts

Unitarity and thresholds

Unitarity of the S-matrix implies that

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$$S = \mathbb{1} + iT$$

$$\Longrightarrow \qquad \operatorname{Im} T = \frac{1}{2} T T^\dagger$$

For the experts:
Assuming (for now) reality of
momenta and Feynman's *iε*

Unitarity and thresholds

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$$S = \mathbb{1} + iT \quad \Longrightarrow \quad \mathrm{Im}\, T = \frac{1}{2} T T^\dagger$$

Positivity manifests, but singularities are not

[Hannesdóttir, Mizera (2022)]

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$$S = \mathbb{1} + iT$$

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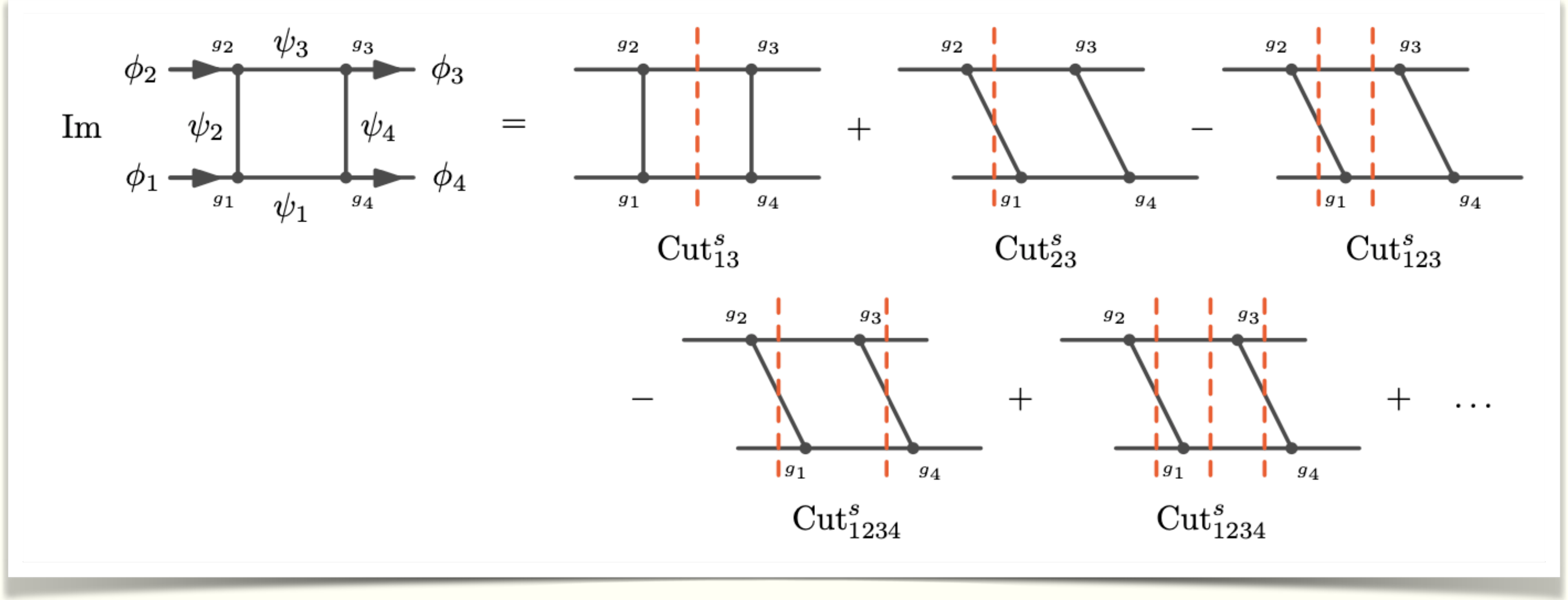
Insert a complete basis of (on-shell) states

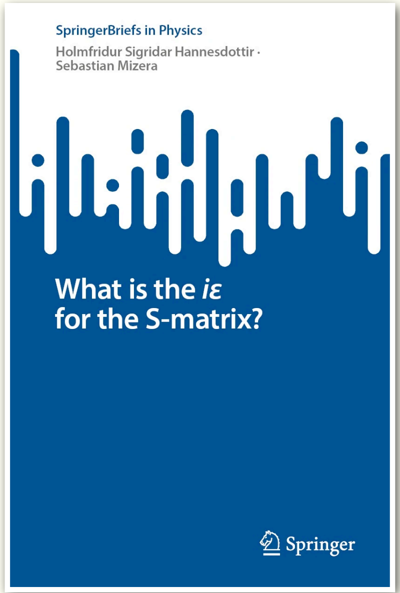
Unitarity and thresholds

At the level of the matrix elements $\mathcal{M}_{\text{in}\to\text{out}} \equiv \langle \text{out}|T|\text{in}\rangle$

$$\text{Im}\,\mathcal{M}_{n_A\to n_B} = \frac{1}{2}\sum\!\!\!\!\!\!\int_X \mathcal{M}_{n_A\to X}\,\mathcal{M}^*_{X\to n_B}$$

In perturbation theory, this gives the *Cutkosky equation*





[Cutkosky (1961), Hannesdóttir, Mizera (2022)]

Unitarity and thresholds

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$$\text{Im}\,\mathcal{M}_{n_A\to n_B} \;=\; \text{Sum over unitarity cuts}$$

The locations at which a cut starts contributing are called *thresholds*

Takeaway point

The imaginary part has support where cuts themselves have support

Unitarity and thresholds

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At these locations the amplitude cannot be real analytic, and we say that it is *singular*

Necessary conditions for singularities (I)

Qualitative necessary conditions

Amplitudes *can* be singular when (i) the phase space of cuts opens up, and (ii) when cuts are singular

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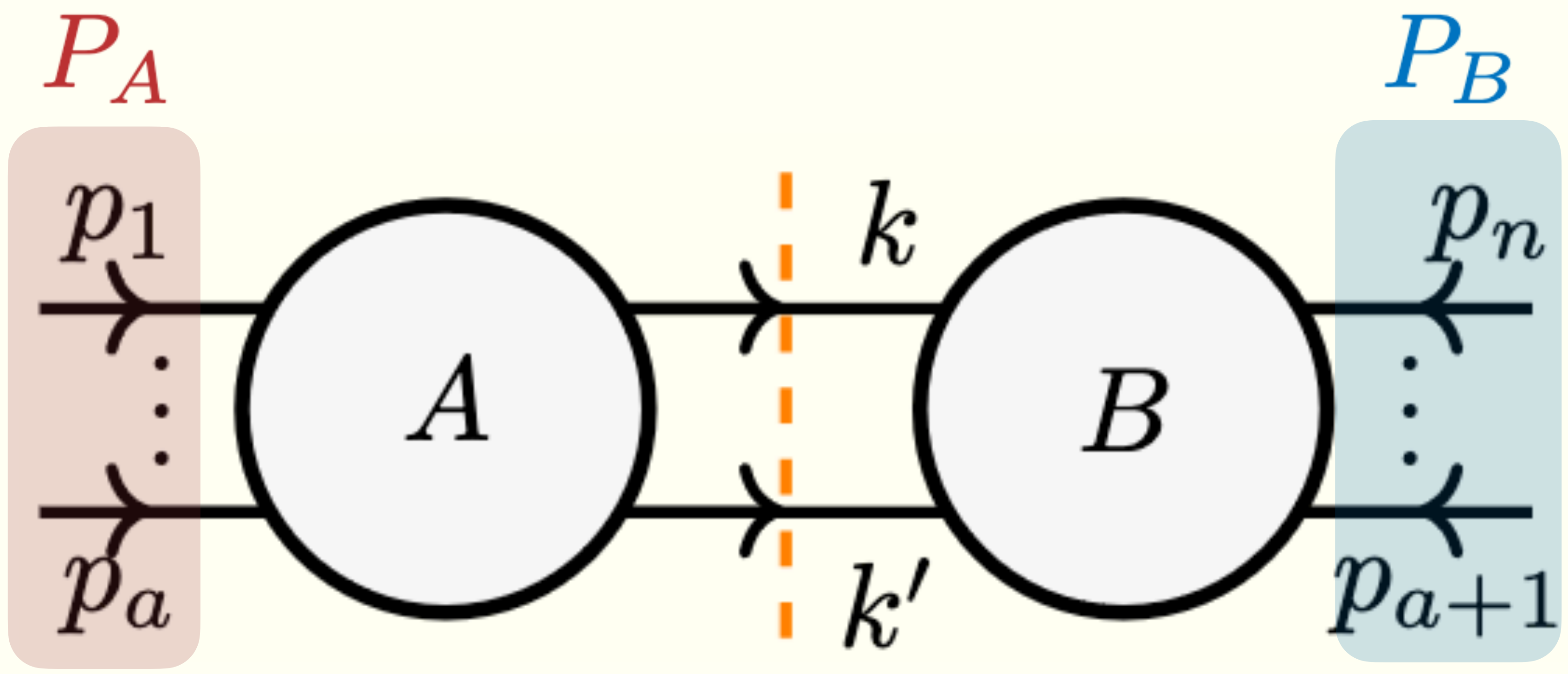
Our focus is on Feynman graphs AB that can be *disconnected* into two subgraphs A and B *two-particle cut*



The invariants on each side are

$$X_\xi = \{q_i \cdot q_j \mid q_\bullet \in \{k\} \cup P_\xi\}$$

$$(\xi = A, B)$$

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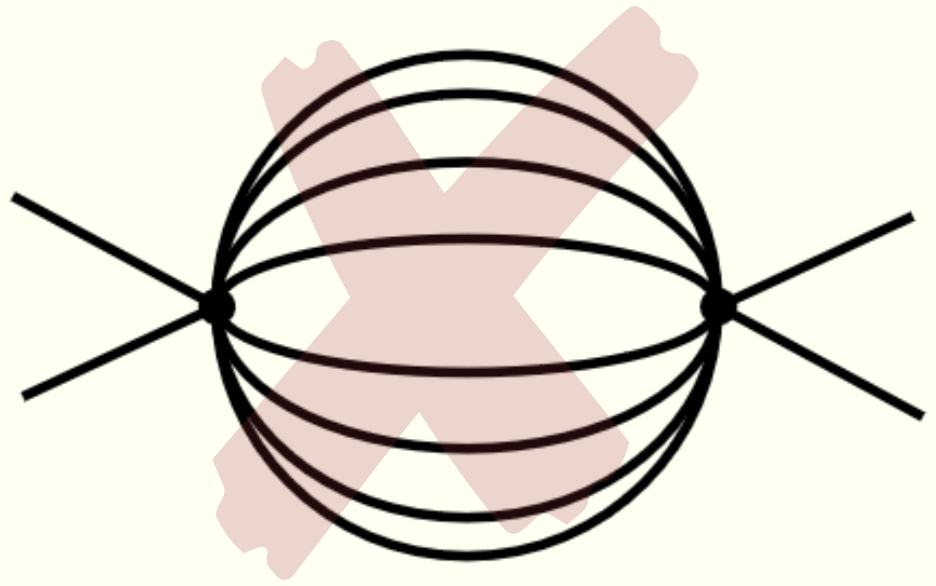
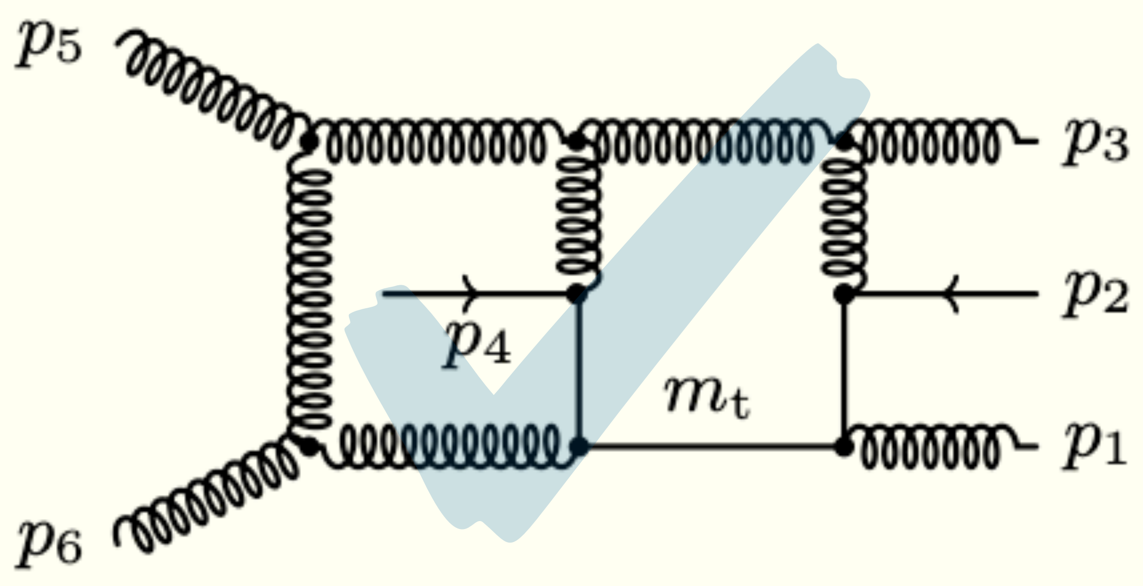


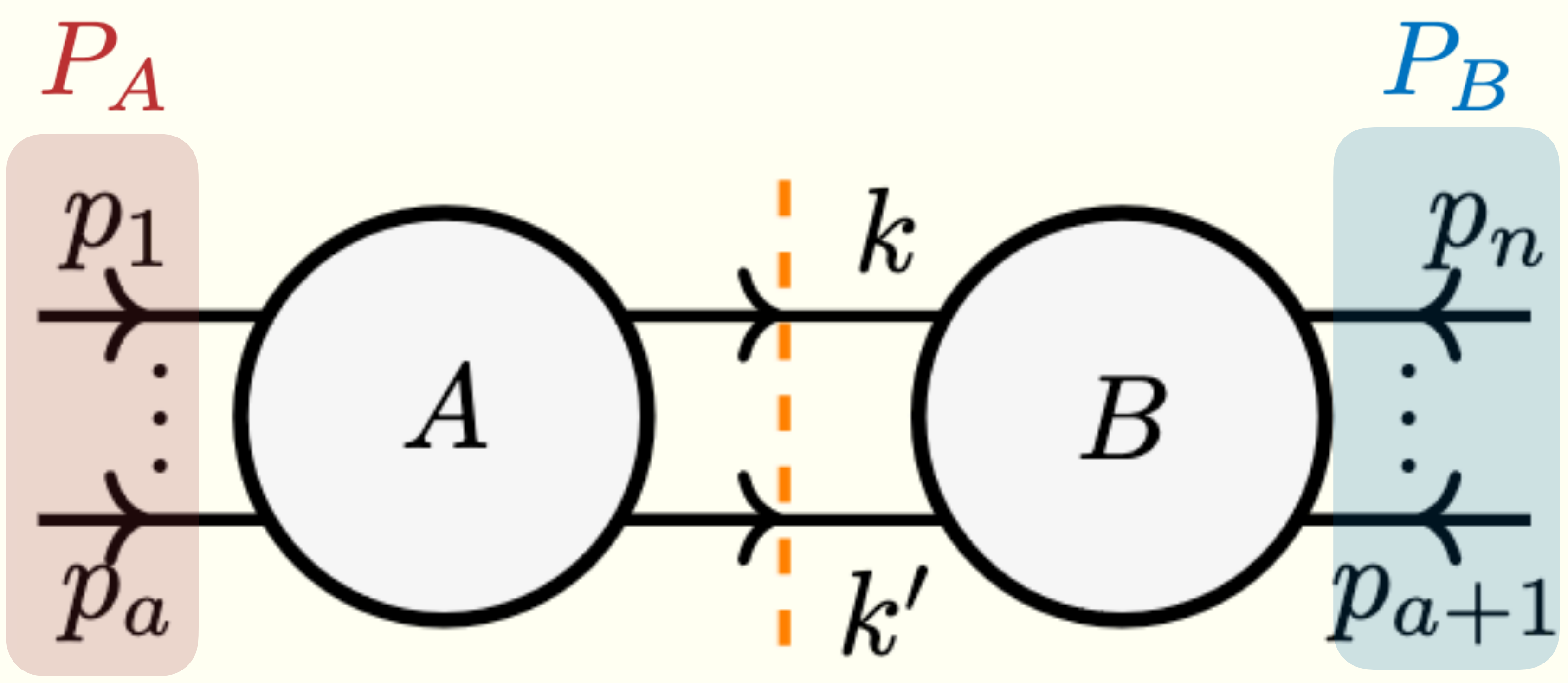


We'll learn how to compute singularities of Standard Model processes like this one!

Two-particle cuts in Baikov form

As an integral over independent the scalar products between loop and external momenta

Ask me later to fill the details!



$$= C \int_\Gamma \mathrm{d}\mu \frac{A(X_A)\, B(X_B)}{\det G(Q)^{\frac{n+1-\mathrm{D}}{2}}}$$

Two-particle cuts in Baikov form

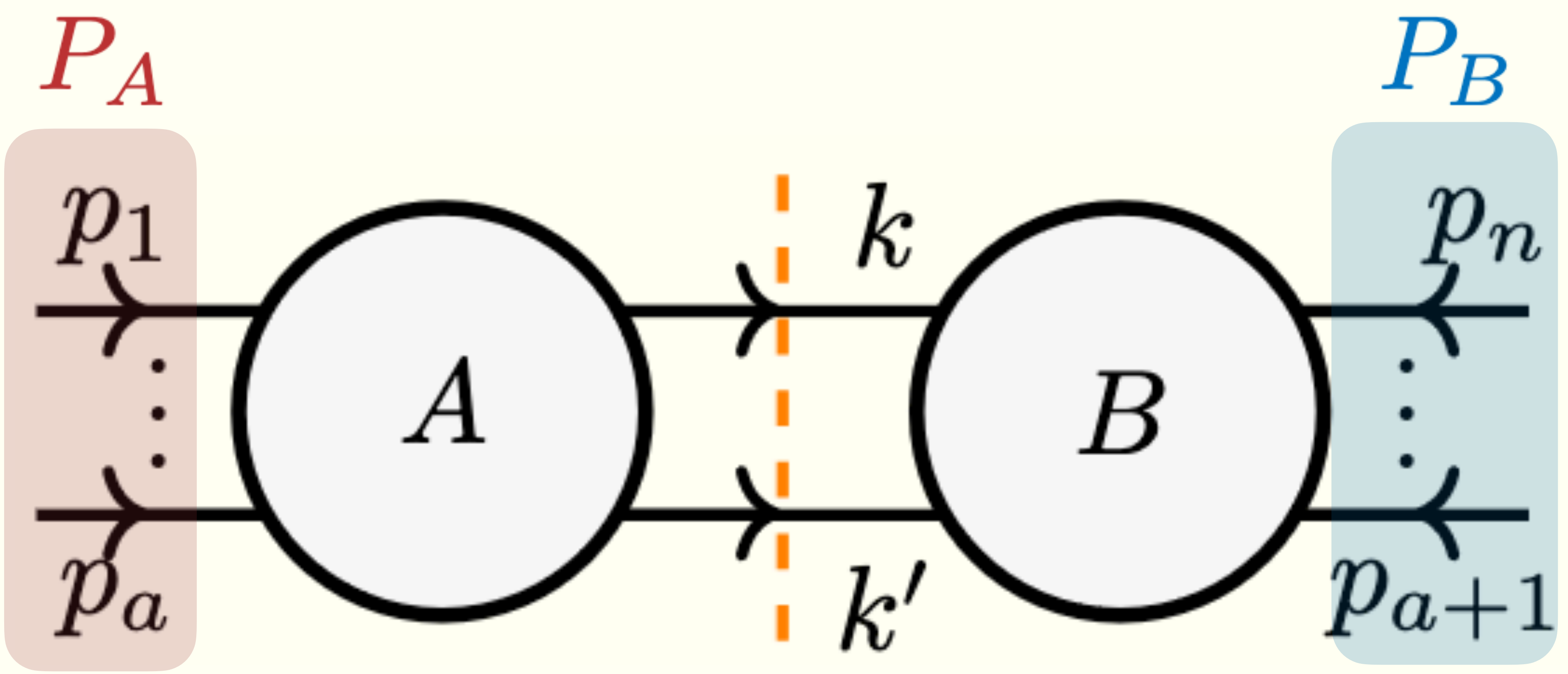
(The details I am skipping over)

Integration measure

$$\mathrm{d}\mu = \prod_{Q_i \in Q} \mathrm{d}(k \cdot Q_i)\ \delta[k^2 - m^2]\ \delta[(k + p_{1\ldots a})^2 - m^2]$$

Set of Baikov variables for the B-blob

$k^2,\ k \cdot P_B,\ P_B^2$



$$= C \int_\Gamma \mathrm{d}\mu \frac{A(X_A)\, B(X_B)}{\det G(Q)^{\frac{n+1-D}{2}}}$$

Gram determinant over

$Q = \{k\} \cup P_A \cup P_B \backslash \{p_n\}$

Normalization

$$C = \frac{\det G(P_A \cup P_B \backslash \{p_n\})^{\frac{n-D}{2}}}{\sqrt{\pi}}$$

Integration contour

$$\Gamma = \left\{ k \cdot Q_i \,\middle|\, \frac{\det G(Q)}{\det G(P_A \cup P_B \backslash \{p_n\})} > 0, Q_i \in Q \right\}$$

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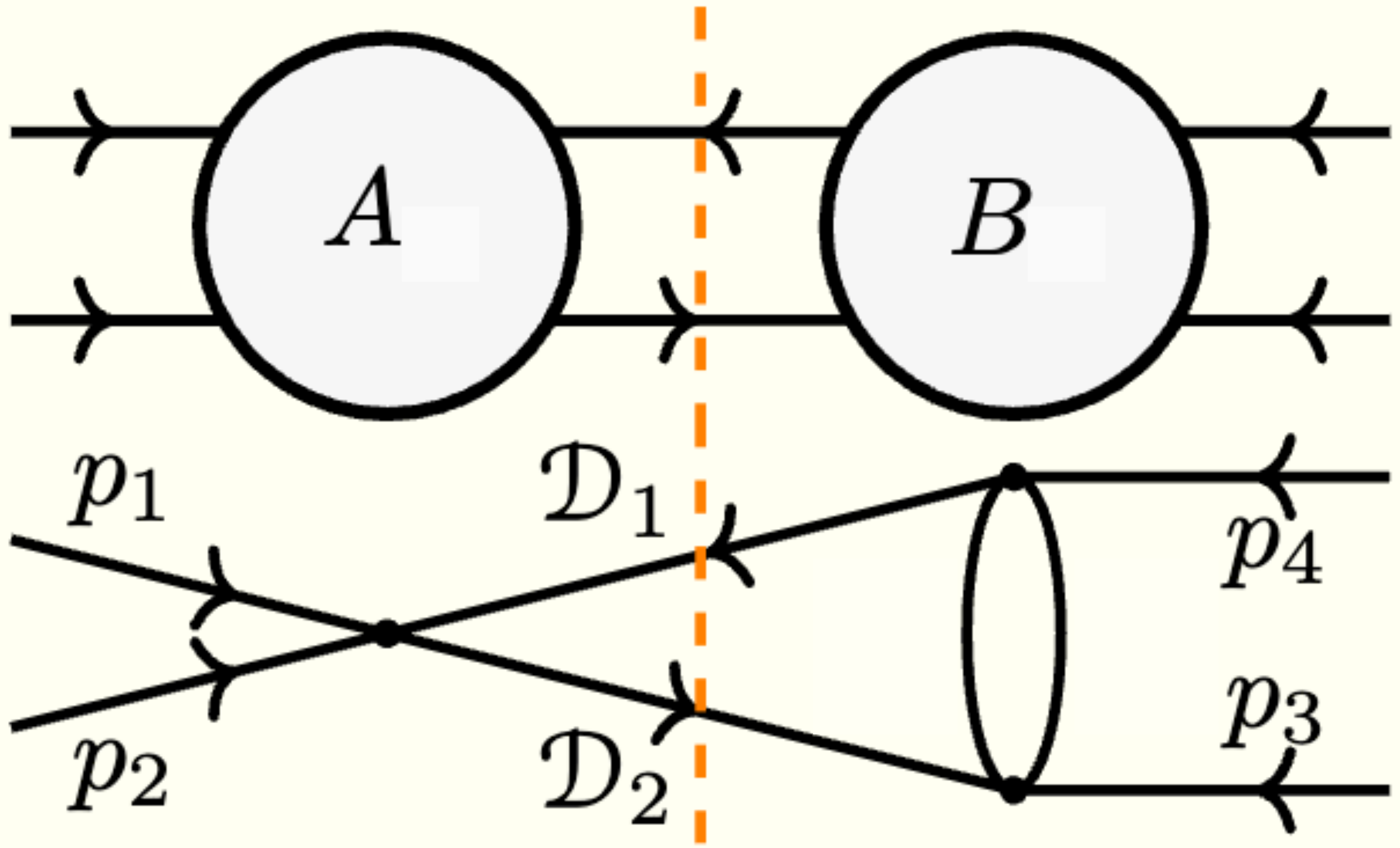

$$\propto \int_\Gamma \frac{\mathrm{d}\mu \;\; A \;\; B}{(\det G \;\;)^\gamma}$$

$$\mathrm{d}\mu \;\; = \mathrm{d}(k_1 \cdot p_{12}) \wedge \mathrm{d}(k_1 \cdot p_3) \wedge \mathrm{d}(k_1^2) \;\; \delta(\mathcal{D}_1)\delta(\mathcal{D}_2)$$

Vertex

Bubble integrand

$$G \;\; = \begin{bmatrix} p_{12}^2 & p_{12} \cdot k_1 & p_{12} \cdot p_3 \\ p_{12} \cdot k_1 & k_1^2 & k_1 \cdot p_3 \\ p_{12} \cdot p_3 & k_1 \cdot p_3 & p_3^2 \end{bmatrix}$$

Necessary conditions for singularities (II)

Qualitative necessary conditions

Amplitudes *can* be singular when (i) the phase space of cuts opens up, and (ii) when cuts are singular

What does it mean for two-particle cut ?

