Bootstrapping $\mathcal{N} = 4$ sYM using integrability.

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Based on a 2207.01615 and an upcoming paper with S. Caron-Huot, F. Coronado

June 10, 2024

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- AdS/CFT: dual to type IIB string theory on $\text{AdS}_5 \times S_5$. Teaches us about the duality and the dual string theory using integrability, conformal bootstrap, susy localization, etc!

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- It has two parameters: the number of colors N_c and the t'Hooft coupling $g^2 = g_{YM}^2 N_c/(16\pi^2)$ (also $\lambda = g_{YM}^2 N_c$).
- The dual parameters in string theory are the string tension, $T = R^2/2\pi\alpha'$ and the string coupling g_{str}:

$$g^2 = T^2/4 \qquad \frac{1}{N_c} = \frac{g_{\text{str}}}{4\pi^2 T^2}$$

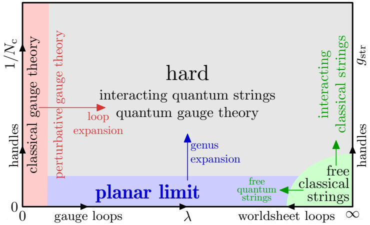

Figure: source: review of integrability in AdS/CFT, 1012.3982.

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- For this, we introduce and use a tailored toolkit for determining the OPE coefficients in the theory through combining Conformal Bootstrap and Integrability:

Outline

Setup

- $\mathcal{N} = 4$ SYM in planar limit and finite t'Hooft coupling

$$g_{\rm YM}^2 \to 0 \qquad N_c \to \infty, \qquad \frac{g_{YM}^2 N_c}{16\pi^2} = g^2$$

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- In the limit, we consider the simplest half-BPS operator, the stress-tensor multiplet:

$$O(x, y) \propto \mathrm{Tr}[(y \cdot \phi(x))^2]$$

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- In our analysis we will focus on multiple couplings ranging from $g = 0.1$ (weak coupling) to $g = 3.7$ (strong couplings).

Correlator

- The 4-pt function of this half-bps operator using superconformal ward identities can be written as:

$$\langle O(x_1, y_1) \dots O(x_4, y_4) \rangle \underset{g^2 \to 0}{=} \text{Free theory} + \frac{1}{c}(z-\alpha)(\bar{z}-\bar{\alpha})(\bar{z}-\alpha)(z-\bar{\alpha})\mathcal{H}(z,\bar{z})$$

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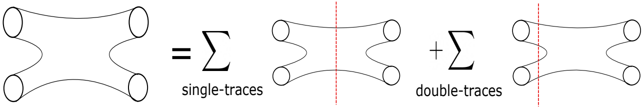
- In the planar limit $\mathcal{H}$ is N_c-independent and it effectively becomes like a correlator of four scalar primaries with $\Delta = 4$ with nice properties.
- The reduced correlator, $\mathcal{H}$, is dual to Virasoro-Shapiro amplitude in $\mathrm{AdS}_5 \times S_5$.

Properties of reduced correlator

- It admits an operator product expansion (OPE) and can be decomposed in sum of conformal blocks.
  - The operators showing up in the OPE are single trace and double trace operators



$$\mathcal{H}(u,v) = \sum_{\text{Single \& Double Traces}} \lambda^2_{\Delta,J} G^{N=4}_{\Delta,J}(u,v)$$

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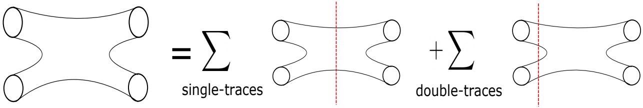

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  - The spectrum of single-trace operators, (Δ, J) is known from integrability for any t'Hooft coupling but the OPE coefficient, $\lambda_{\Delta,J}$ is not known.