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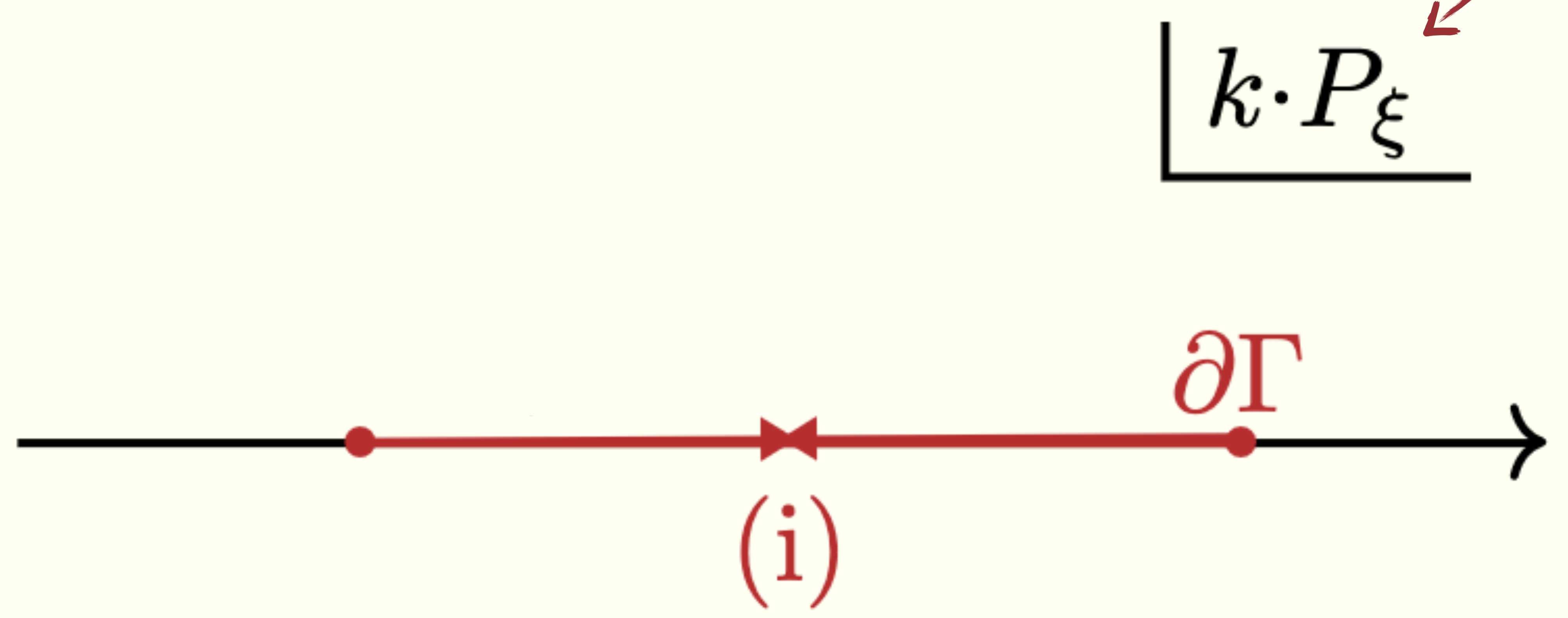
What does it mean for two-particle cut ?

(i) At thresholds, the phase space Γ *closes down* to a single isolated point (only classical scattering is possible)

Left over integration variables on the two-particle cut



$(\xi = A, B)$

Boundary $\partial\Gamma = \{\det G = 0\}$ collapses to a point
(i.e., from all directions)

NECESSARY CONDITIONS FOR SINGULARITIES (II)

Qualitative necessary conditions

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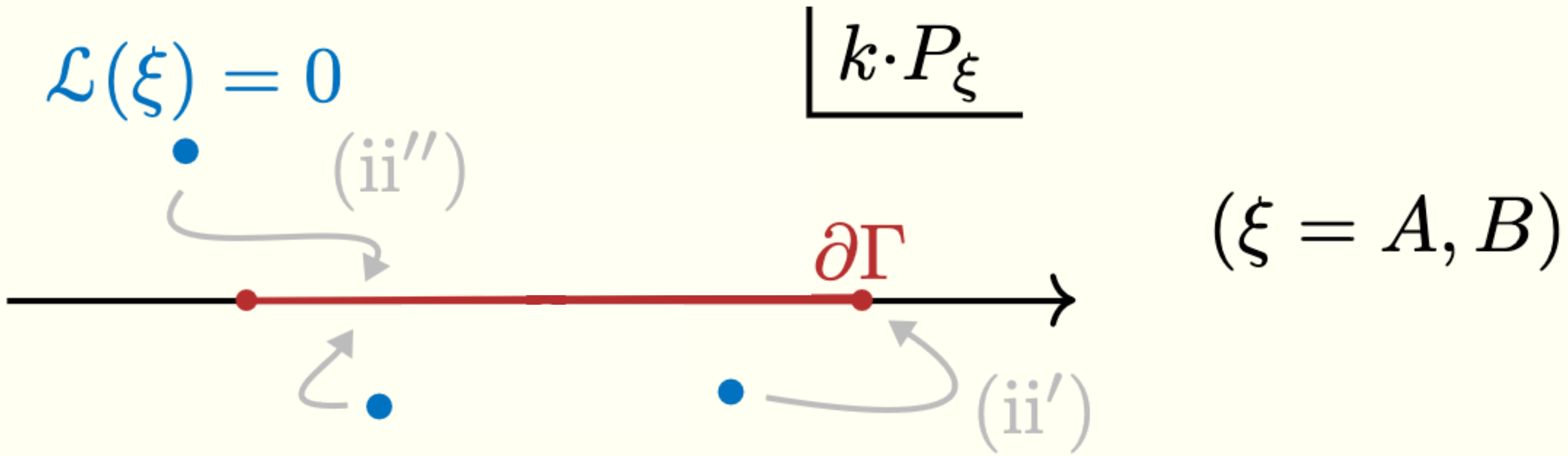
What does it mean for two-particle cut ?

(ii) Double discontinuities happen where the singular locus of A (or B) pinches Γ or hits $\partial\Gamma$



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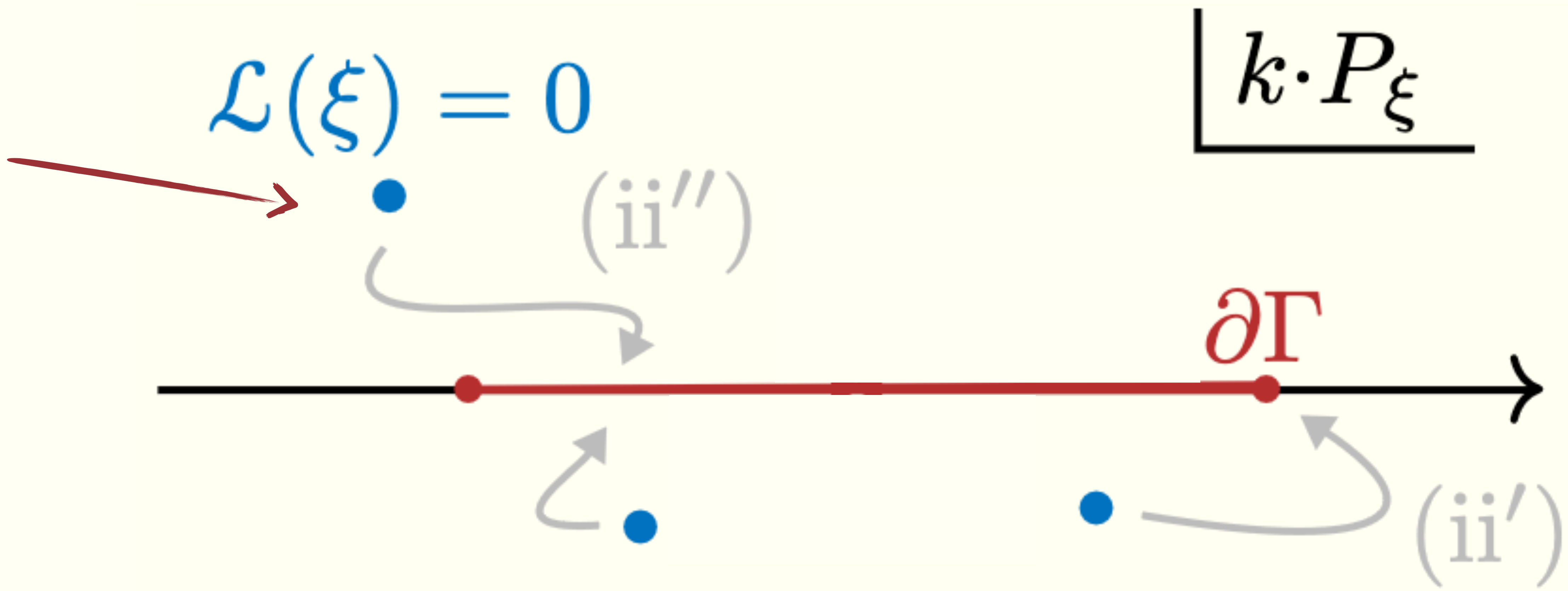
Never expected to happen in momentum space without $\det G = 0$ (*Landau*: on a singularity k is a linear combination of external momenta)



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Necessary conditions for singularities (III)

Algebraic necessary conditions for (i) and (ii′) can be uniformly obtained as follows:

1) Pick a (possibly empty) subset $\mathcal{S} \subset \mathcal{L}(A) \cup \mathcal{L}(B)$ of singularities on the left and right

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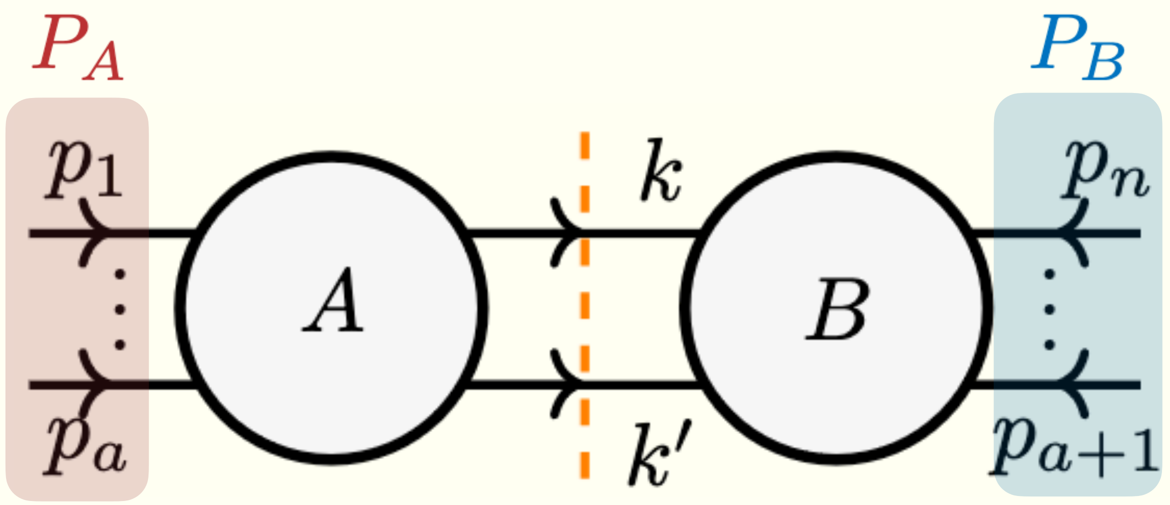
2) Localizing on the two-particle cut *and* on the loci $\{\mathcal{S}_i = 0\}$ fixes $\dim(\mathcal{S})$ variables from



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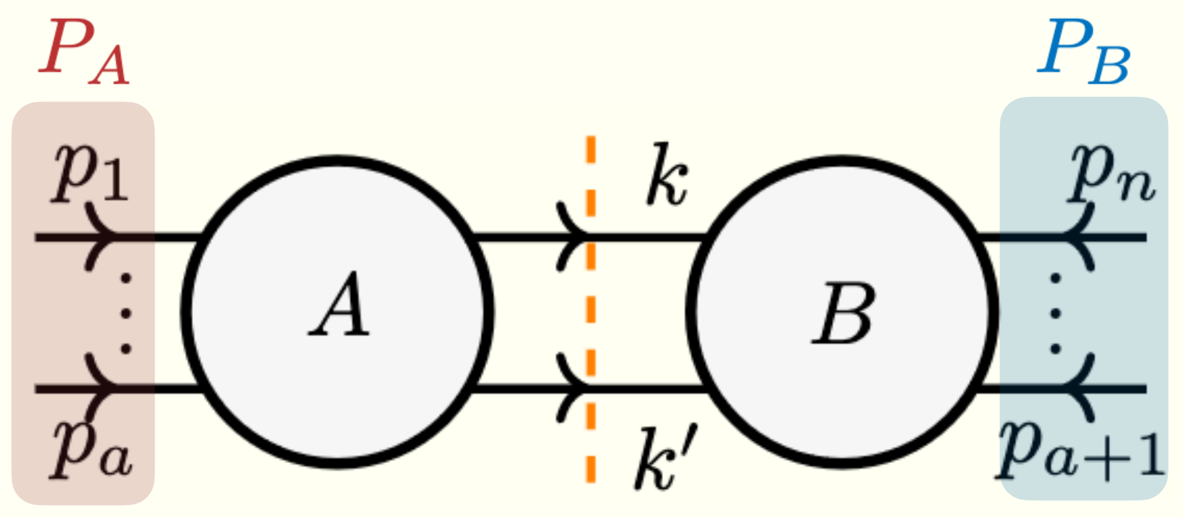


3) This leaves a set $X_{\mathcal{S}}$ of independent variables in terms of which $\partial\Gamma$ is

$$0 = \det \tilde{G}(X_{\mathcal{S}}) \equiv \det G|_{\{\mathcal{S}_i=0\}}$$

Necessary conditions for singularities (III)

Algebraic necessary conditions for (i) and (ii′) can be uniformly obtained as follows:

To ensure that there are no direction along which we could deform the contour to avoid the singularity, we have

$$\mathcal{L}(AB)_{\mathcal{S}} : \left[\begin{array}{l} \det \tilde{G} = 0 \\ \dfrac{\partial \det \tilde{G}}{\partial (k \cdot p_i)} = 0 \end{array}\right. \qquad \text{for} \quad k{\cdot}p_i \in X_{\mathcal{S}}$$

There is always one more equation than unknowns and so this system yields an algebraic constraint *on kinematic space*

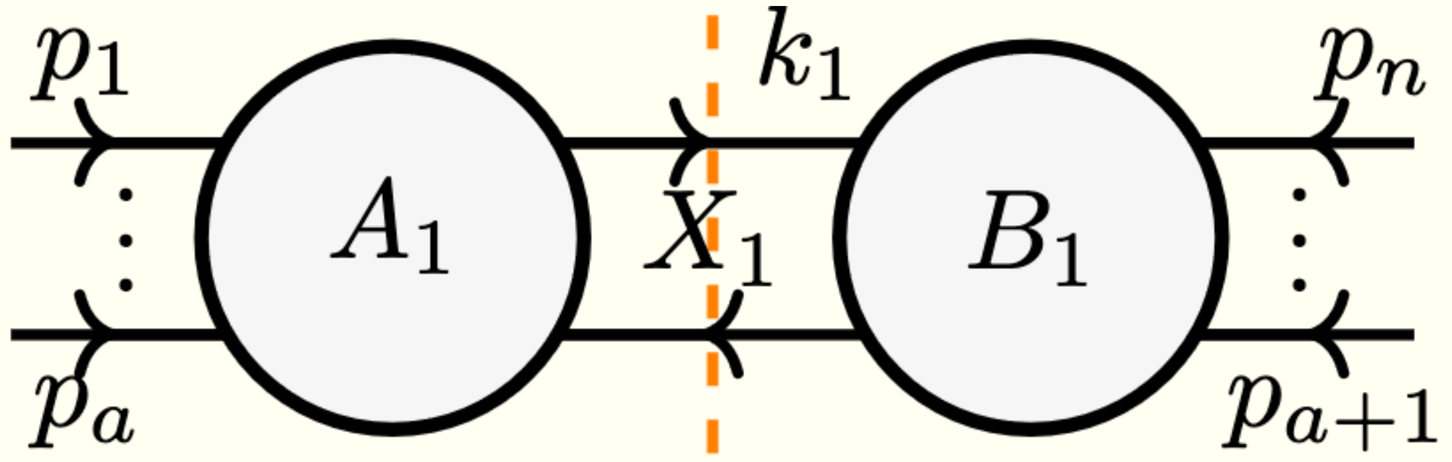
$$\mathcal{L}(AB)_{\mathcal{S}} = 0$$

Necessary conditions for singularities (III)

To find the remaining singularities of AB contained in a two-particle cut, we loop over all sets $\mathcal{S}$ of subamplitudes singularities (including subtopologies)

The focus of today is on the full graph
(leading singularities)

Recursion via unitarity



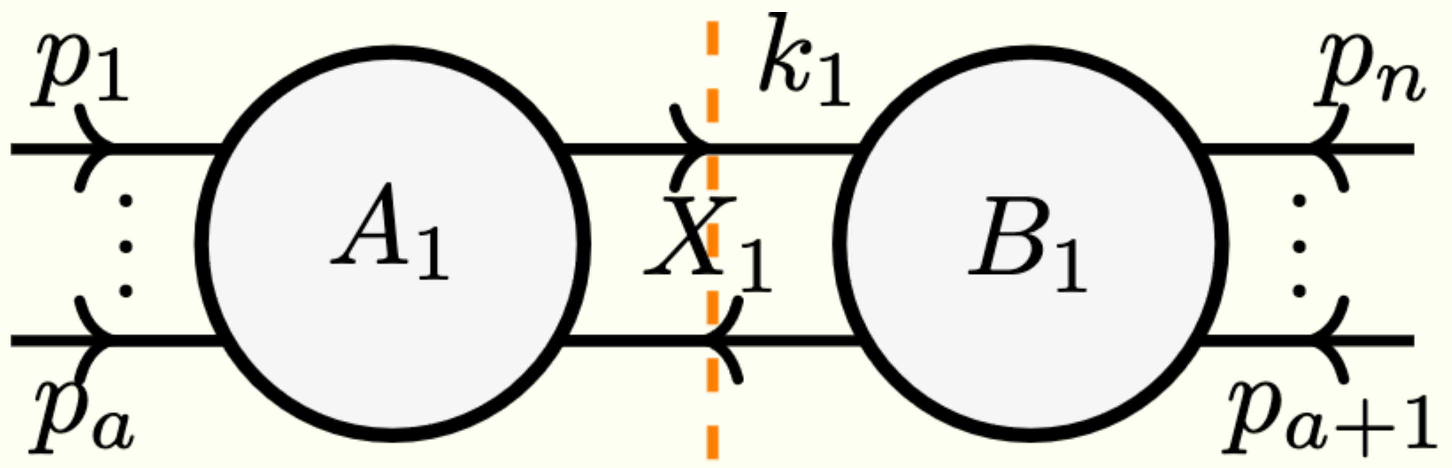
$$= C_1 \int_{\Gamma_1} \frac{\mathrm{d}\mu_1 A_1\, B_1}{(\det G_1)^{\frac{n+1-\mathrm{D}}{2}}}$$

The necessary conditions for (e.g., leading) singularities require to *know*

$$\mathcal{L}(A_1) = \mathcal{L}(B_1) = 0$$

Can these be constructed recursively ?

Recursion via unitarity



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The necessary conditions for (e.g., leading) singularities require to *know*

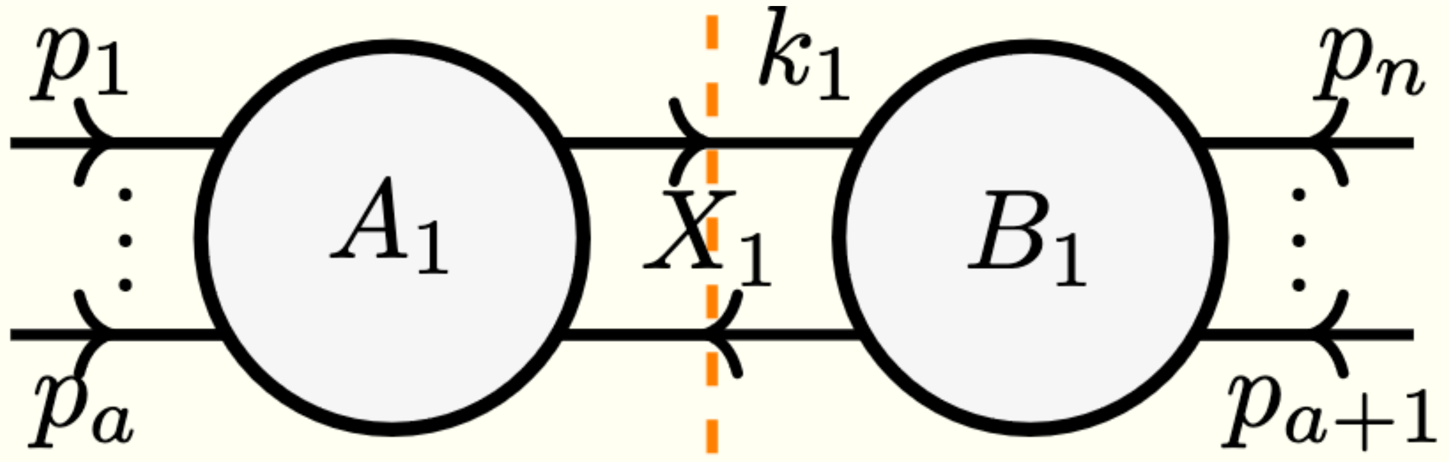
$$\mathcal{L}(A_1) = \mathcal{L}(B_1) = 0$$

Can these be constructed recursively ?

If either is two-particle-reducible, yes

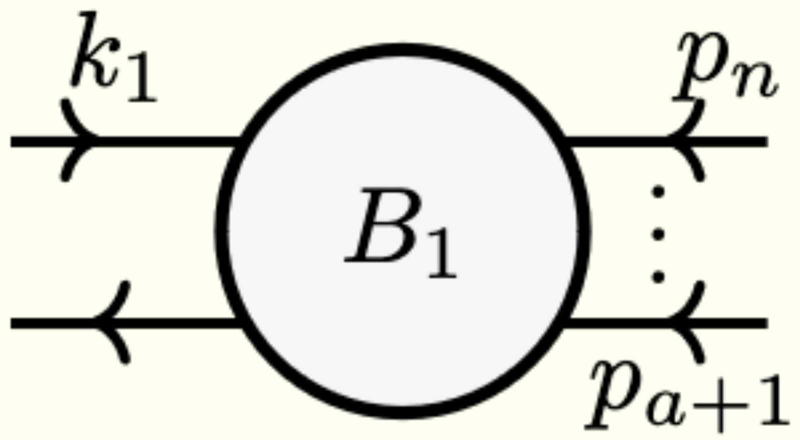
(just repeat the same argument over the blobs!)

Recursion via unitarity

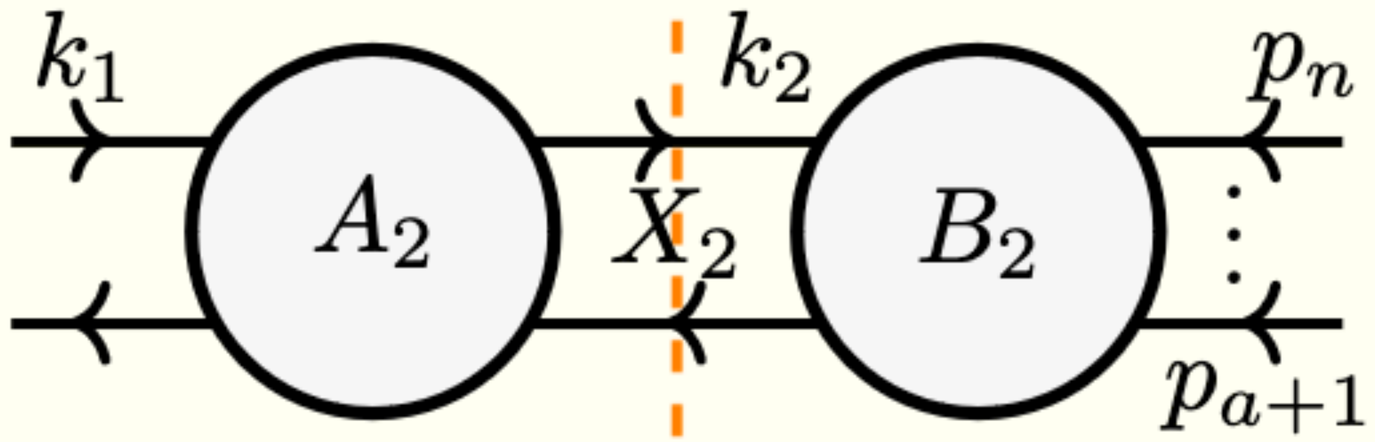


$$= C_1 \int_{\Gamma_1} \frac{\mathrm{d}\mu_1 A_1\, B_1}{(\det G_1)^{\frac{n+1-\mathrm{D}}{2}}}$$

If B_1 is two-particle-reducible,
just repeat the same argument



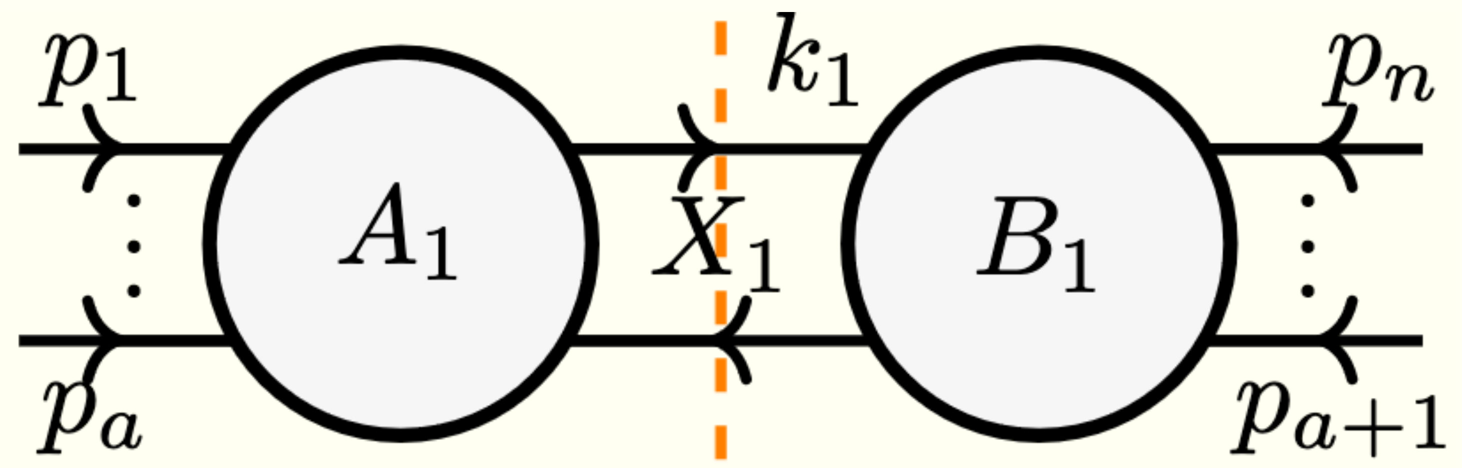
$\mapsto$



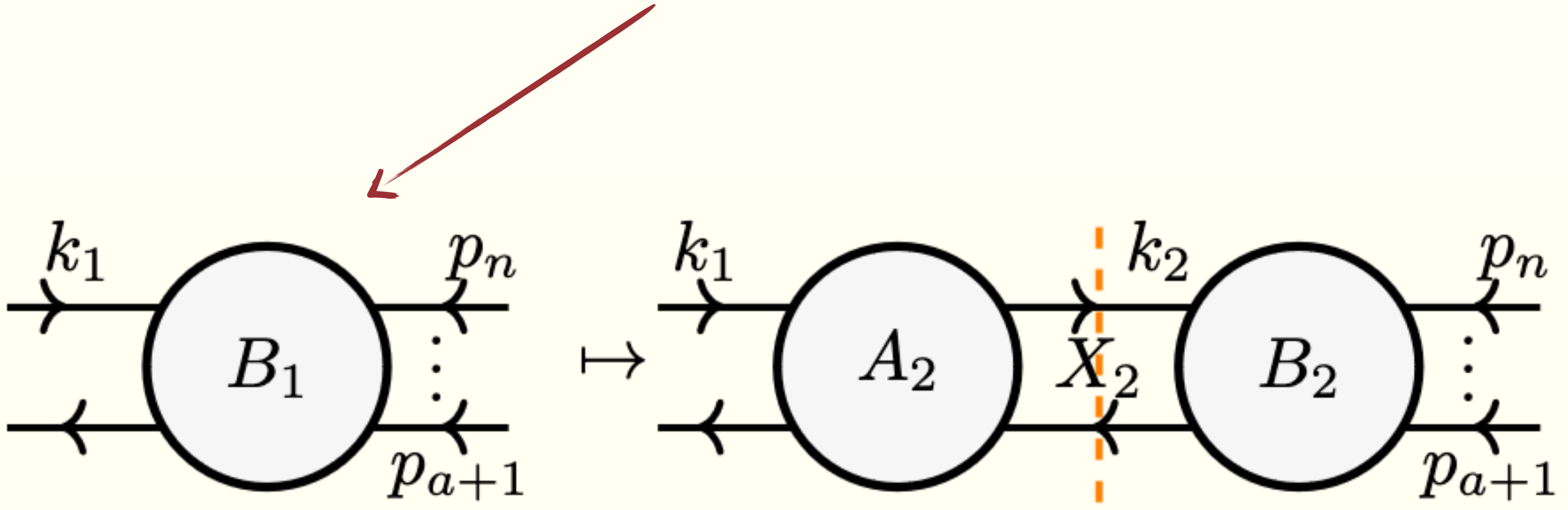
$$= C_2 \int_{\Gamma_2} \frac{\mathrm{d}\mu_2 A_2\, B_2}{(\det G_2)^{\cdots}}$$

Means we take another
two-particle cut

Recursion via unitarity



$$= C_1 \int_{\Gamma_1} \frac{\mathrm{d}\mu_1 A_1\, B_1}{(\det G_1)^{\frac{n+1-\mathrm{D}}{2}}}$$

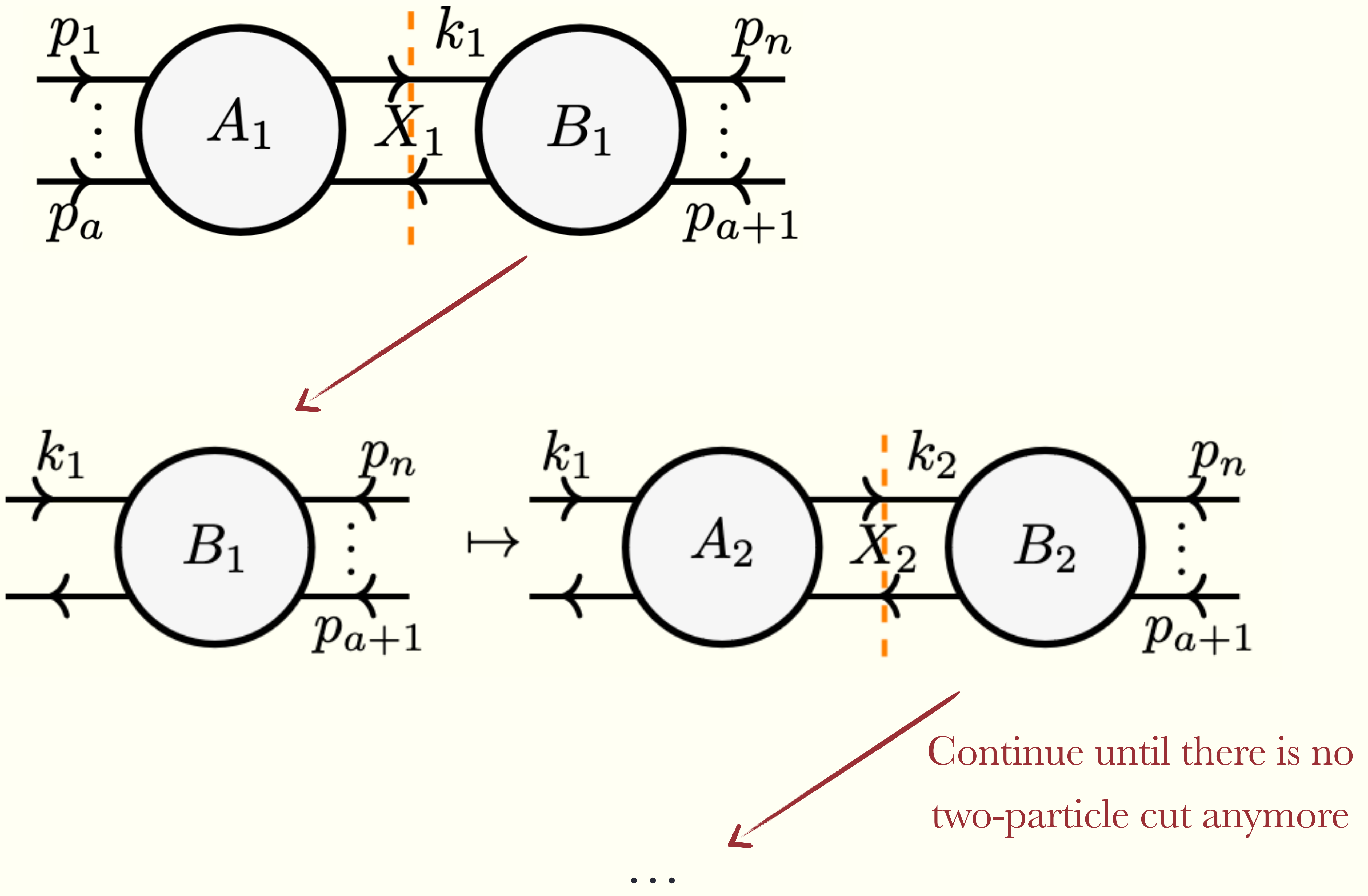


$$= C_2 \int_{\Gamma_2} \frac{\mathrm{d}\mu_2 A_2\, B_2}{(\det G_2)\ \cdots}$$

Singular locus of B_1 is given by solving

$$\mathcal{L}(B_1)_S : \left[\begin{array}{l} \det \tilde{G}_2 = 0 \\ \dfrac{\partial \det \tilde{G}_2}{\partial (k_2 \cdot p_i)} = 0 \end{array}\right. \qquad \text{for } k_2 \cdot p_i \in X_S$$

Recursion via unitarity



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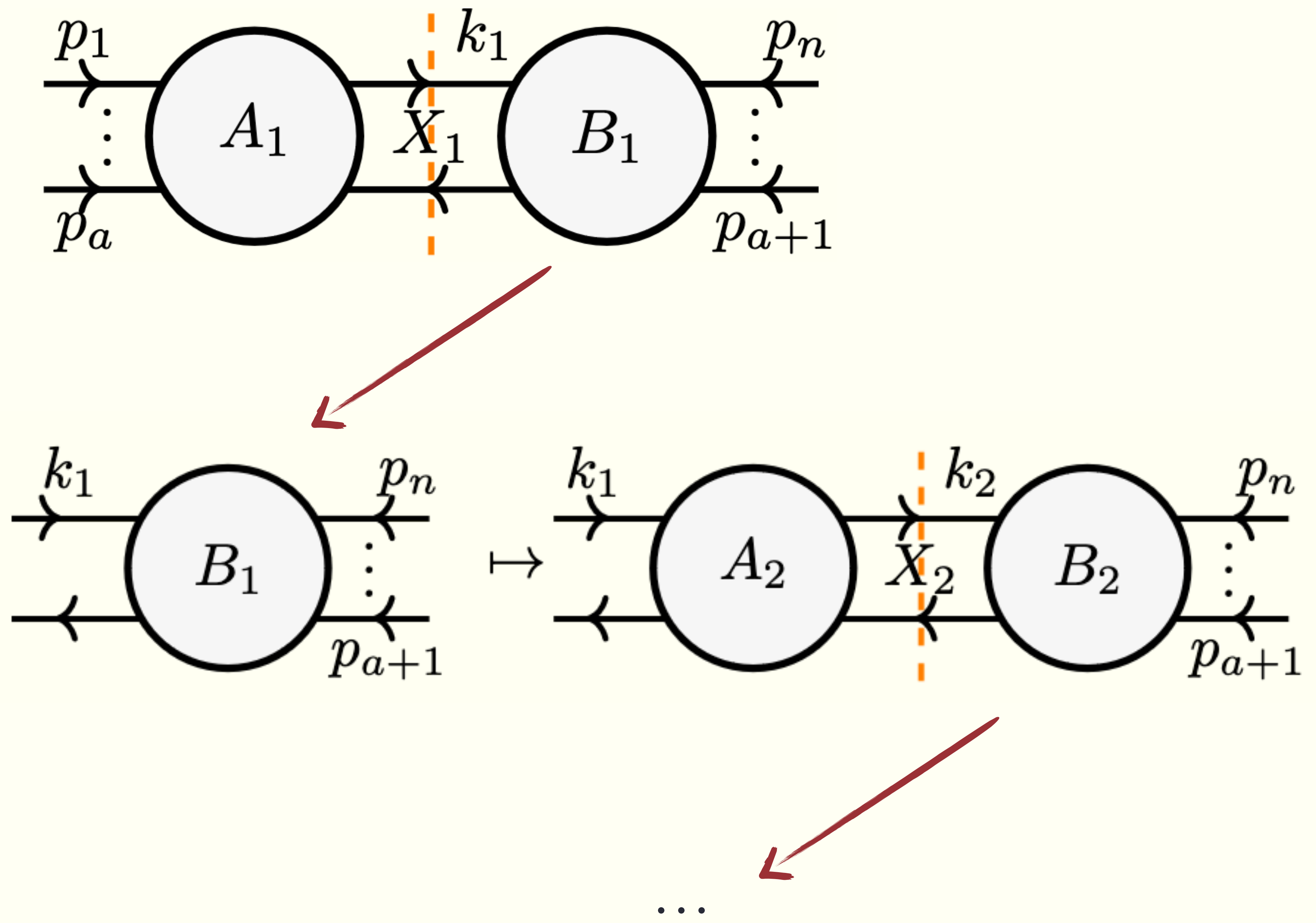


At the *end* of the recursion, we are left with either:

(1) A collection of tree-level subgraphs [easy/systematic]

Not the focus today

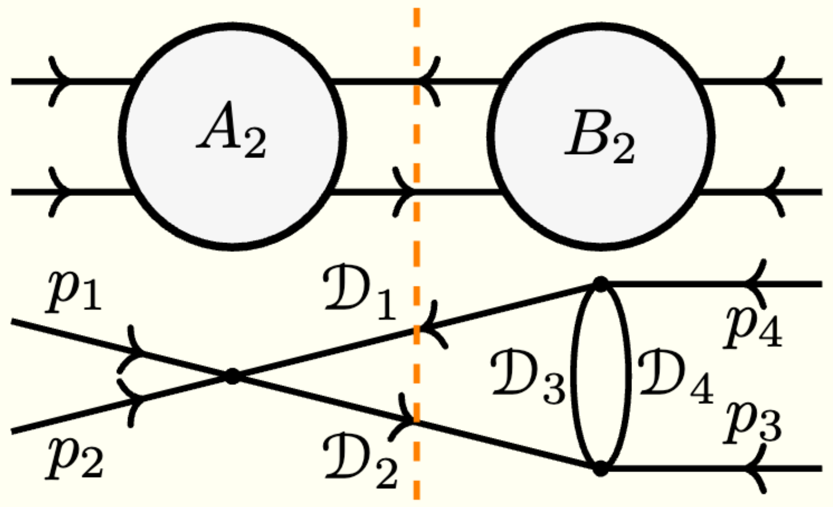
(2) A collection of subgraphs contains loop(s) [harder]
(may need external inputs for non-2PR subgraphs)

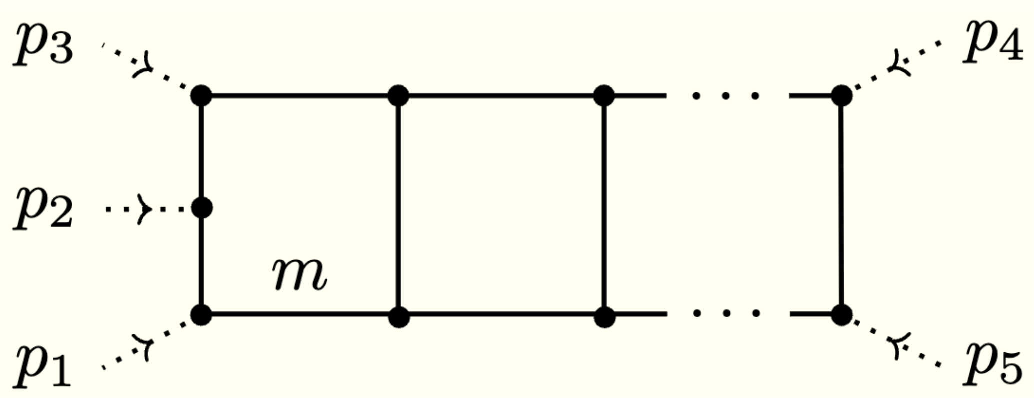
Outline

Recursion via unitarity

Proof of principle examples:

Recursively finding singularities





Checks and new analytic predictions:

Leading singularities

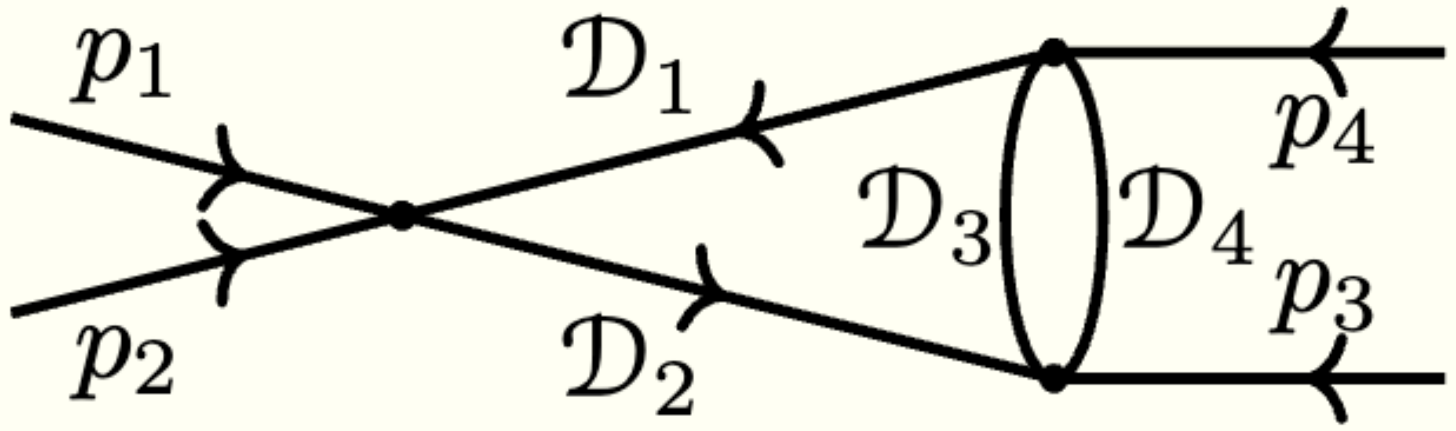
(Generic kinematic pentabox)

(Three-loop QED+QCD boX)

(Non-planar massive hexabox)

Recursively finding singularities

The generic kinematic parachute graph



$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
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$$p_i^2 = M_i^2$$
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What are the candidate leading singularities?

Recursively finding singularities

The generic kinematic parachute graph

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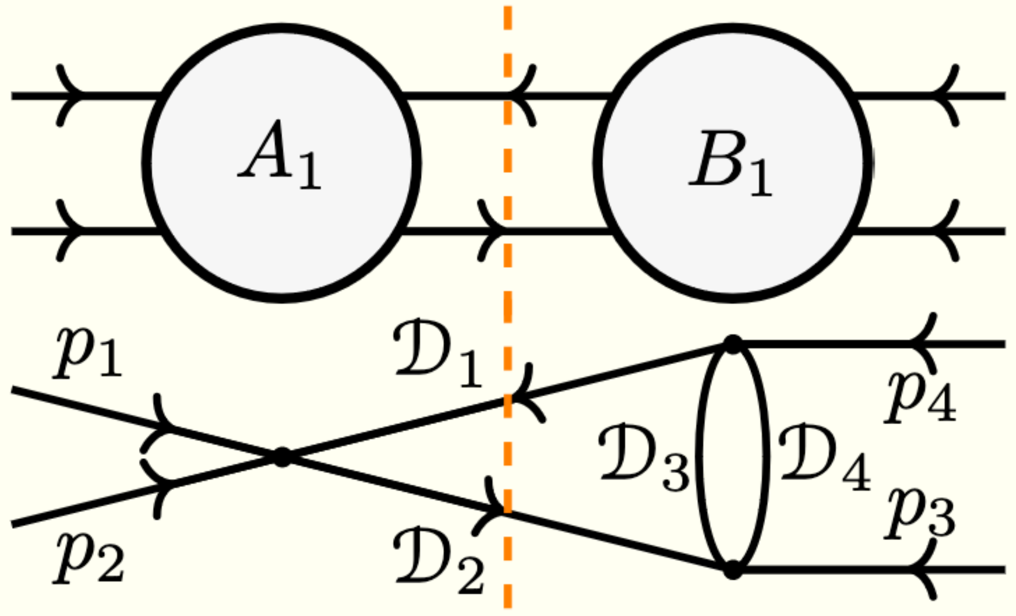
Let's look at a first two-particle cut



$$= C_{\text{par}} \int_{\Gamma_1} \frac{\mathrm{d}\mu_1 A_1\, B_1}{(\det G_1)^{\frac{4-\mathrm{D}}{2}}}$$

$$\propto \left(\det[p_i \cdot p_j]_{i,j=12,3}\right)^{\frac{3-\mathrm{D}}{2}}$$

Recursively finding singularities

The generic kinematic parachute graph

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
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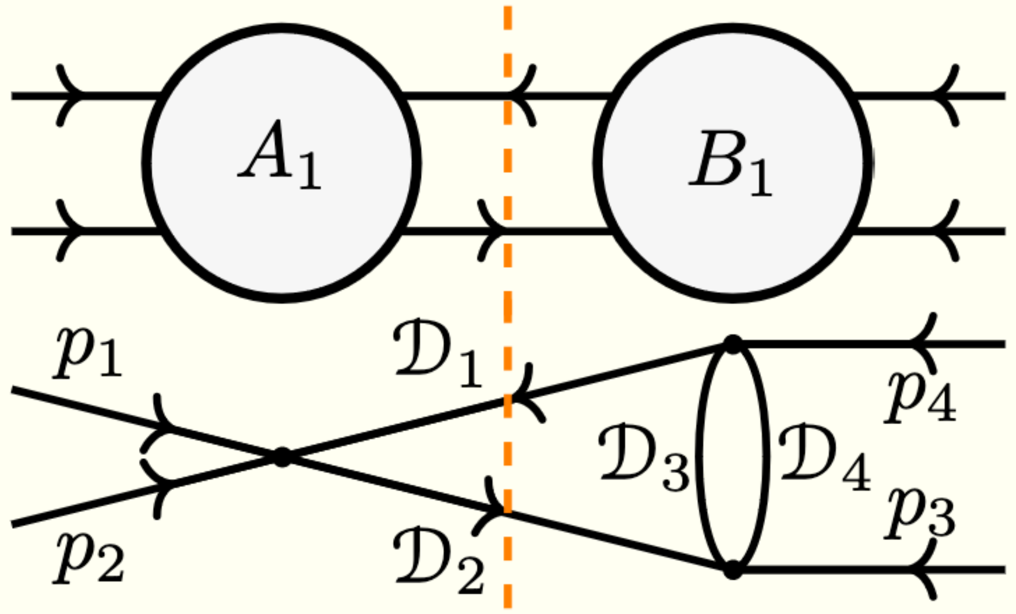
Let's look at a first two-particle cut

$=\mathrm{d}(k_1 \cdot p_{12})\mathrm{d}(k_1 \cdot p_3)\mathrm{d}(k_1^2)\delta[\mathcal{D}_1]\delta[\mathcal{D}_2]$