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- Regge Limit: as $z, \bar{z} \to \infty$ with $\frac{z}{\bar{z}} =$ fixed, we have: $\mathcal{H} \sim z^{J_*-4}$
Assumption from bound of chaos: $J_* \leq 2$

Our Question

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- Our goal: use bootstrap to bound the OPE coefficient of the lightest unprotected single-trace operators in the spectrum (Konishi operator).
- What is known:
  - in weak coupling regime: perturbatively to five loops, independently from bootstrap+localization

    [Eden, Paul, Goncalves, Fleury, Pereira . . . ;Chester, Dempsey, Pufu]
  - in strong coupling regime (dual to massive string mode): known from bootstrapping AdS Virasoro-Shapiro amplitude (analytically)

    [Alday, Hansen, Silva;2022-2023]

No results on the OPE away from the weak and strong coupling!

Outline

Attempts to Bootstrap (Conventional Method):

- Bootstrap method is a way of imposes consistency constraints such as crossing at the level of OPE to bound spectrum and OPE coefficients of the operators. For example $s \leftrightarrow u$-channel crossing gives:

$$\mathcal{H}(u,v) - u^{-4}\mathcal{H}(\frac{1}{u},\frac{v}{u}) = 0$$

$$\sum_{\text{ST and DT}} \lambda^2_{\Delta,J} F_{\Delta,J}(u,v) + F^{\text{protected}}(u,v) = 0$$

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- We can then think of $\vec{F}_{\Delta,\ell}$ as infinite dimensional vectors in the infinite dimensional vector space of functions of u and v.

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$$\sum_{\text{DT \& ST}\neq\text{Konishi}} \lambda^2_{\Delta,J}\alpha[\vec{F}_{\Delta,\ell}] = \mp\lambda^2_{\text{Konishi}} - \alpha[F^{\text{protected}}] \geq 0$$

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- But this does not work: double-traces do not enter with positive coefficient (Leading order OPE coefficient of DTs enters in the free part!)

For DT: $\lambda^2_{\Delta,J} \ngeq 0$

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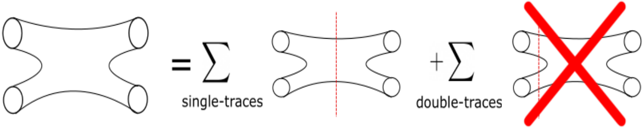

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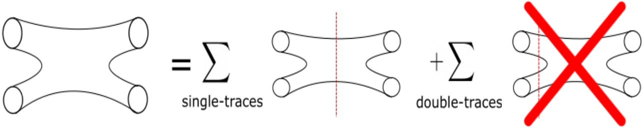

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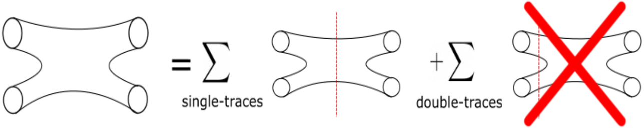

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Below are the summary of steps we take to set up the bootstrap problem:

- Build sum rules that acts on this OPE expansion and are only sensitive to the single-trace operators in the expansion: single trace enter in OPE with positive coefficients and their scaling dimensions are computable using integrability.
- Get the spectrum of single-trace data from integrability.
- Use these sum rules in numerical bootstrap to get bounds on OPE coefficients.

Outline

How to kill the double-trace: double discontinuity

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We can do this by defining the dDisc of the correlator (similar to imaginary part of amplitudes).

$$\text{dDisc}_s \mathcal{H}(z,\bar{z}) = \mathcal{H}(z_+,\bar{z}_-) - \frac{1}{2}\mathcal{H}(z_-,\bar{z}_-) - \frac{1}{2}\mathcal{H}(z_+,\bar{z}_+)$$

with $z_\pm = z \pm i0$
dDisc systematically kills all the double traces in the OPE sum since it acts on the blocks as:

$$\text{dDisc}_s G_{\Delta,J}(z,\bar{z}) = 2\sin^2\left(\frac{\Delta - J - 2\Delta_{\text{exteral}}}{2}\pi\right) G_{\Delta,J}(z,\bar{z})$$