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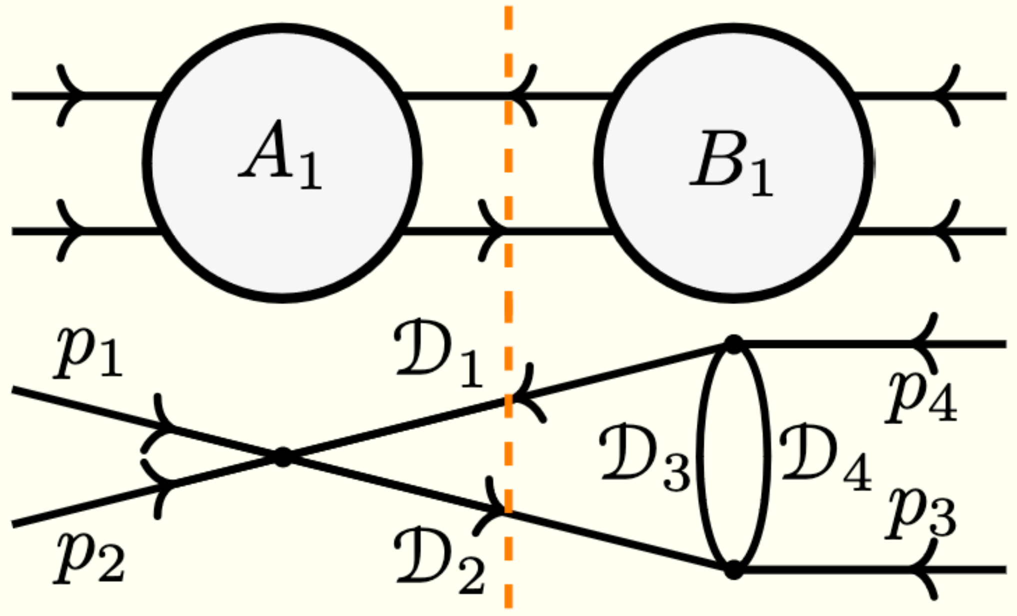

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$A_1 = -i\lambda$

$C_{\text{par}} \propto \left(\det[p_i \cdot p_j]_{i,j=12,3}\right)^{\frac{3-\mathrm{D}}{2}}$

Recursively finding singularities

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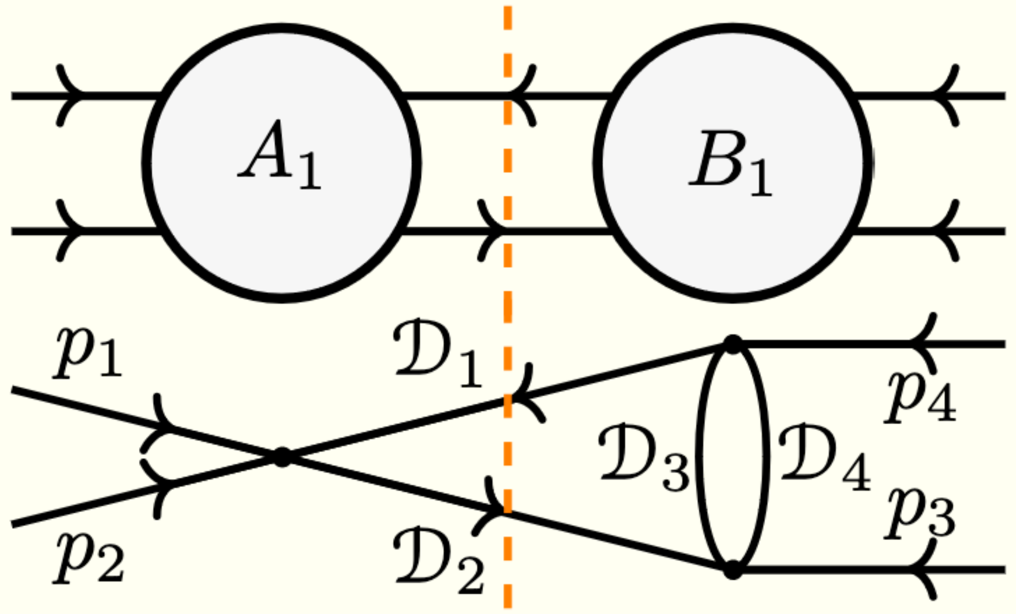

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Recursively finding singularities

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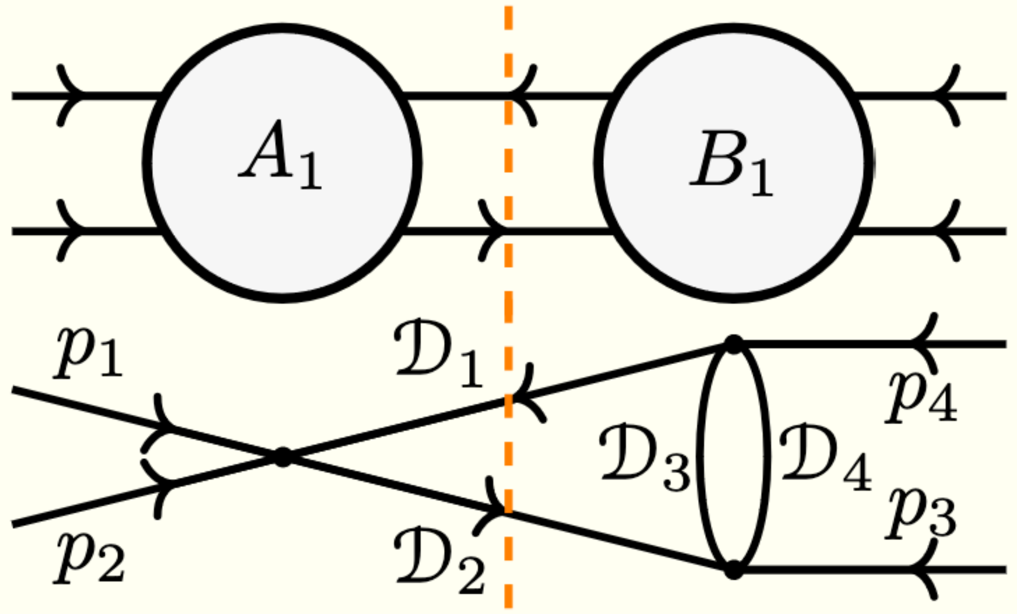

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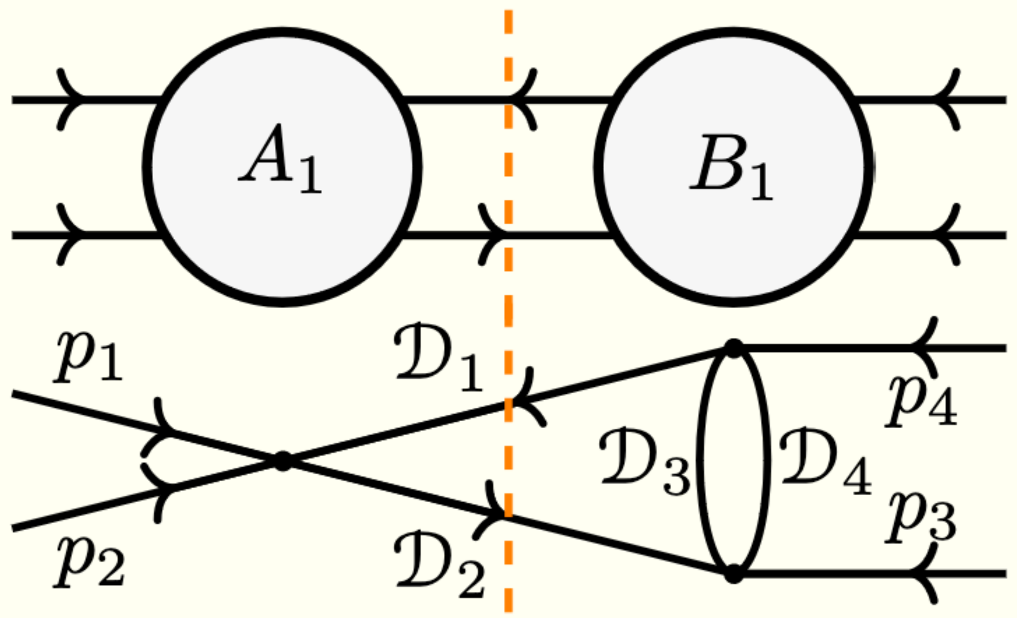

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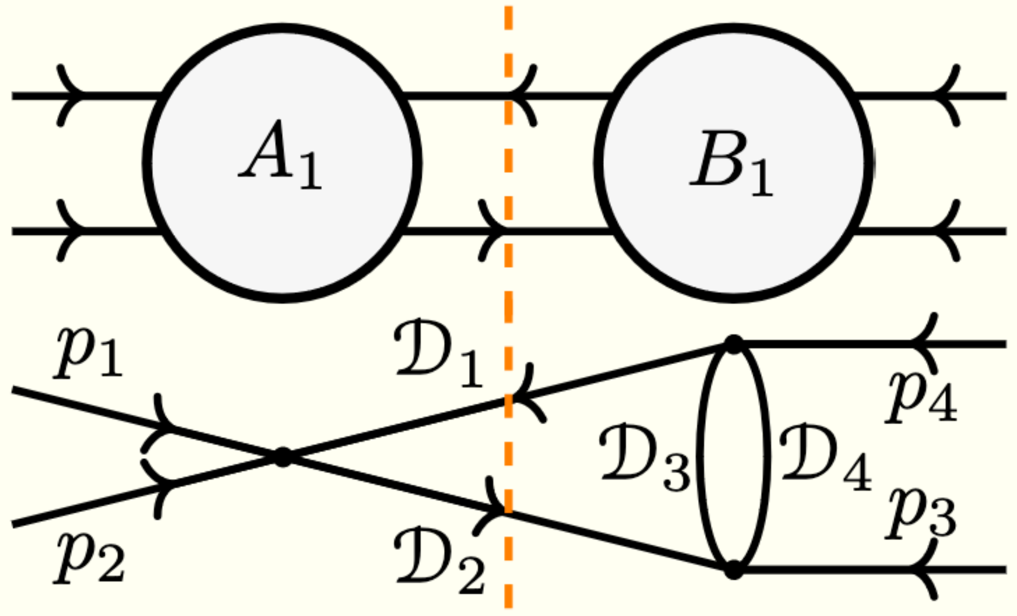

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Fixed by the singular locus of B_1

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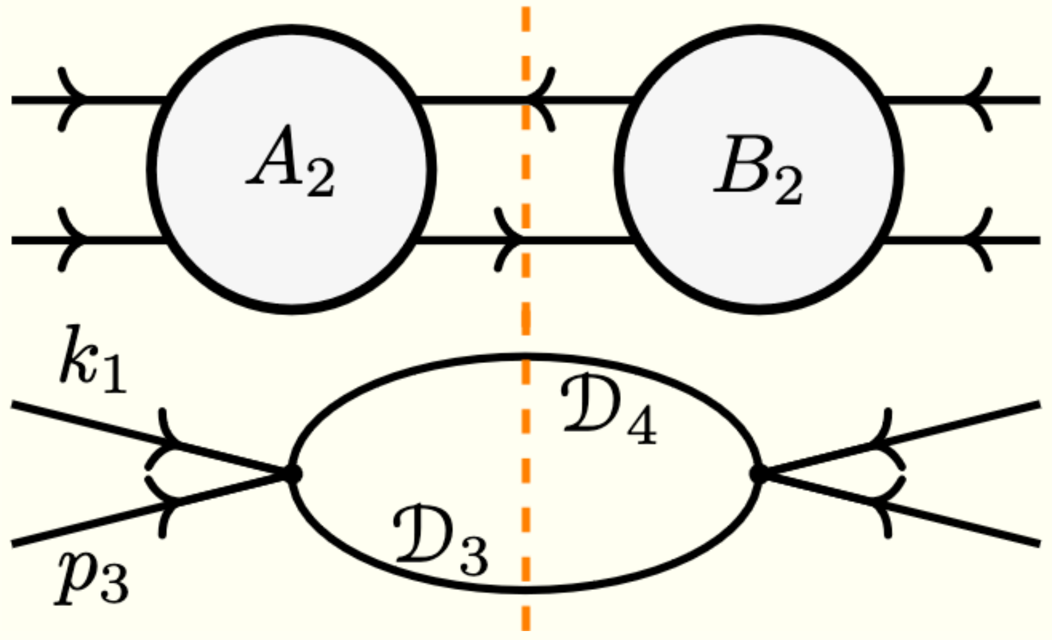
Singular locus of B_1 is given by repeating the *same* argument over the bubble



$$= C_{\text{bub}} \int_{\Gamma_2} \frac{\mathrm{d}\mu_2 A_2\, B_2}{(\det G_2)^{\frac{3-\mathrm{D}}{2}}}$$

$C_{\text{bub}} \propto (\Lambda^2)^{\frac{2-\mathrm{D}}{2}}$

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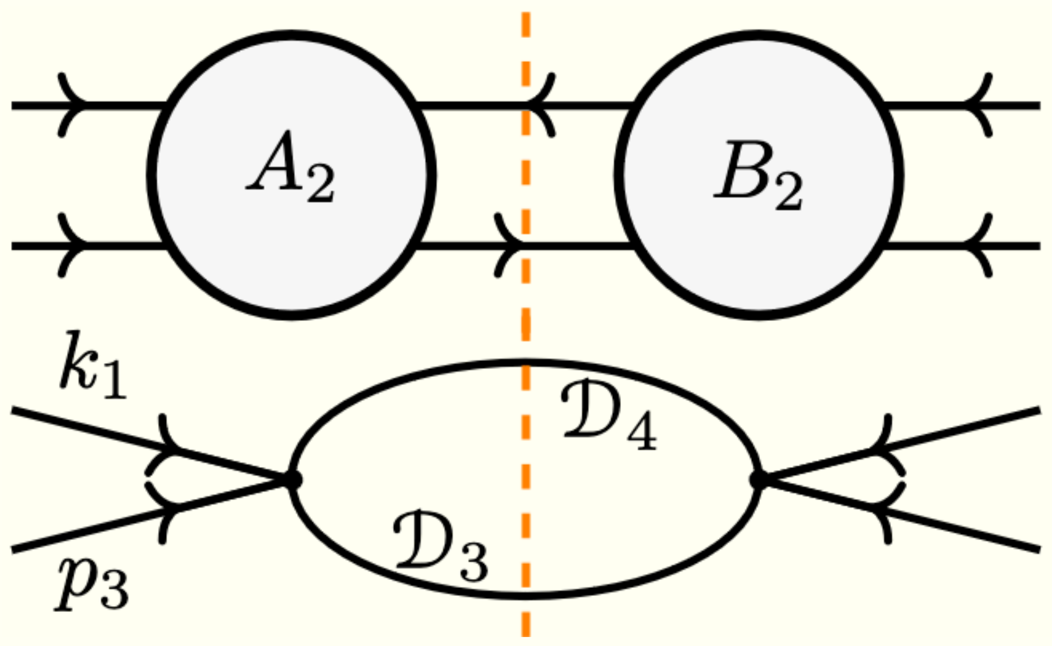

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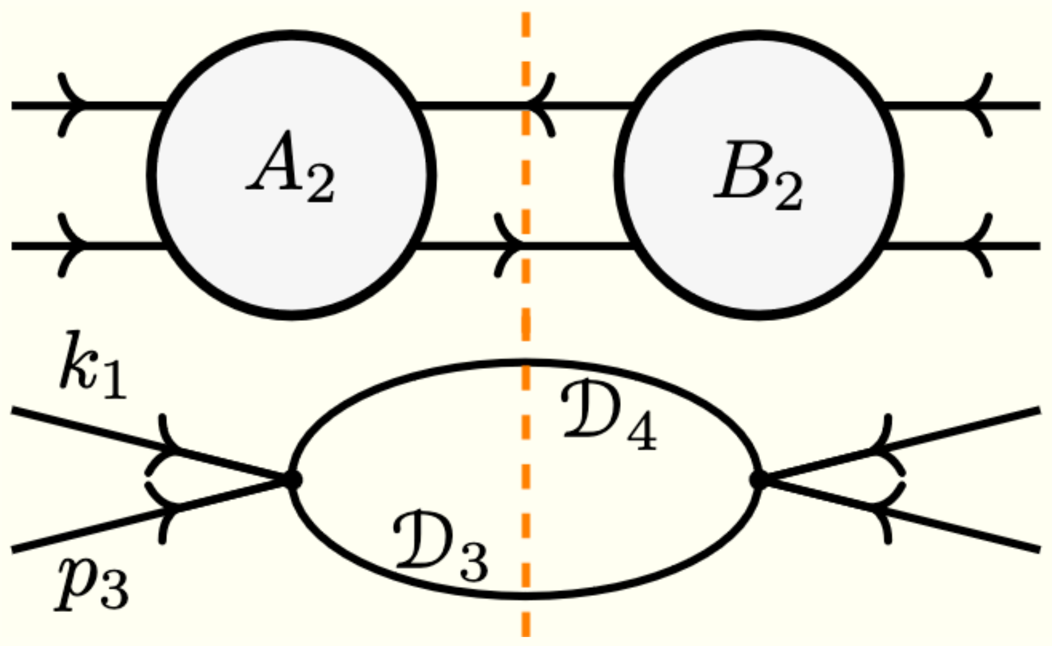

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Trivial vertices

$\propto (\Lambda^2)^{\frac{2-\mathrm{D}}{2}}$

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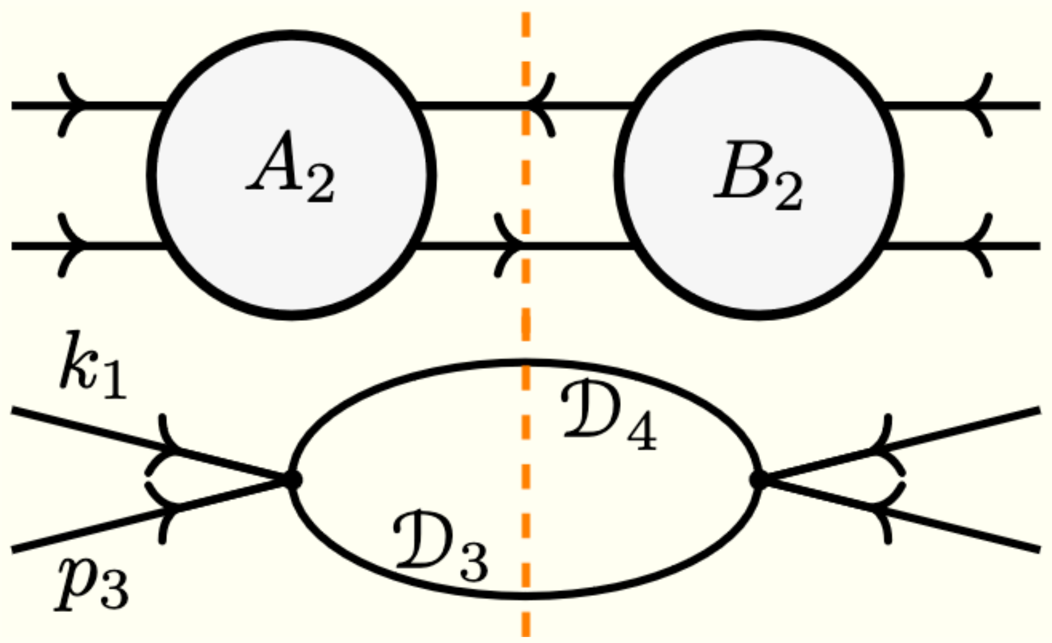

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$A_2\, B_2$: Trivial vertices

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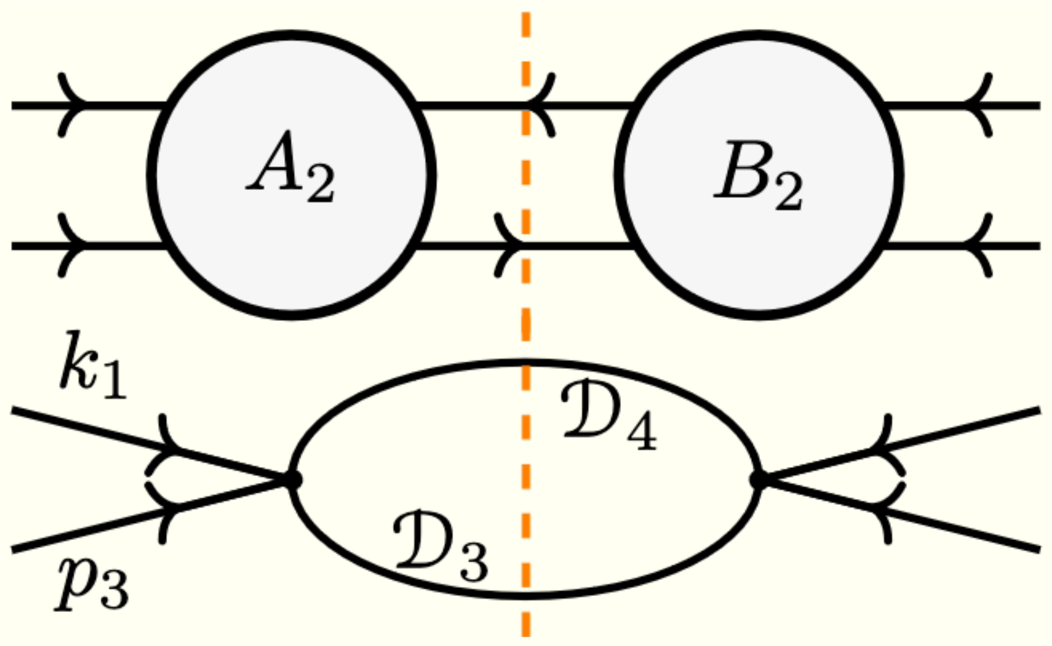

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Trivial vertices

$\propto (\Lambda^2)^{\frac{2-\mathrm{D}}{2}}$

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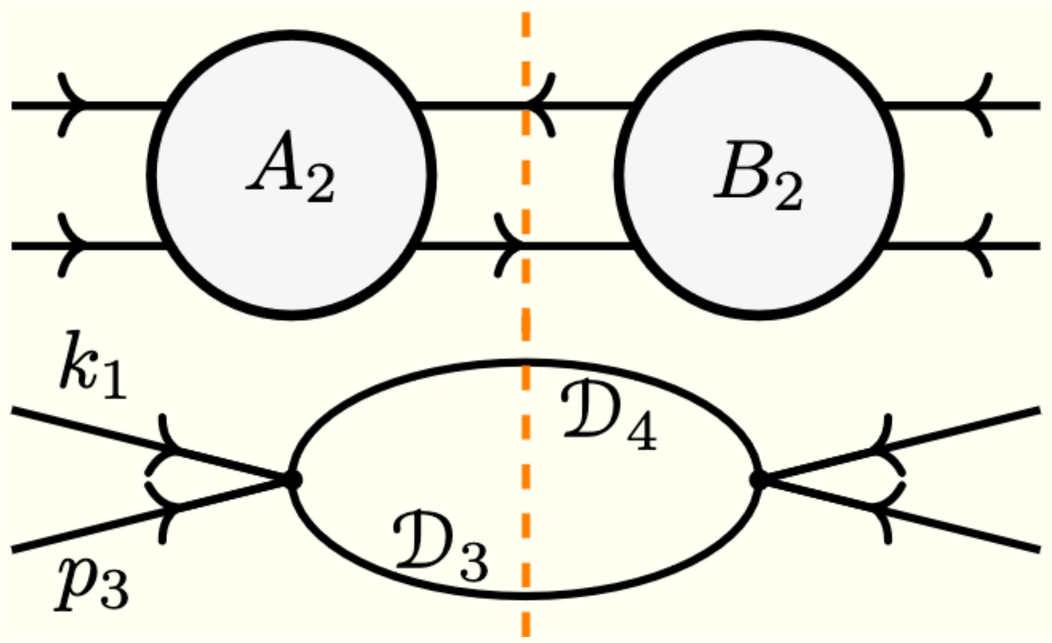
Singular locus of B_1 is given by repeating the *same* argument over the bubble

Imposing $\det \tilde{G}_2 = 0$ gives $\mathcal{L}(B_1)_1 = 0$

$$k_1 \cdot p_3 = \frac{1}{2}\left[(m_3 \pm m_4)^2 - m_2^2 - M_3^2\right]$$



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$= \mathrm{d}(k_2 \cdot \Lambda)\mathrm{d}(k_2^2)\delta[\mathcal{D}_3]\delta[\mathcal{D}_4]$

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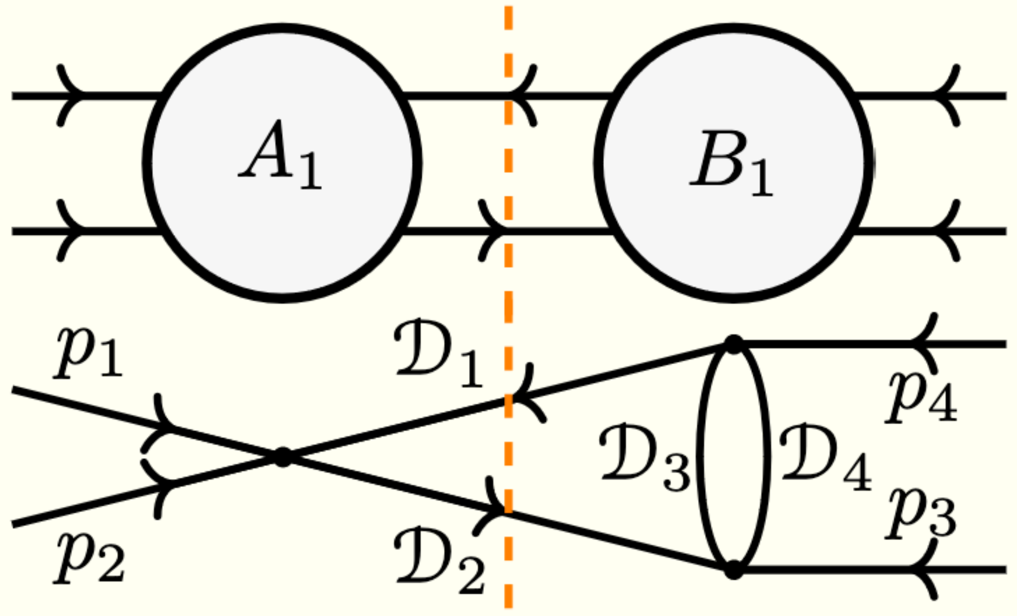
What are the candidate leading singularities ?



$$= C_{\text{par}} \int_{\Gamma_1} \frac{\mathrm{d}\mu_1 A_1\, B_1}{(\det G_1)^{\frac{5-\mathrm{D}}{2}}}$$

Setting $\mathcal{S} = \{\mathcal{L}(B_1)_1 = 0\}$ fixes the remaining invariant:

$$k_1 \cdot p_3 = \frac{1}{2}\left[(m_3 \pm m_4)^2 - m_2^2 - M_3^2\right]$$

$$= \begin{bmatrix} p_{12}^2 & p_{12} \cdot k_1 & p_{12} \cdot p_3 \\ p_{12} \cdot k_1 & k_1^2 & k_1 \cdot p_3 \\ p_{12} \cdot p_3 & k_1 \cdot p_3 & p_3^2 \end{bmatrix}$$

Recursively finding singularities

The generic kinematic parachute graph

$$p_i^2 = M_i^2$$

$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$

$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$

$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

What are the candidate leading singularities ?

$$\begin{vmatrix} s & \frac{m_2^2 - m_1^2 + s}{2} & \frac{M_4^2 - M_3^2 - s}{2} \\ \frac{m_2^2 - m_1^2 + s}{2} & m_2^2 & \frac{(m_4 \pm m_3)^2 - m_2^2 - M_3^2}{2} \\ \frac{M_4^2 - M_3^2 - s}{2} & \frac{(m_4 \pm m_3)^2 - m_2^2 - M_3^2}{2} & M_3^2 \end{vmatrix} = 0$$

Recursively finding singularities

The generic kinematic parachute graph

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$
$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

What are the candidate leading singularities ?

Matches with `PLD.jl`!

[Fevola, Mizera, Telen (2023)]

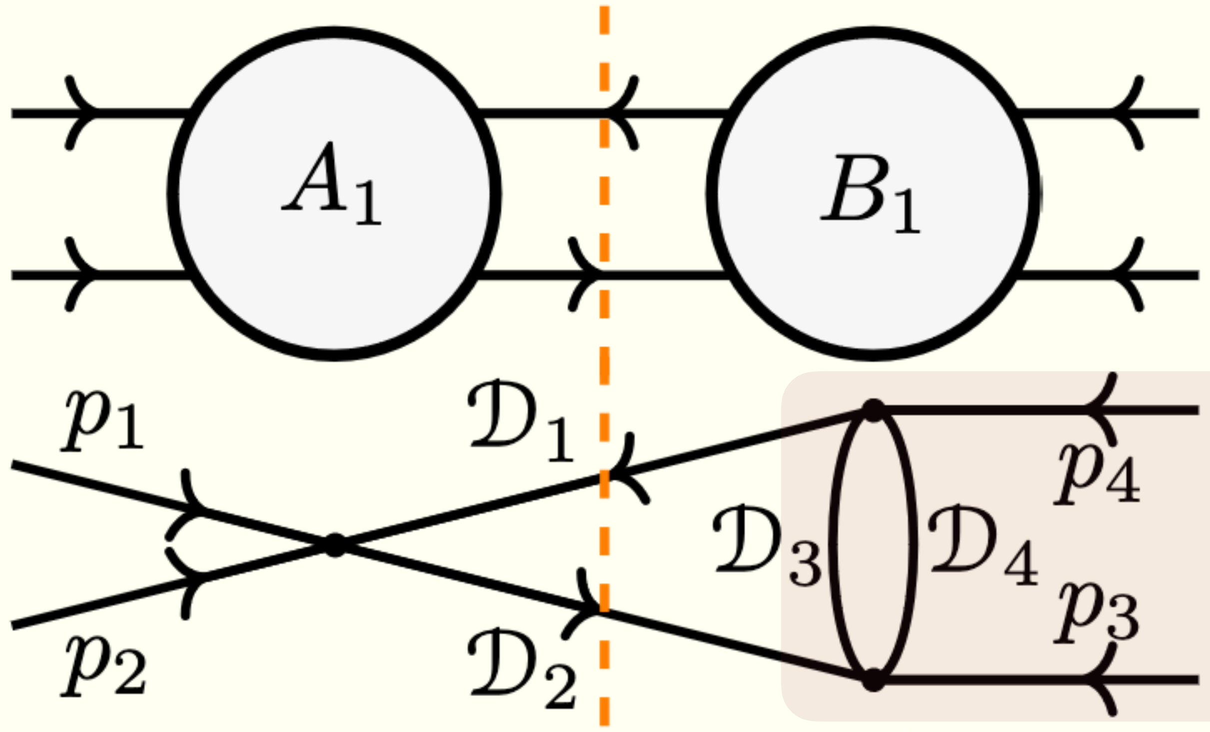
What about other singularities ?

On the previous slide, we localized G_1 on the bubble *leading* singularity

$\mathcal{S} = \{\mathcal{L}(B_1)_1 = 0\}$ fixed the remaining invariant:

$$k_1 \cdot p_3 = \frac{1}{2}\left[(m_3 \pm m_4)^2 - m_2^2 - M_3^2\right]$$

$$\begin{bmatrix} p_{12}^2 & p_{12} \cdot k_1 & p_{12} \cdot p_3 \\ p_{12} \cdot k_1 & k_1^2 & k_1 \cdot p_3 \\ p_{12} \cdot p_3 & k_1 \cdot p_3 & p_3^2 \end{bmatrix}$$



$$\overset{D=3}{=} \frac{\sqrt{\pi}}{\sqrt{\Lambda^2}} \log\left[\frac{m_3 + m_4 + \sqrt{\Lambda^2}}{m_3 + m_4 - \sqrt{\Lambda^2}}\right]$$

$\uparrow$

$\Lambda^\mu = (p_3 + k_1)^\mu$

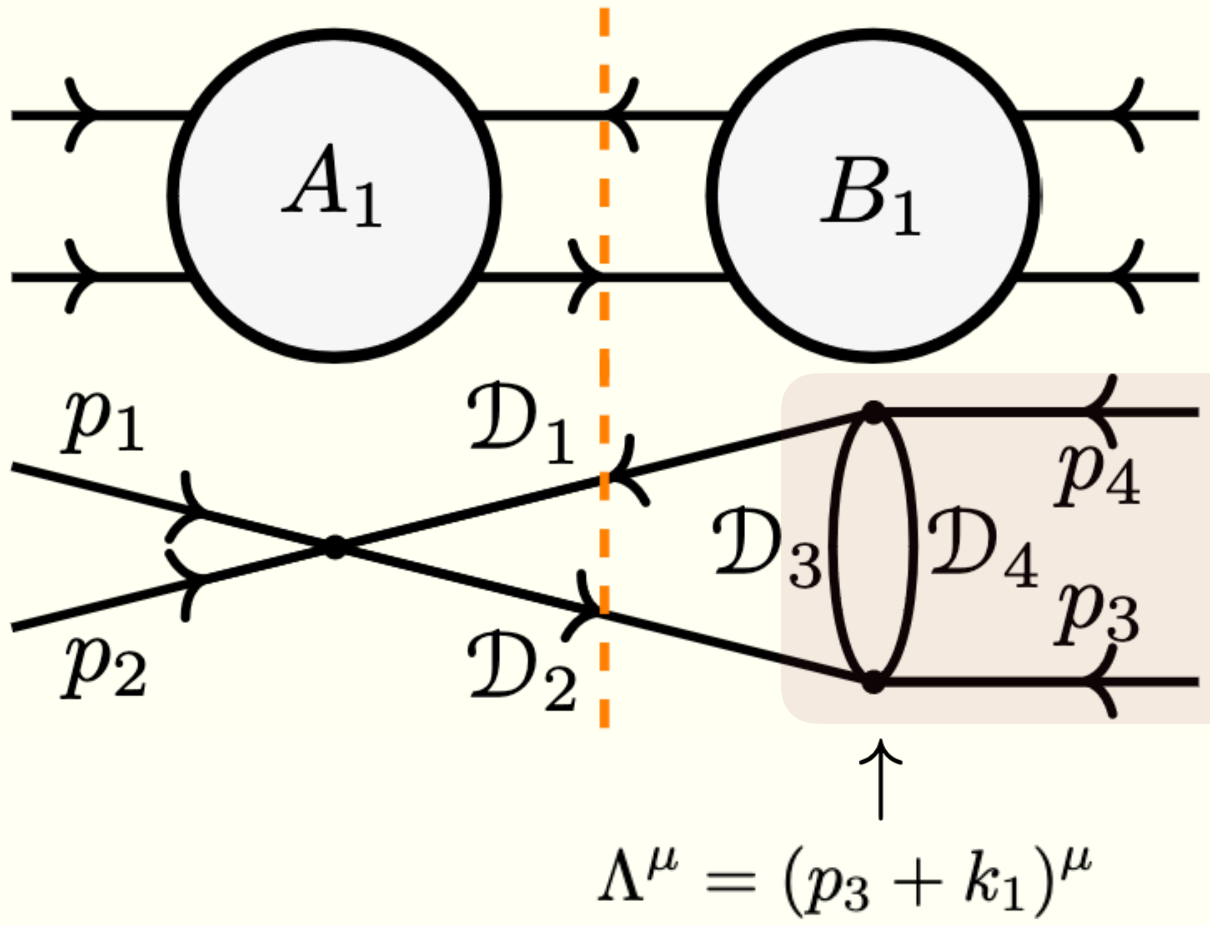
What about other singularities ?

But nothing stops us to localize on *other* singularities of B_1 (e.g., second-type singularity at $\Lambda^2 = 0$)

$\mathcal{S} = \{\mathcal{L}(B_1)_2 = 0\}$ fixes the remaining invariant:

$$k_1 \cdot p_3 = \frac{1}{2}\left[-m_2^2 - M_3^2\right]$$

$$\begin{bmatrix} p_{12}^2 & p_{12}\cdot k_1 & p_{12}\cdot p_3 \\ p_{12}\cdot k_1 & k_1^2 & k_1\cdot p_3 \\ p_{12}\cdot p_3 & k_1\cdot p_3 & p_3^2 \end{bmatrix}$$



$$\overset{D=3}{=} \frac{\sqrt{\pi}}{\sqrt{\Lambda^2}} \log\left[\frac{m_3 + m_4 + \sqrt{\Lambda^2}}{m_3 + m_4 - \sqrt{\Lambda^2}}\right]$$

$$\begin{vmatrix} s & \frac{m_2^2 - m_1^2 + s}{2} & \frac{M_4^2 - M_3^2 - s}{2} \\ \frac{m_2^2 - m_1^2 + s}{2} & m_2^2 & -\frac{m_2^2 + M_3^2}{2} \\ \frac{M_4^2 - M_3^2 - s}{2} & -\frac{m_2^2 + M_3^2}{2} & M_3^2 \end{vmatrix} = 0$$

(accessible on the max-cut)

Matches with `HyperInt`!

[Berghoff, Panzer (2022); eq. (6.15)]

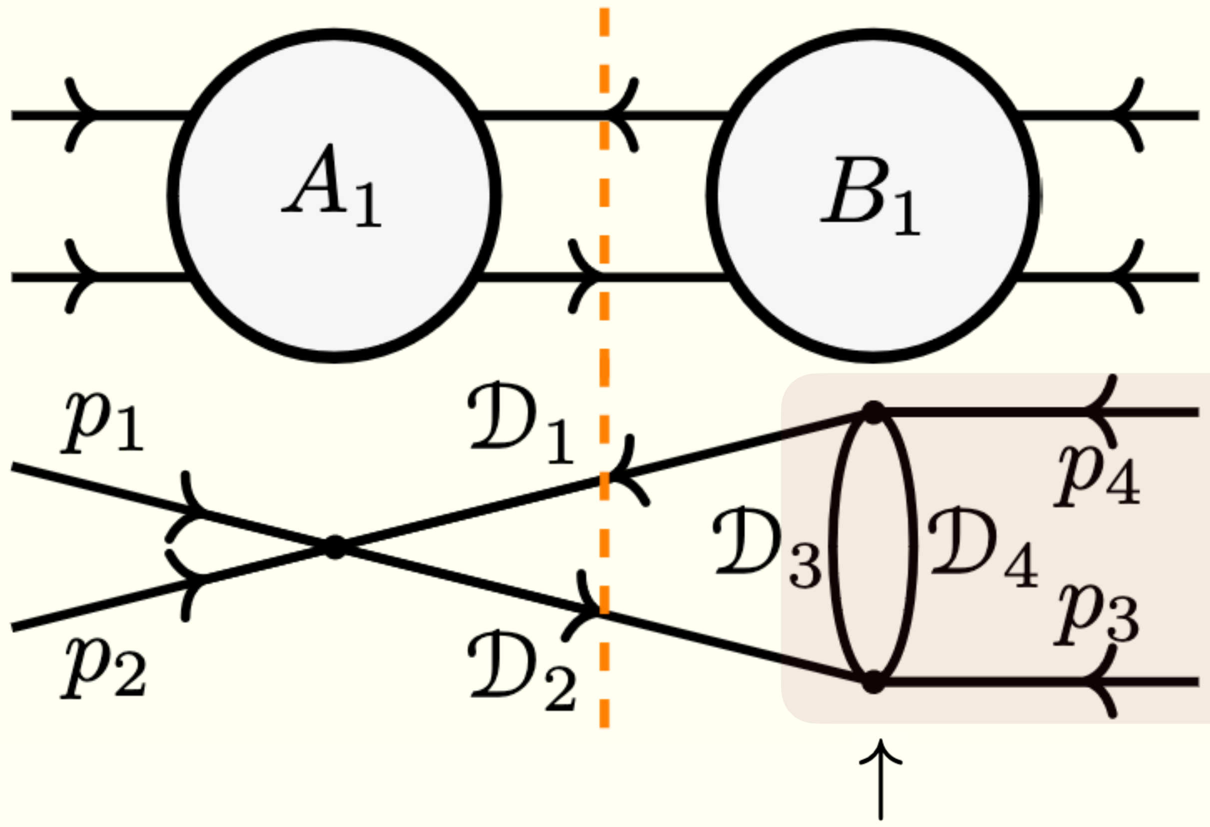
What about other singularities ?

But nothing stops us to localize on *other* singularities of B_1 (e.g., second-type singularity at $\Lambda^2 = 0$)

$\mathcal{S} = \{\mathcal{L}(B_1)_2 = 0\}$ fixes the remaining invariant:

$$k_1 \cdot p_3 = \frac{1}{2}\left[-m_2^2 - M_3^2\right]$$

$$\begin{bmatrix} p_{12}^2 & p_{12}\cdot k_1 & p_{12}\cdot p_3 \\ p_{12}\cdot k_1 & k_1^2 & k_1\cdot p_3 \\ p_{12}\cdot p_3 & k_1\cdot p_3 & p_3^2 \end{bmatrix}$$



$$\Lambda^\mu = (p_3 + k_1)^\mu$$

$$\overset{D=3}{=} \frac{\sqrt{\pi}}{\sqrt{\Lambda^2}} \log\left[\frac{m_3 + m_4 + \sqrt{\Lambda^2}}{m_3 + m_4 - \sqrt{\Lambda^2}}\right]$$

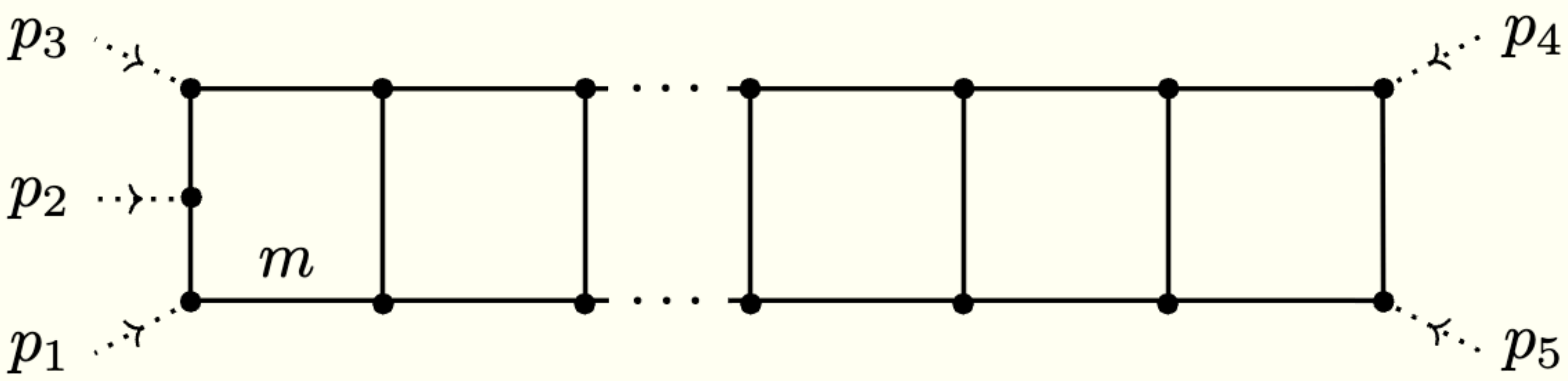
```
##################################
# Component 3
##################################

D[3] = M[3]^2*m[1] - M[3]*M[4]*m[1] - M[3]*M[4]*m[2] + M[3]*M[4]*s + M[3]*m[1]^2 - M[3]*m[1]*m[2] - M[3]*m[1]*s + M[4]^2*m[2] -
M[4]*m[1]*m[2] + M[4]*m[2]^2 - M[4]*m[2]*s + m[1]*m[2]*s
χ[3] = 18
weights[3] = []
computed_with[3] = ["HyperInt"]
```

Matches with `HyperInt`!

[Berghoff, Panzer (2022); eq. (6.15)]

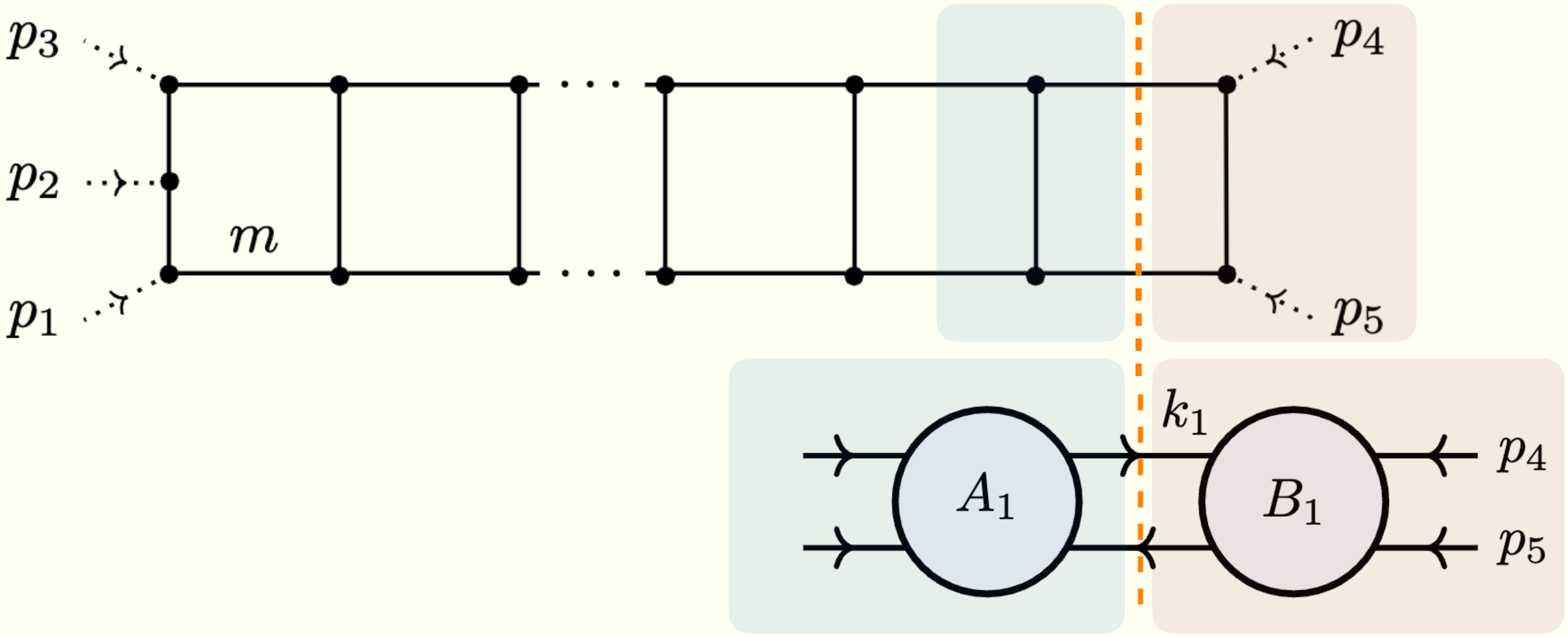
L-LOOP RESULTS

The massive penta-ladder



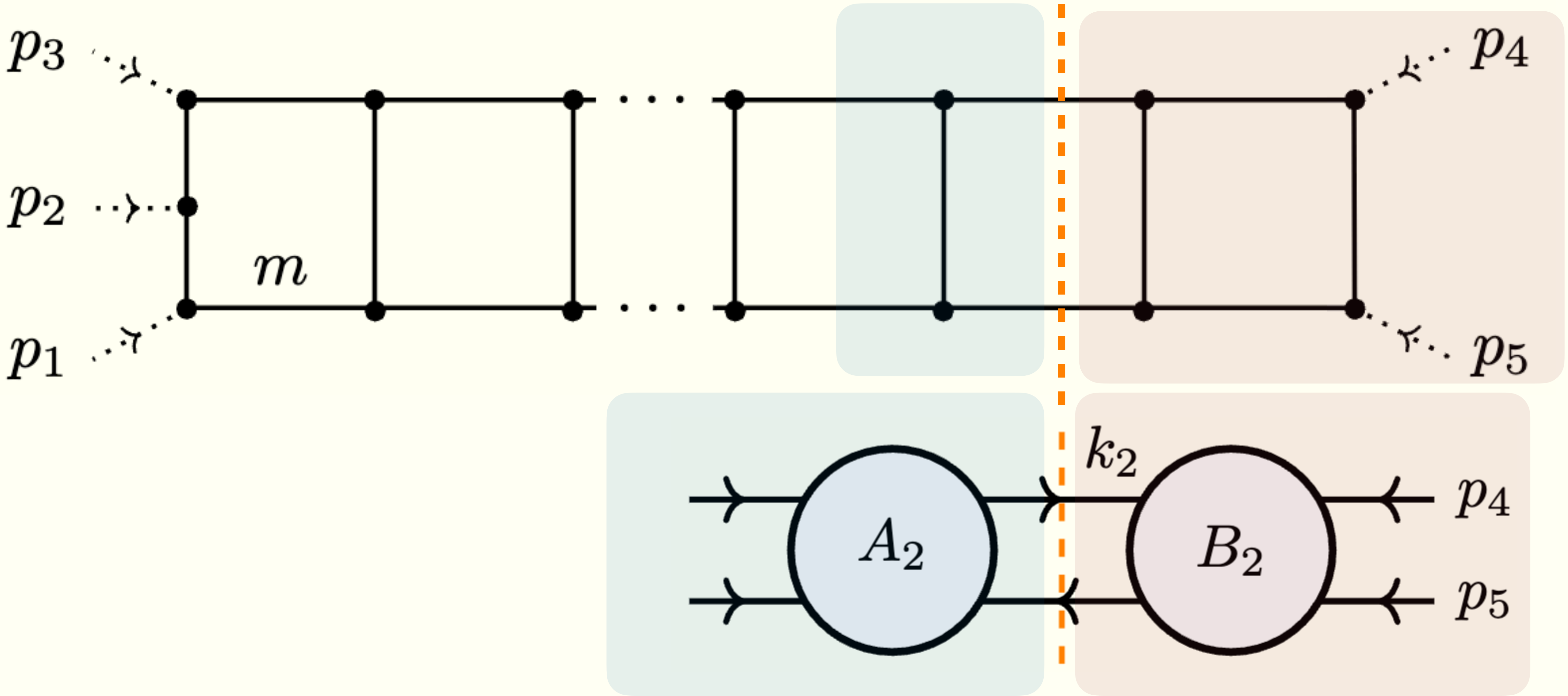
L-LOOP RESULTS

The massive penta-ladder



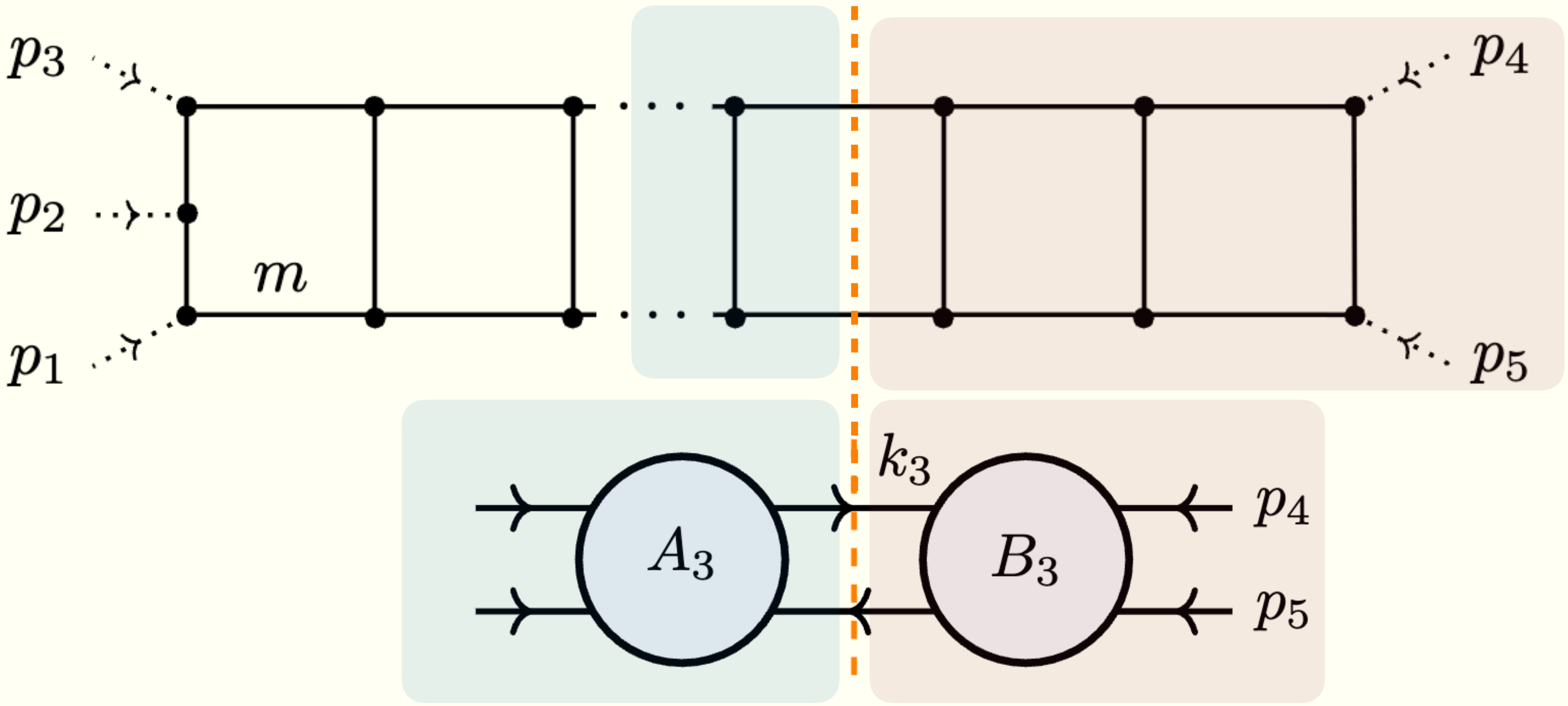
L-LOOP RESULTS

The massive penta-ladder



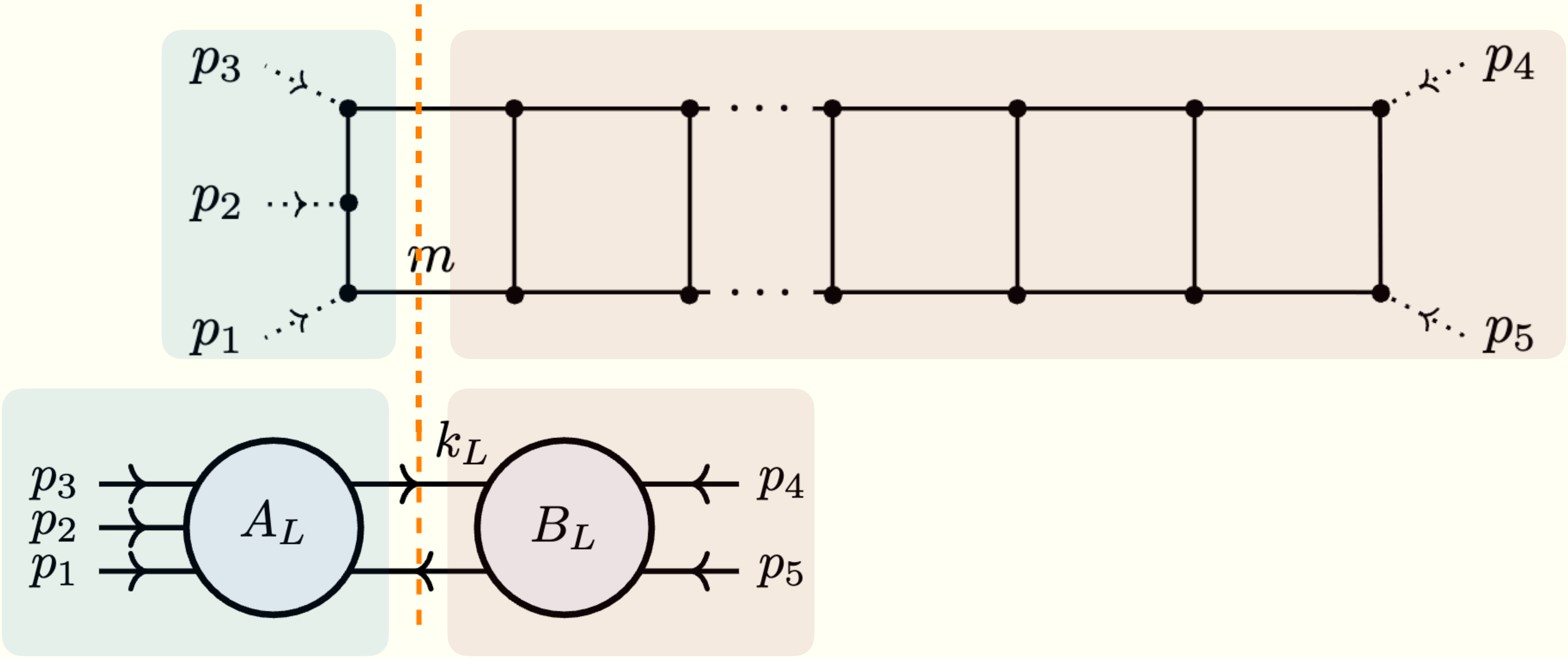
L-LOOP RESULTS

The massive penta-ladder



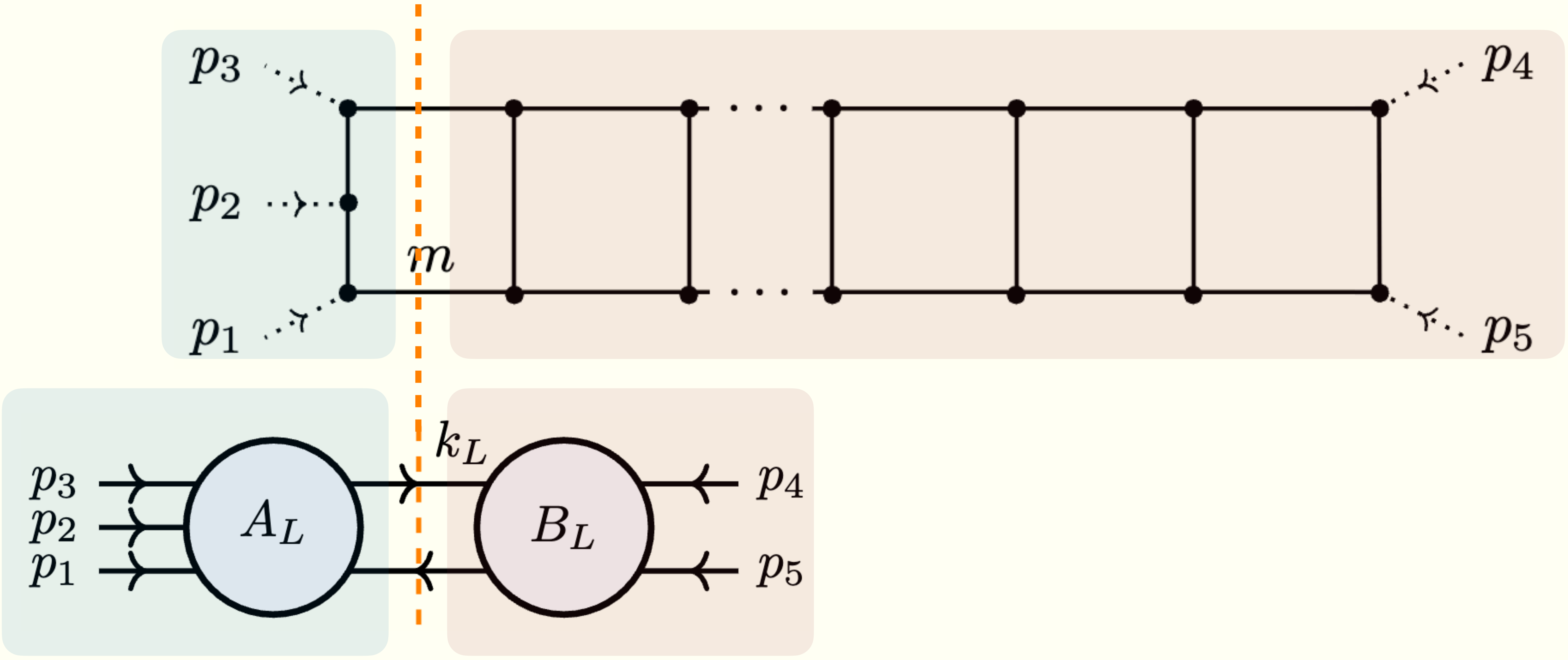
L-loop results

The massive penta-ladder



L-LOOP RESULTS

The massive penta-ladder



The leading singularity of the L-loop penta-ladder is the same as for the ladder when t is replaced by

$$\lambda(Z_{m,m,m,m})^{L-1}\lambda(Z_{m,0,0,m}) - \lambda(Z_{m,0,0,\sqrt{t}}) = 0$$

$$\left[\begin{array}{l} \lambda(z) = z + \sqrt{z^2-1} \\ Z_{a,b,c,d} = \dfrac{\sqrt{s_{45}}(s_{45}+2d^2-2a^2-b^2-c^2)}{\sqrt{s_{45}-4a^2}\sqrt{s_{45}-(b+c)^2}\sqrt{s_{45}-(b-c)^2}} \end{array}\right.$$

[Correia, Sever, Zhiboedov (2020)]

$$\begin{aligned} &m^4 s_{12}s_{23}(s_{12}+s_{23}-s_{45})+s_{12}s_{23}\big[t^2(s_{12}+s_{23}-s_{45}) \\ &-s_{15}s_{34}s_{45}+t(s_{12}(s_{23}-s_{15}) - s_{23}s_{34}+(s_{15}+s_{34})s_{45})\big] \\ &+m^2\big[s_{12}^2(s_{15}^2-2ts_{23}-s_{15}s_{23})+(s_{23}s_{34}+(s_{15}-s_{34})s_{45})^2 \\ &+s_{12}(s_{23}s_{34}(s_{45}-s_{23})-2ts_{23}(s_{23}-s_{45})-2s_{15}^2s_{45} \\ &+s_{15}(2s_{34}s_{45}+s_{23}(2s_{34}+s_{45})))\big] = 0 \end{aligned}$$

[Caron-Huot, Correia, Giroux (2024)]

Outline

Recursion via unitarity

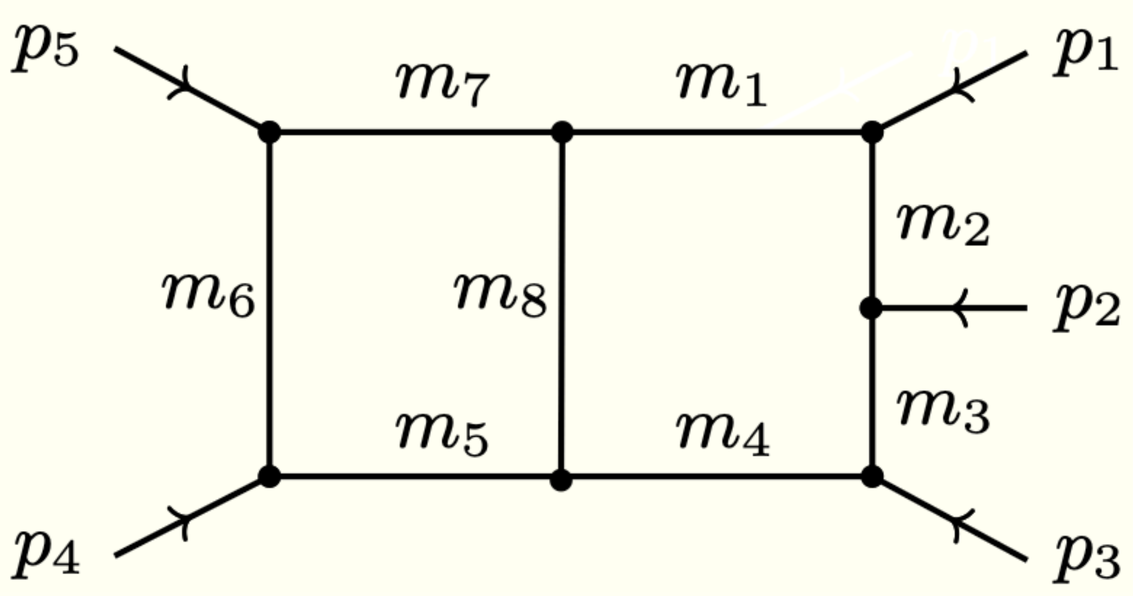
Proof of principle examples:

Recursively finding singularities

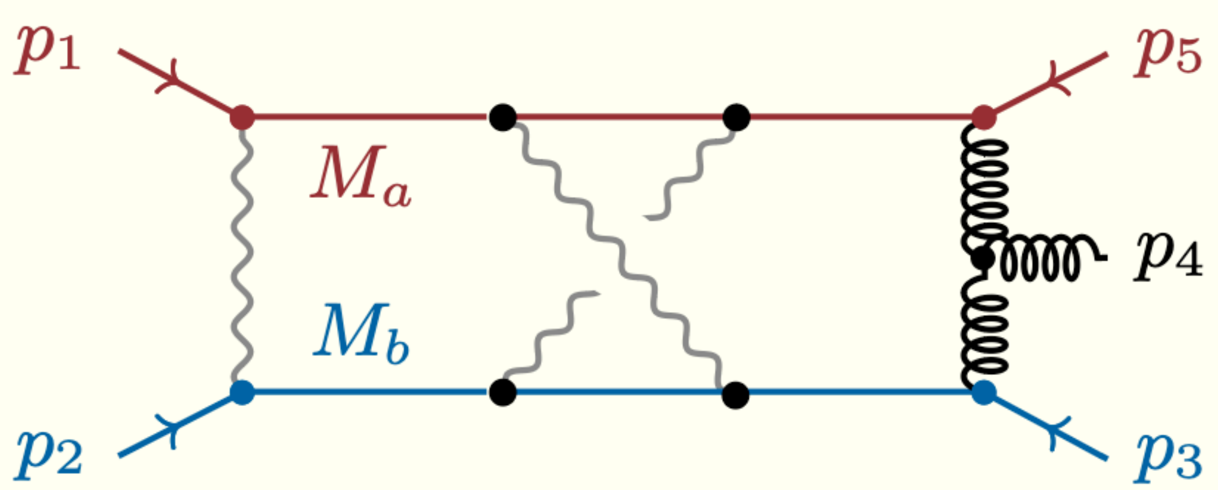
Checks and new analytic predictions:

Leading singularities

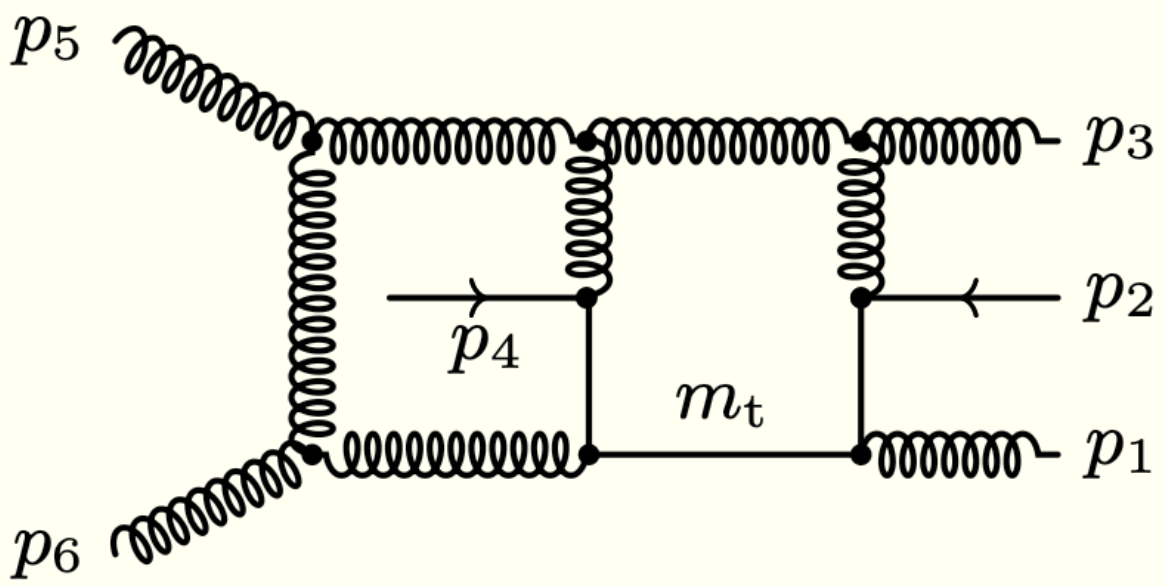
(Generic kinematic pentabox)



(Three-loop QED+QCD boX)



(Non-planar massive hexabox)



Explicit checks

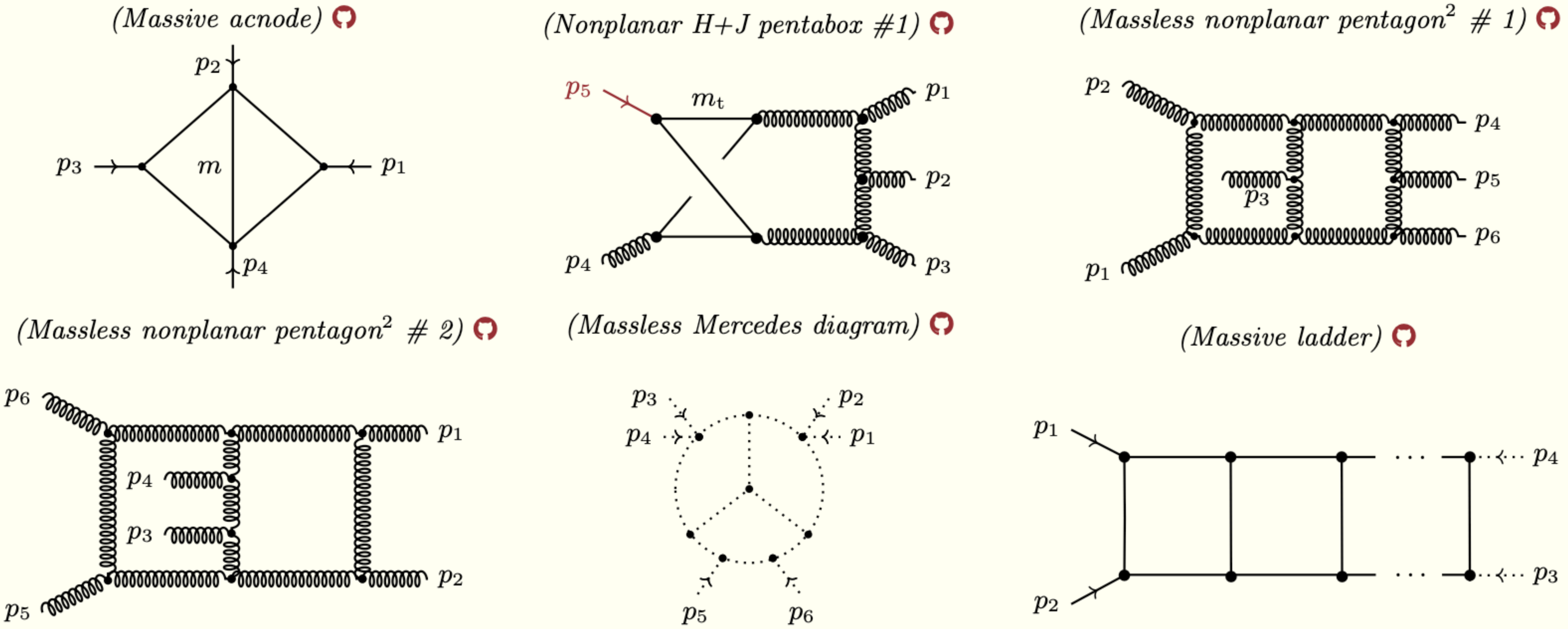
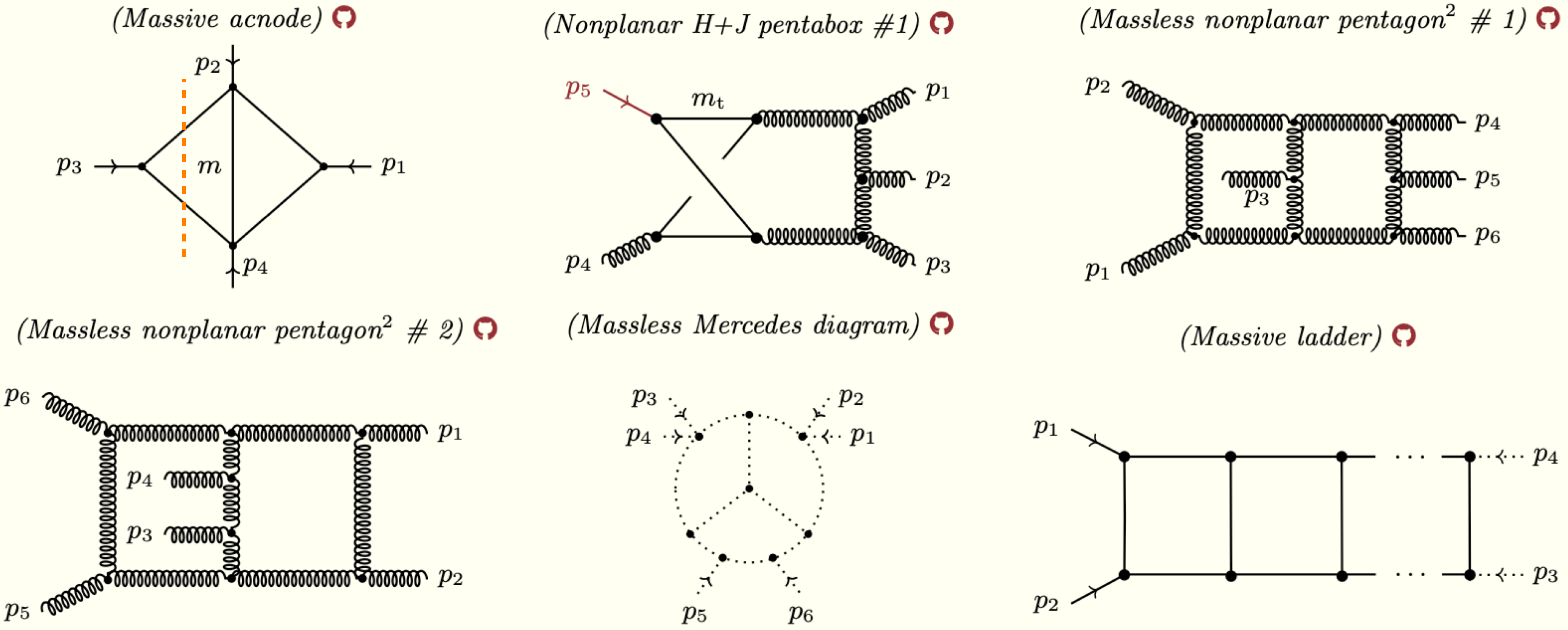


Figure 2. A list of nontrivial examples checked against `PLD.jl` and [19] (for the massive ladder).

Explicit checks



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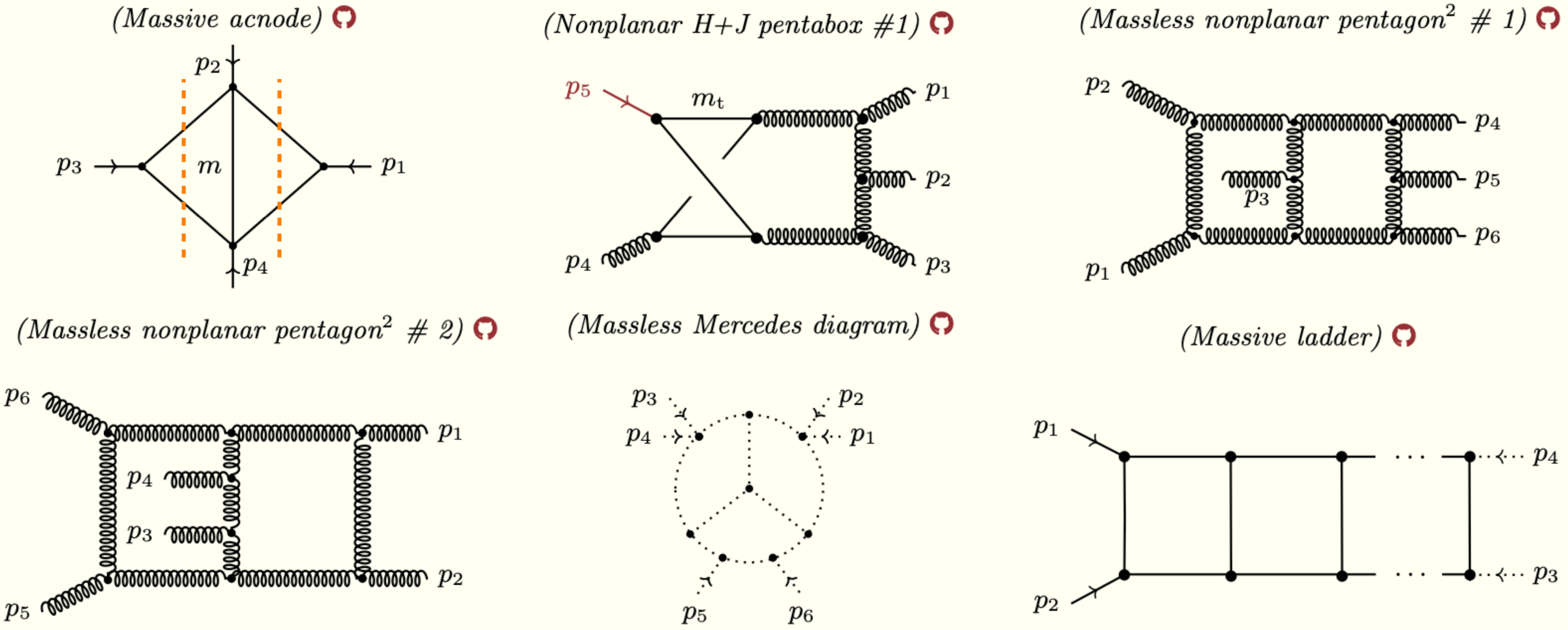
Explicit checks



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Explicit checks

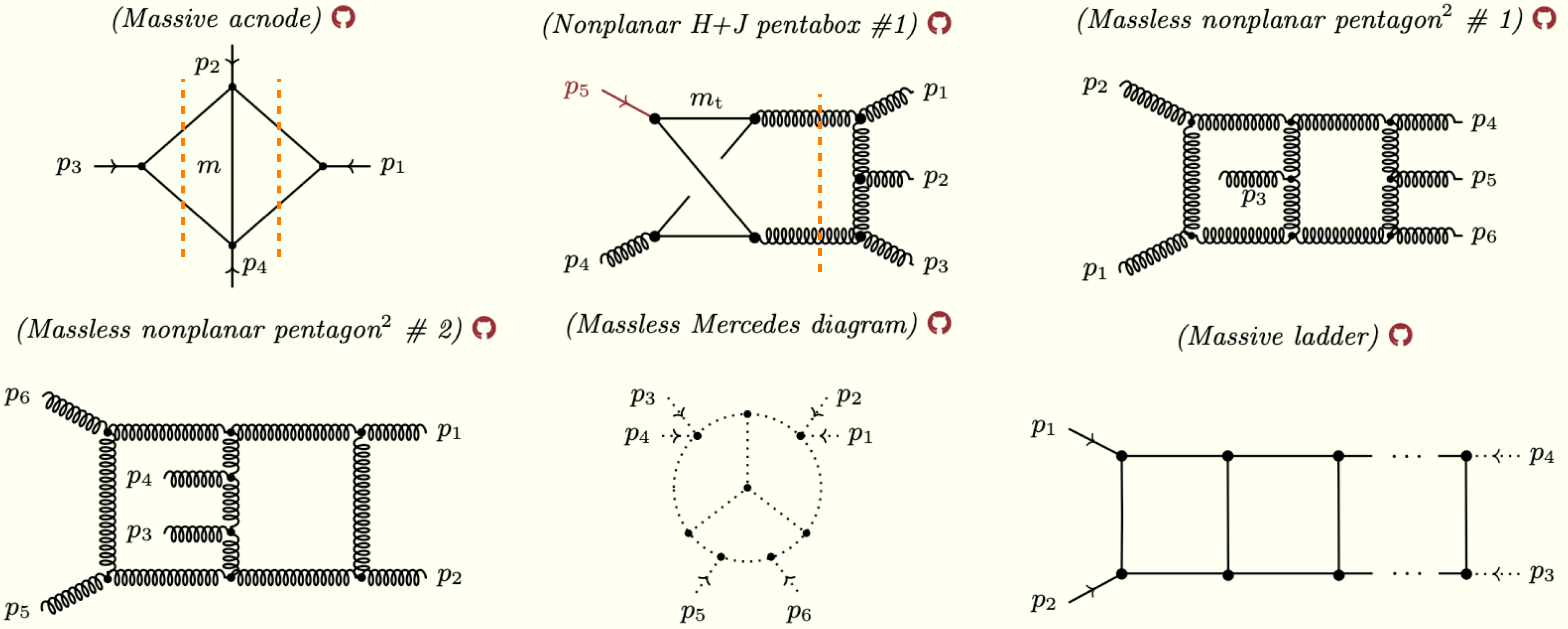


Figure 2. A list of nontrivial examples checked against `PLD.jl` and [19] (for the massive ladder).

Explicit checks

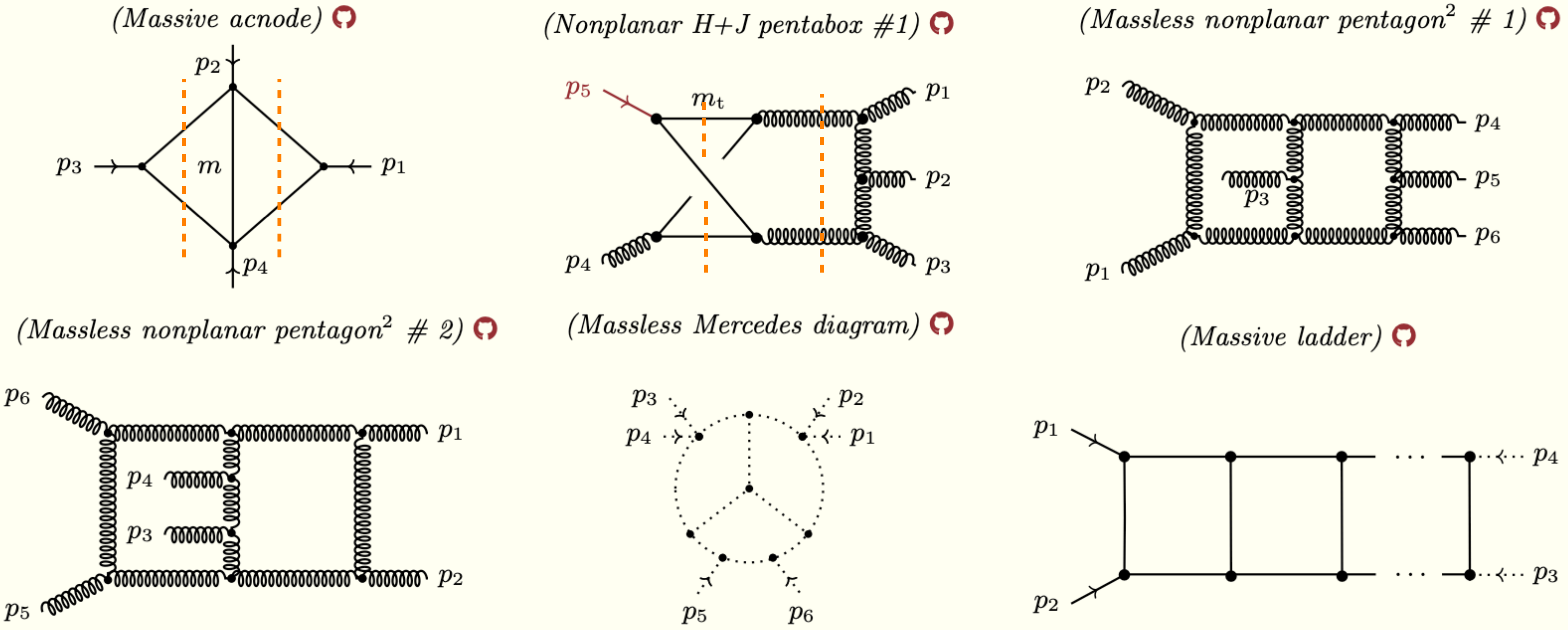


Figure 2. A list of nontrivial examples checked against `PLD.jl` and [19] (for the massive ladder).

Explicit checks

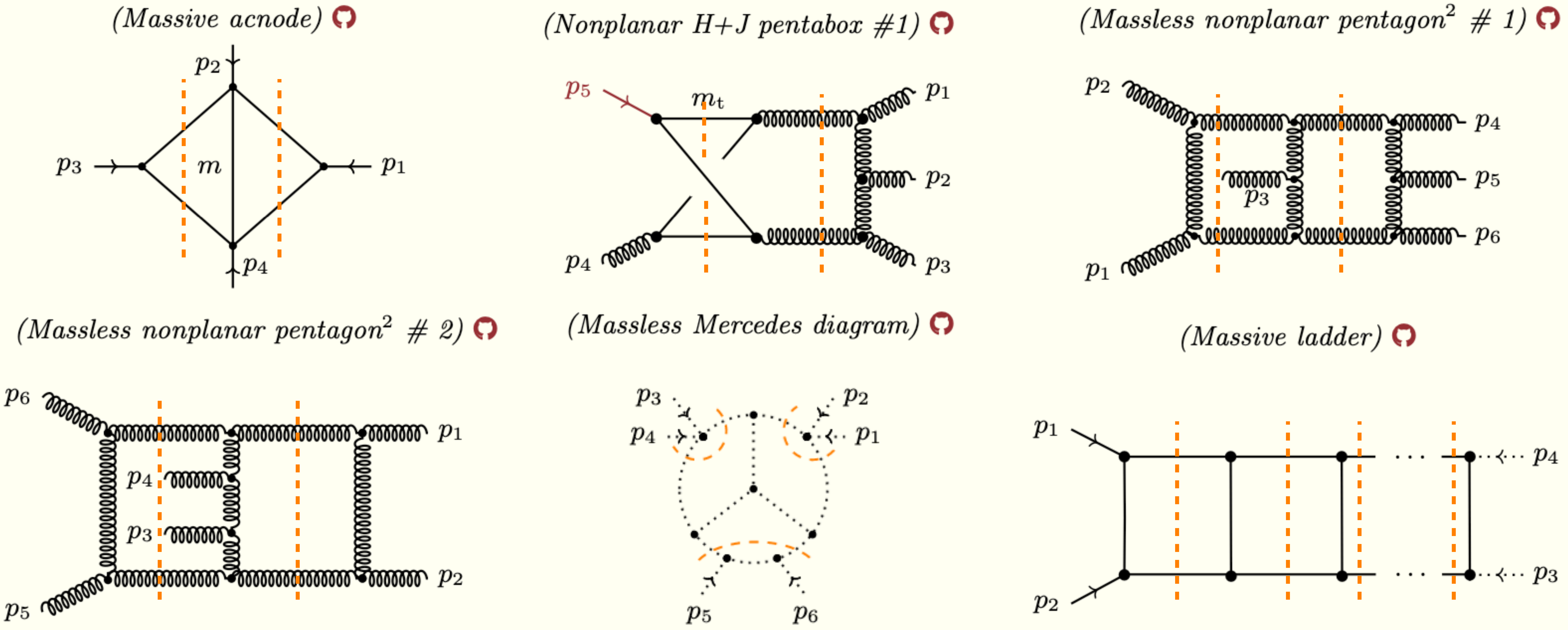
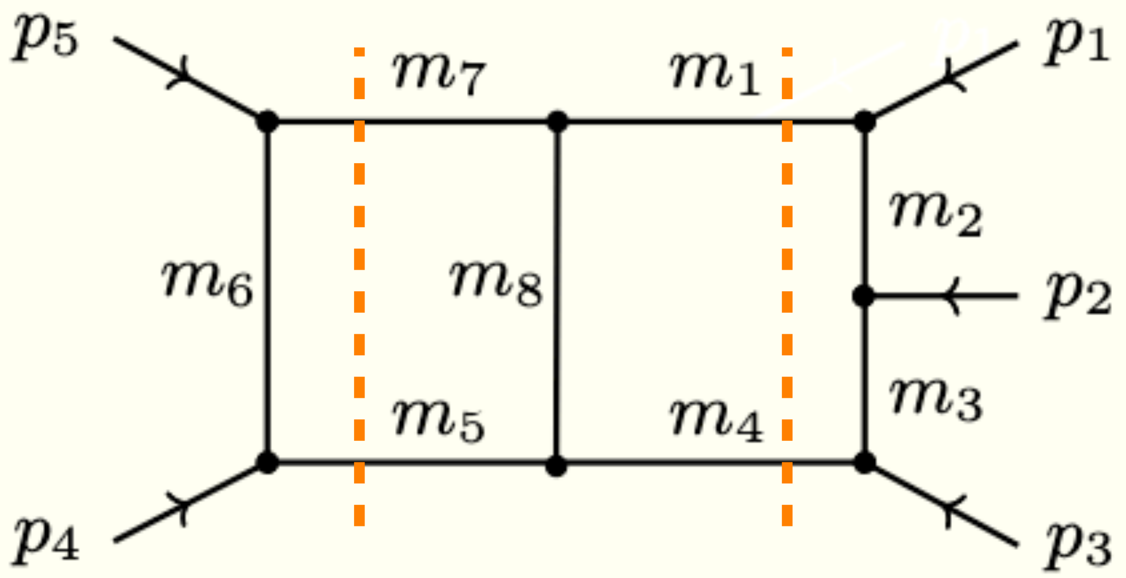


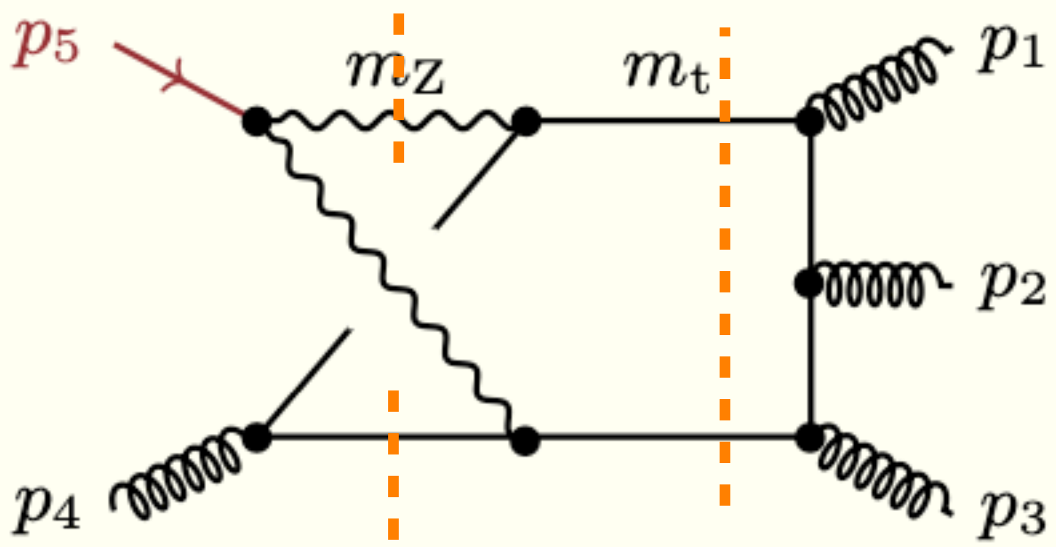
Figure 2. A list of nontrivial examples checked against `PLD.jl` and [19] (for the massive ladder).

New predictions

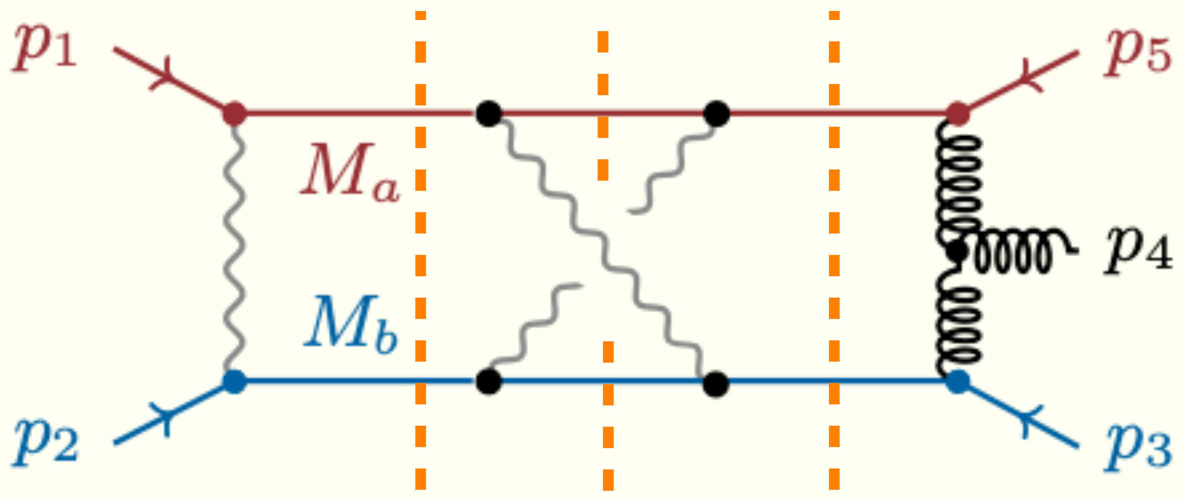
(Generic kinematic pentabox)



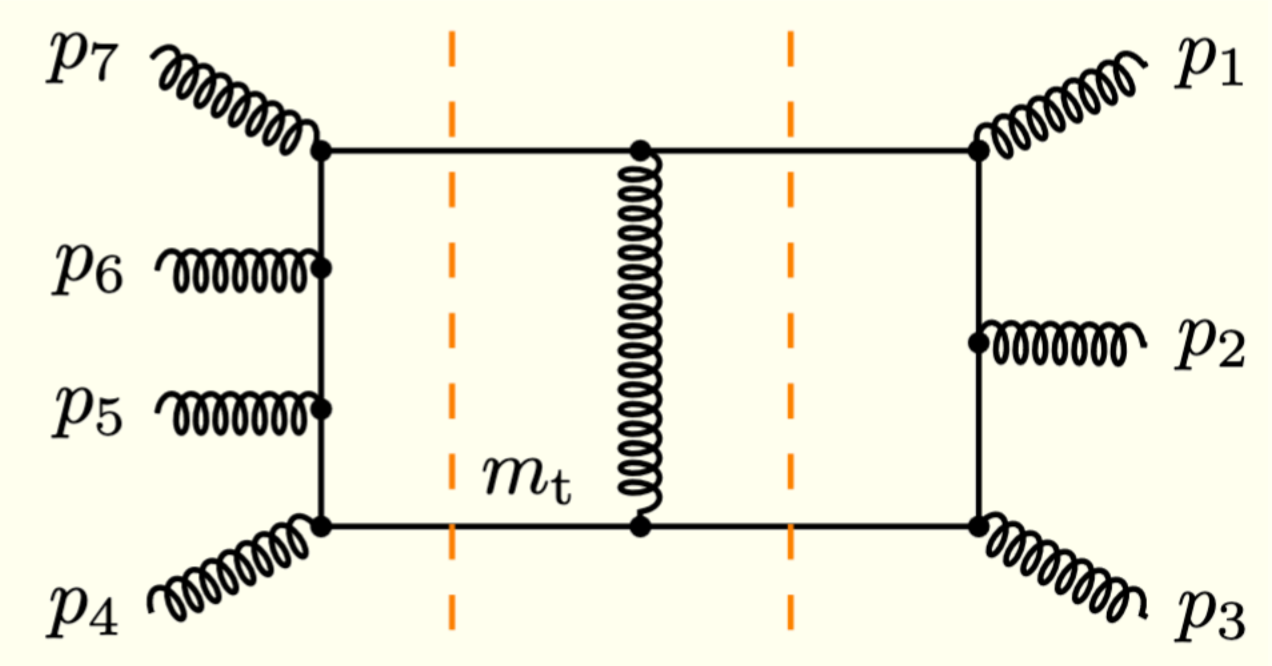
(Nonplanar H+J pentabox #2)



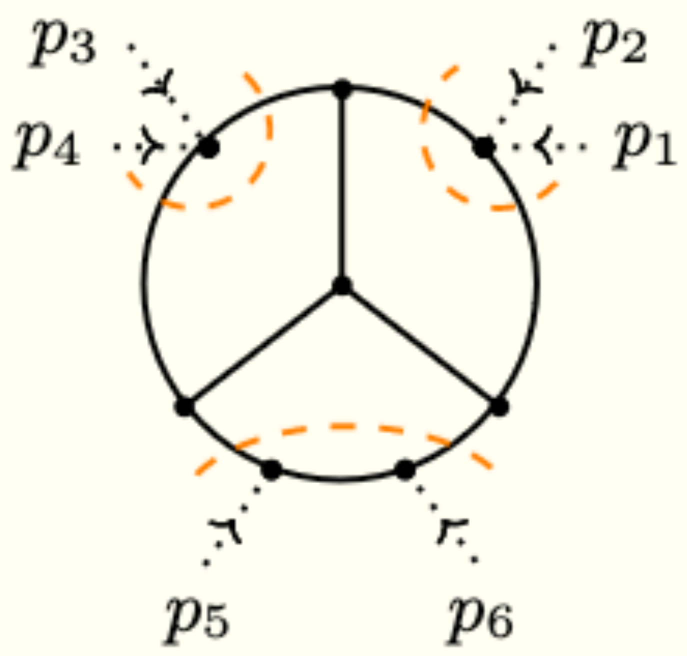
(Three-loop QED+QCD boX)



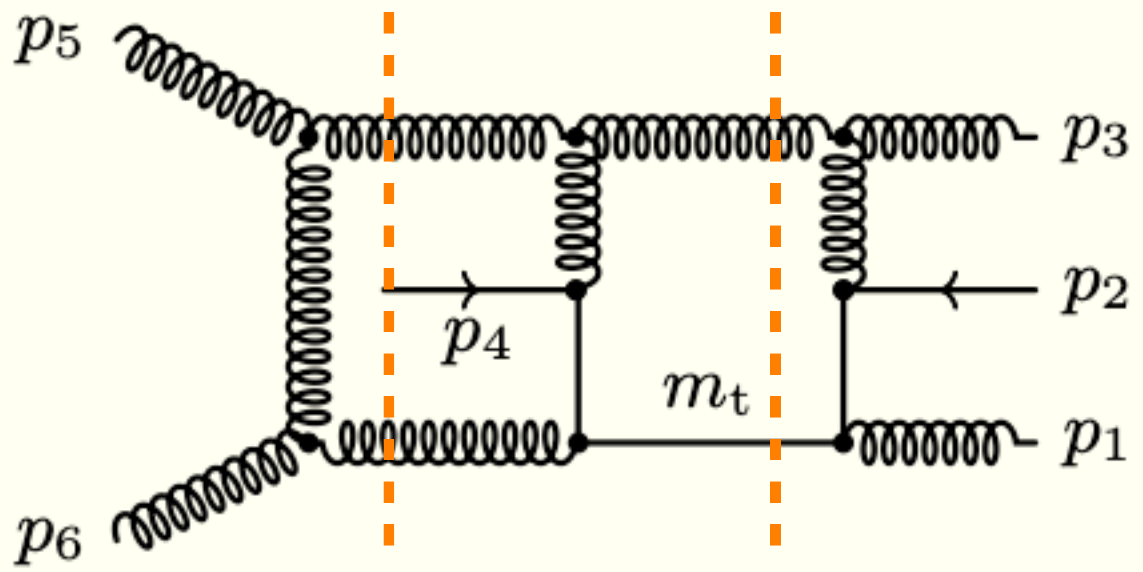
(Massless hexapentagon)



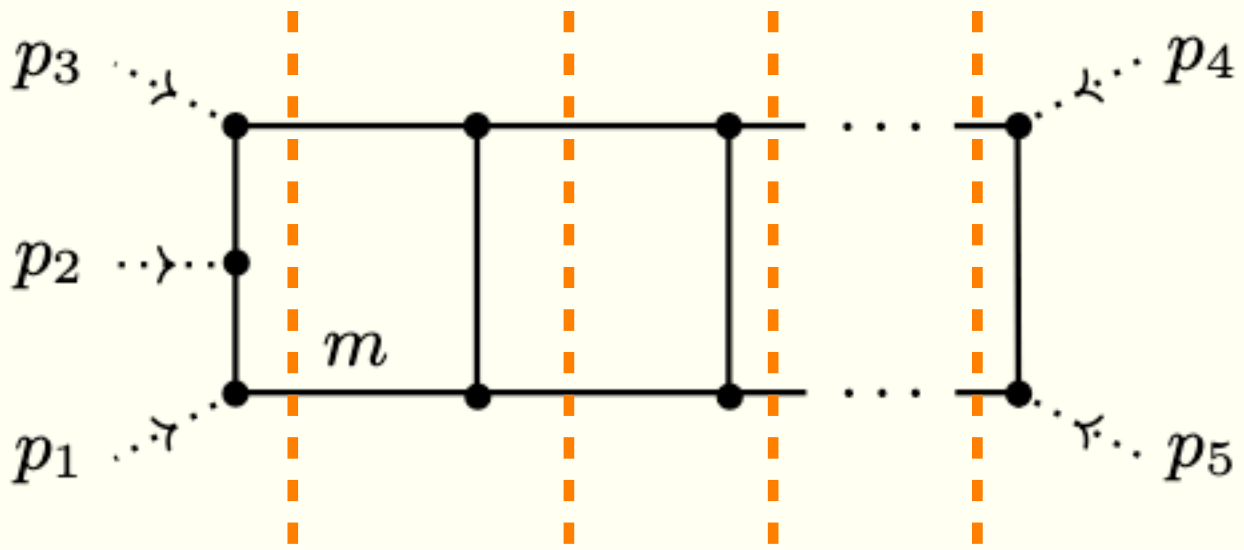
(Massive Mercedes diagram)



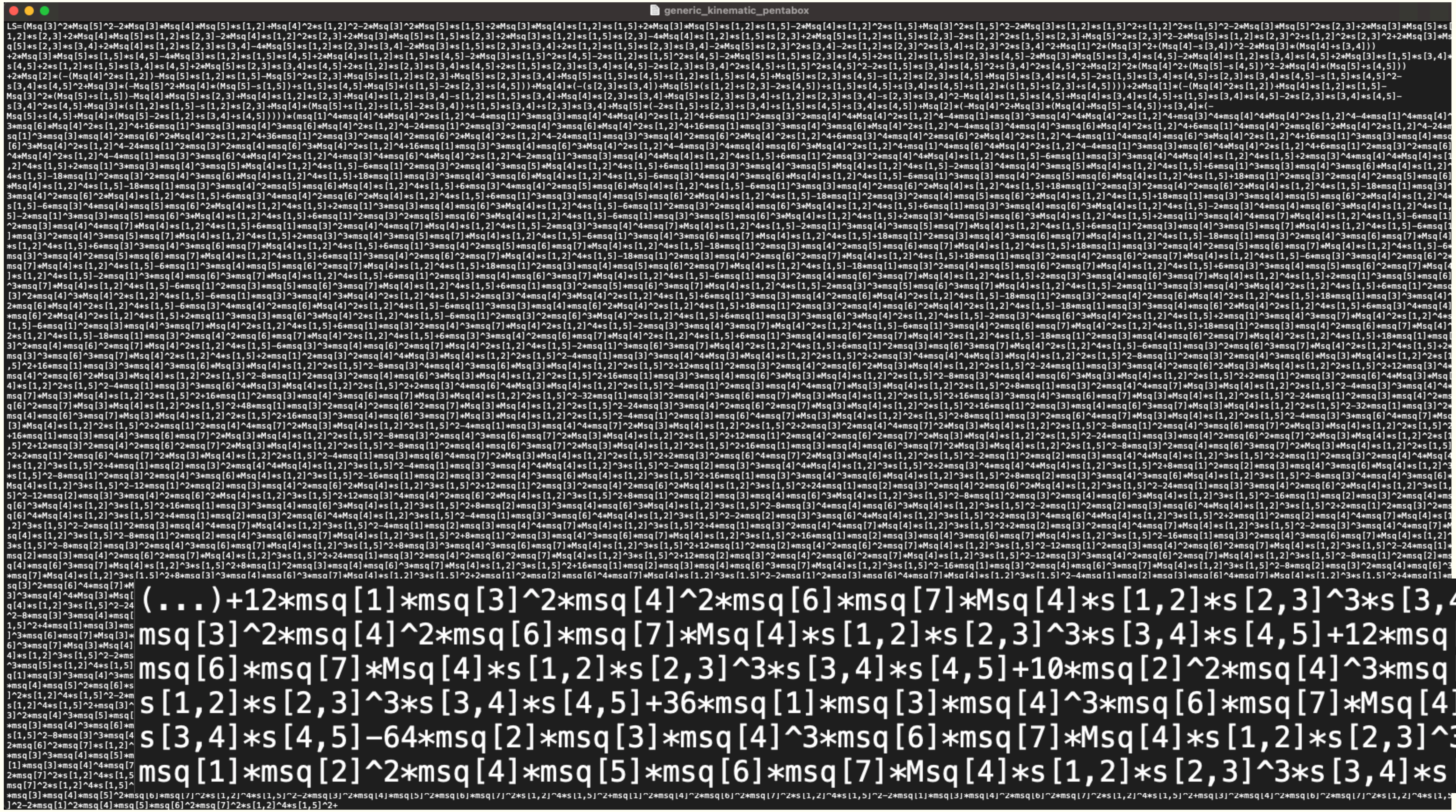
(Non-planar massive hexabox)



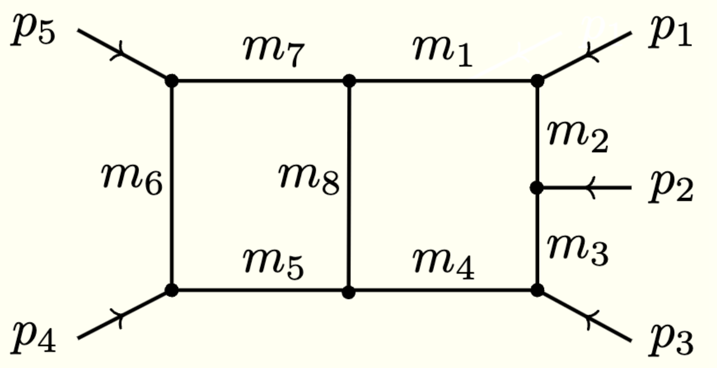
(Massive pentaladder)



Leading singularities can get quite complicated



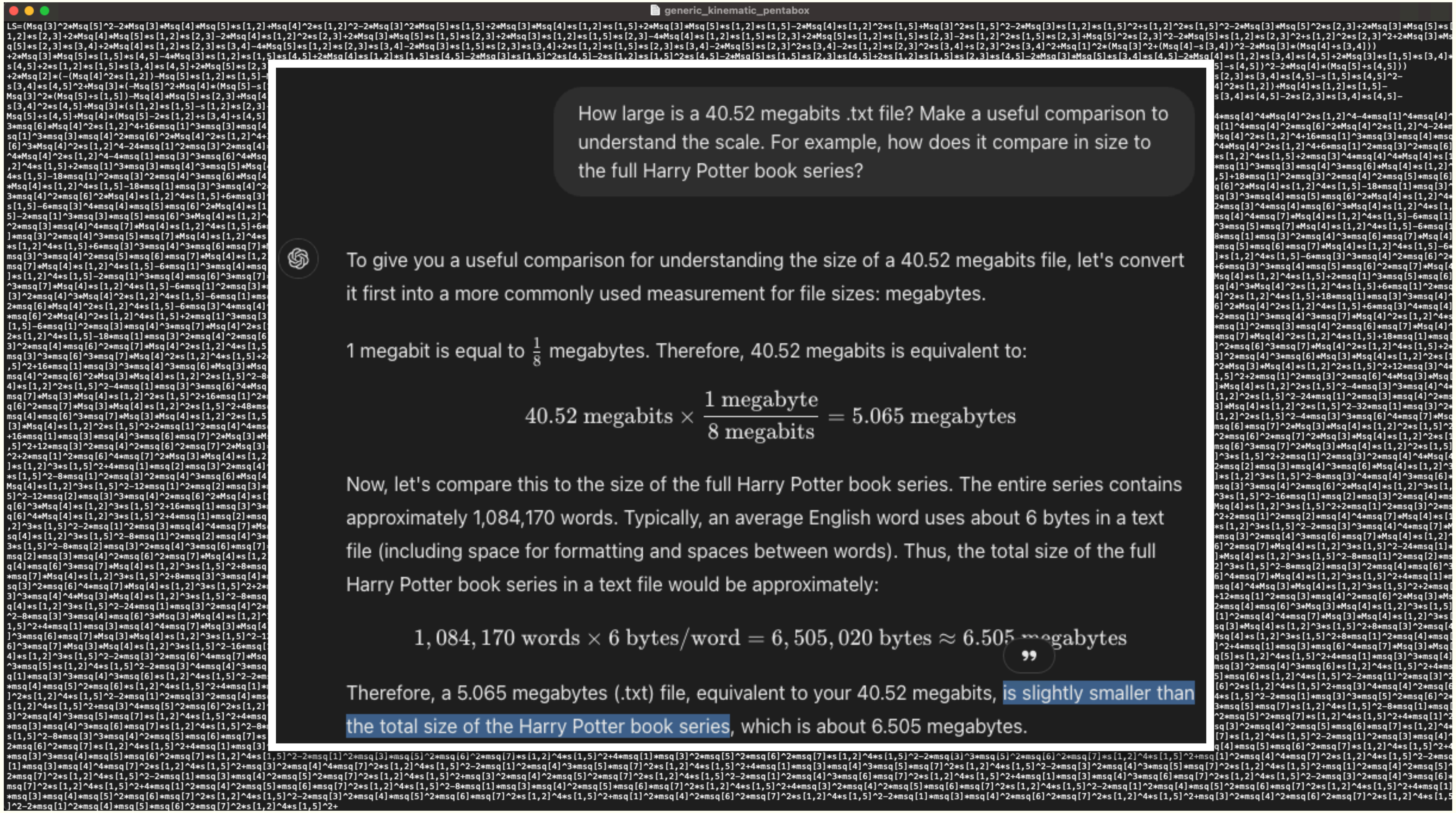

```
(...)+12*msq[1]*msq[3]^2*msq[4]^2*msq[6]*msq[7]*Msq[4]*s[1,2]*s[2,3]^3*s[3,4]*s[4,5]+32*msq[2]*
msq[3]^2*msq[4]^2*msq[6]*msq[7]*Msq[4]*s[1,2]*s[2,3]^3*s[3,4]*s[4,5]+12*msq[1]*msq[2]*msq[4]^3*
msq[6]*msq[7]*Msq[4]*s[1,2]*s[2,3]^3*s[3,4]*s[4,5]+10*msq[2]^2*msq[4]^3*msq[6]*msq[7]*Msq[4]*
s[1,2]*s[2,3]^3*s[3,4]*s[4,5]+36*msq[1]*msq[3]*msq[4]^3*msq[6]*msq[7]*Msq[4]*s[1,2]*s[2,3]^3*
s[3,4]*s[4,5]-64*msq[2]*msq[3]*msq[4]^3*msq[6]*msq[7]*Msq[4]*s[1,2]*s[2,3]^3*s[3,4]*s[4,5]+12*
msq[1]*msq[2]^2*msq[4]*msq[5]*msq[6]*msq[7]*Msq[4]*s[1,2]*s[2,3]^3*s[3,4]*s[4,5]-(...)
```

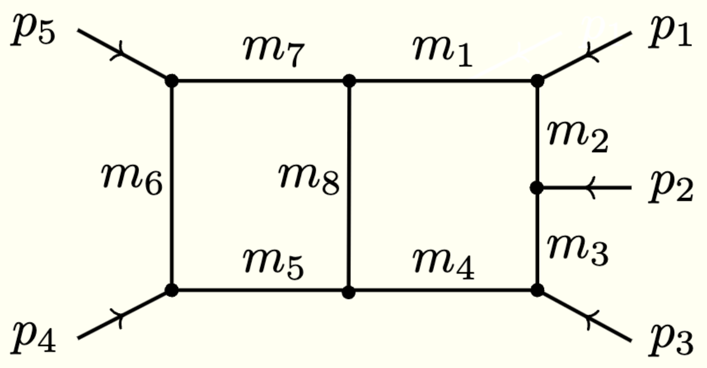


$+\mathcal{O}(10^6)$ terms

[40.52 Mb polynomial]

Leading singularities can get quite complicated





$+\mathcal{O}(10^6)$ terms

[40.52 Mb polynomial]

Conclusion

We introduced an efficient unitarity-based method to extract singularities of Feynman integrals

Stress-tested the method against cutting-edge tools like `HyperInt` and `PLD.jl`

Made new predictions for multi-loop processes, including many examples in the Standard Model

Outlook

Many future directions… here are some pressing ones

Systematic way to include higher-cut subgraphs into the recursion without knowing *a priori* their singularities ?

Recursively reconstructing Schwinger parameters ?

Can we *prove* that this procedure captures all singularities of a diagram (and its subtopologies) ?

Get better at solving systems of high-degree polynomial equations

Thank you!

Dirac at the IAS on his way to cut (actual) trees
[Credit: *Shelby White and Leon Levy Archives Center*]

Extra slides

TYPES OF SOLUTIONS

Leading or subleading singularities

When all or a subset of propagators are set on-shell

[Bjorken, Landau, Nakanishi (1954)]

Second- or mixed-type singularities

When all or a subset of loop momenta diverge ($\ell_i \to \infty$)

[Cutkosky (1960), Fairlie, Landshoff, Nuttall, Polkinghorne (1962)]

[Drummond (1963), Boyling (1967)]

Beyond the standard classification singularities

When a subset of loop momenta diverge ($\ell_i \to \infty$) at different rates

[Berghoff, Panzer (2022), Fevola, Mizera, Telen (2023)]

Higher-cuts diagrams

Examples of (sub)graphs whose singularities cannot be resolved *systematically* by the two-particle cut recursion (may need to use, e.g., `PLD.jl`)

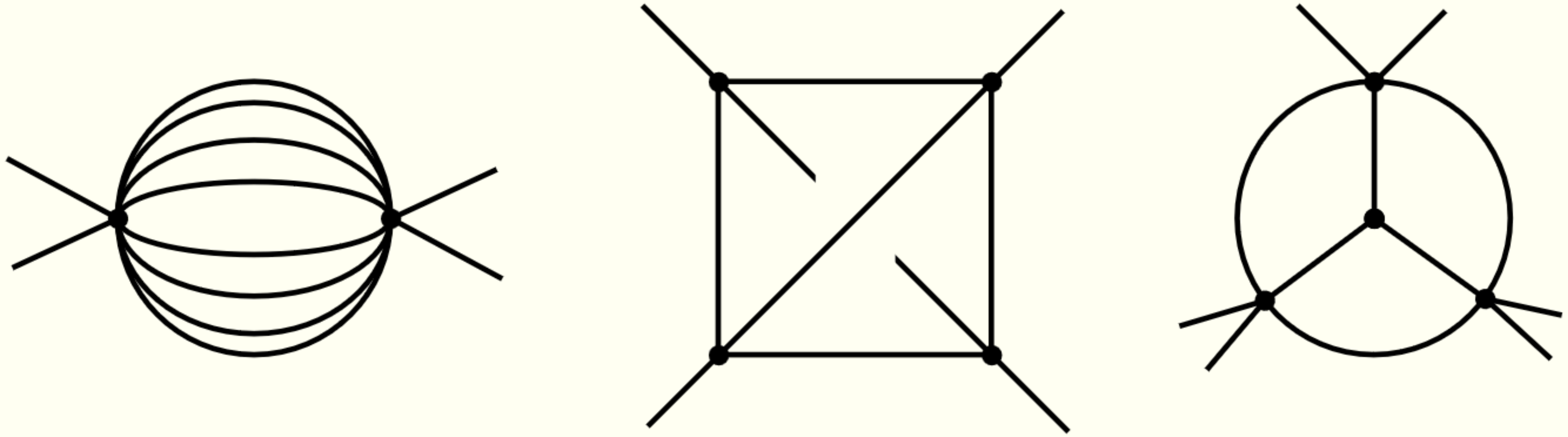


Figure 3. Examples of diagrams with *no* two-particle cuts splitting the graph in two disjoint subgraphs.

Recursively finding singularities

But wait! `PLD.jl` flags another leading singularity :

```
##################################
# Component 2
##################################

D[2] = M[3]^2 - 2*M[3]*M[4] - 2*M[3]*s + M[4]^2 - 2*M[4]*s + s^2
Ï‡[2] = 16
weights[2] = [[-1, -1, -1, -1], [0, 0, 0, 0]]
computed_with[2] = ["PLD_num", "HyperInt"]
```

Where is it in our approach ?

The singularity depends solely on *external* invariants

$$\begin{vmatrix} s & \frac{m_2^2-m_1^2+s}{2} & \frac{M_4^2-M_3^2-s}{2} \\ \frac{m_2^2-m_1^2+s}{2} & m_2^2 & \frac{(m_4\pm m_3)^2-m_2^2-M_3^2}{2} \\ \frac{M_4^2-M_3^2-s}{2} & \frac{(m_4\pm m_3)^2-m_2^2-M_3^2}{2} & M_3^2 \end{vmatrix} = 0$$

It is the expected (from C_{bub}) *collinear divergence* between p_{12} and p_3

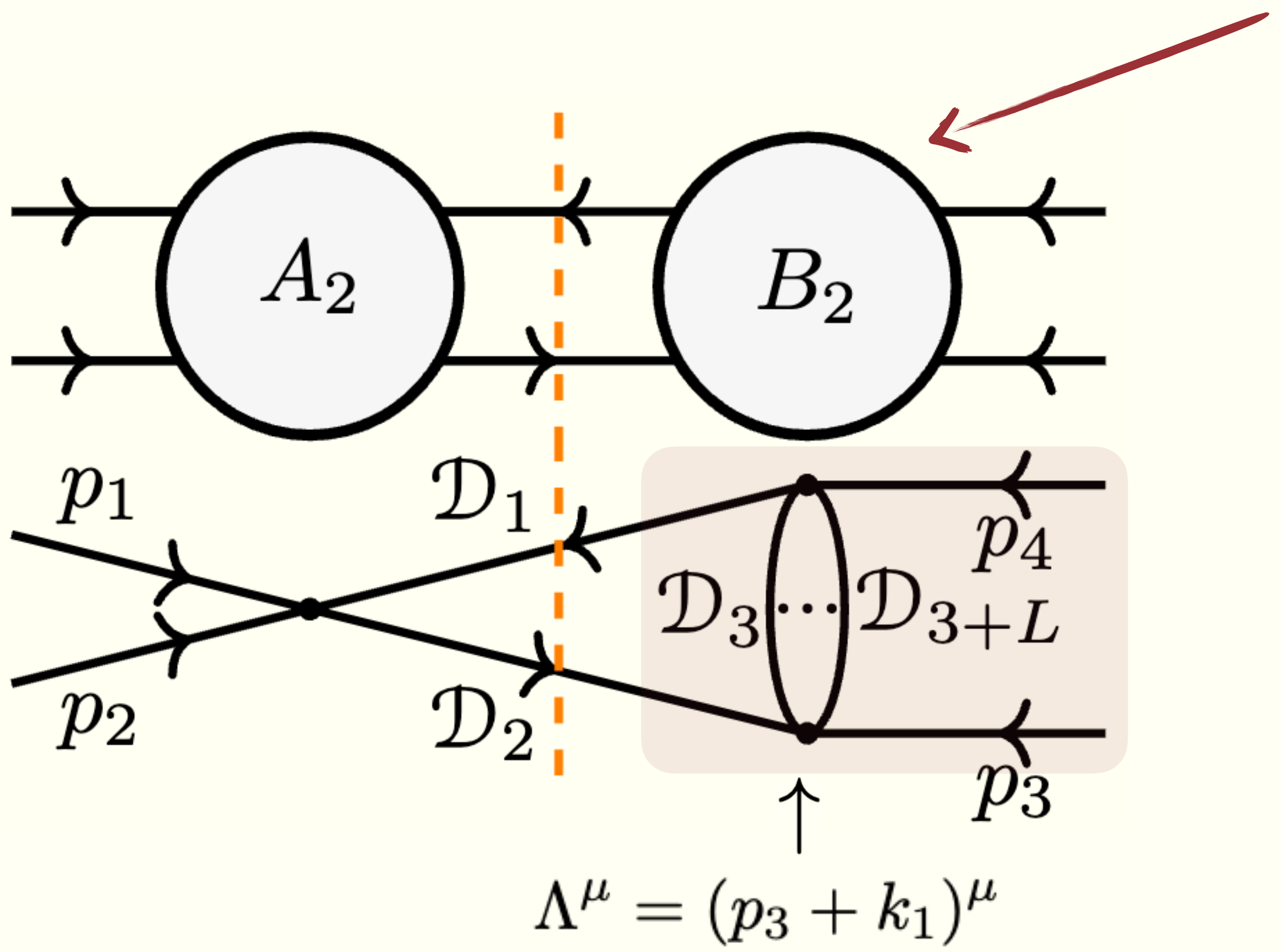
(supported even on the maximal cut)

L-LOOP RESULTS

Some times, this method makes it easy to make L-loop statements

$$\begin{bmatrix} p_{12}^2 & p_{12}\cdot k_1 & p_{12}\cdot p_3 \\ p_{12}\cdot k_1 & k_1^2 & k_1\cdot p_3 \\ p_{12}\cdot p_3 & k_1\cdot p_3 & p_3^2 \end{bmatrix}$$

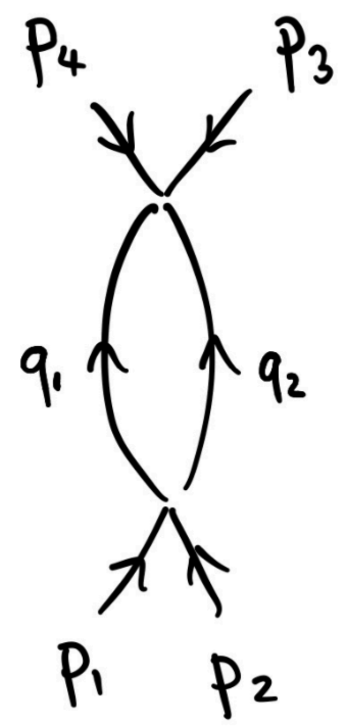
Replace the bubble by L-loop banana graph



Although the banana subgraph does not have a two-particle cut,
we can still find the parachute singularities because the analytic structure of the banana is known *beforehand*

$$k_1 \cdot p_3 = \frac{1}{2}\left[(m_3 \pm m_4 \pm \ldots \pm m_{3+L})^2 - m_2^2 - M_3^2\right]$$

Bubble diagram



$s = (p_1 + p_2)^2$

momentum conservation

$$p_1^\mu + p_2^\mu = q_1^\mu + q_2^\mu = -p_3^\mu - p_4^\mu$$

locality

$$\alpha_1 q_1^\mu = \alpha_2 q_2^\mu$$

on-shellness

$$q_1^2 - m_1^2 = 0$$
$$q_2^2 - m_2^2 = 0$$

$$q_1^\mu = \ell^\mu, \qquad q_2^\mu = p_1^\mu + p_2^\mu - \ell^\mu$$

$$\ell^\mu = (p_1^\mu + p_2^\mu)\frac{\alpha_2}{\alpha_1+\alpha_2}$$

$$s\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)^2 - m_1^2 = 0, \qquad s\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^2 - m_2^2 = 0$$

Lorentz invariant

The solutions are

$$(\alpha_1 : \alpha_2) = \left(\frac{1}{m_1} : \pm\frac{1}{m_2}\right) \qquad s = (m_1 \pm m_2)^2$$

Projective invariance in Schwinger parameters and kinematic variables separately

+ normal threshold

− pseudo-normal threshold

[Slides from Sebastian Mizera]