For DT $\qquad \Delta - J = 2\Delta_{\text{external}} + \frac{\gamma_{\Delta,J}}{N_C}$

For DT $\qquad \text{dDisc}_t G_{\Delta,J}(z,\bar{z}) \propto \frac{1}{N_c^2}$

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- OPE expanding $\mathcal{H}(u',v')$ inside the dDisc gives a new expansion including only stress-tensor multiplet and single traces.

$$\mathrm{dDisc}[\mathcal{H}(u,v)] = \mathrm{dDisc}[\mathcal{H}^{\mathrm{protected}}(u,v)] + \sum_{\substack{(\Delta,J) \\ \text{Long ST}}} \lambda^2_{\Delta,J} 2\sin^2\left(\pi \frac{\Delta - J}{2}\right) G^{N=4}_{\Delta,J}(u,v)$$

- We define the Polyakov-Regge blocks which are dispersive transforms of a single blocks:

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- In addition, we have $\mathrm{dDisc}[\mathcal{H}^{\mathrm{protected}}(u',v')] = \mathrm{dDisc}[\mathcal{H}^{\mathrm{strong}}]$ since at $g \to \infty$, all non-protected single traces become heavy and decouple.

We can now express the correlator as:

$$\mathcal{H}(u,v) = \mathcal{H}^{\mathrm{strong}}(u,v) + \sum_{\substack{(\Delta,J) \\ \text{Long ST}}} \lambda_{\Delta,J}^2 \mathcal{P}_{u,v}^{N=4}[\Delta, J]$$

This is the key formula in our analysis. We can now use this expansion of the correlator to obtain various sum rules that are only sensitive to single-trace data.

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- Sum rules to imposes crossing symmetry (both in position space and Mellin space)
- Sum rules to impose consistency between dispersion relations with different kernels (both in position space and Mellin space).
- sum rules to impose the integrated constraints: relate certain integrations of the correlator to the known functions of the couplings.

Integrated Sum Rules

- Integrated sum rules relate the integrated stress-tensor correlators to derivatives of the free energy of the mass deformed theory placed on a sphere, $\partial_m^4 F(m,\tau,\bar{\tau})$ and $\partial_m^2 \partial_\tau \partial_{\bar{\tau}} F(m,\tau,\bar{\tau})$ (here $\tau = \frac{\theta}{2\pi} + i\frac{4\pi^2}{g_{YM}^2}$).
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- In the planar limit, schematically they look:

$$\text{Integrated}(\mathcal{H}(u,v)) = C(g)$$

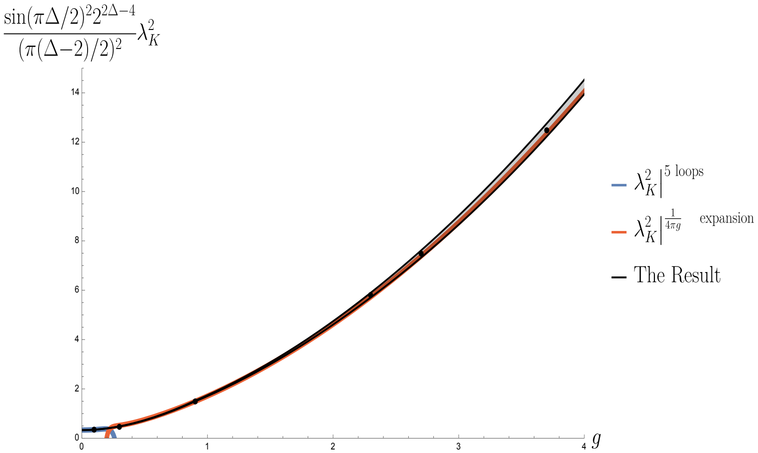
$C(g)$ is calculated from supersymmetry localization and in the planar limit is given in the literature.

[Binder, Chester, Pufu, Wang;2019], [Chester, Pufu; 2020], [Dorigoni,Green, Wen; 2021], [Wen, Zhang; 2022]

Outline

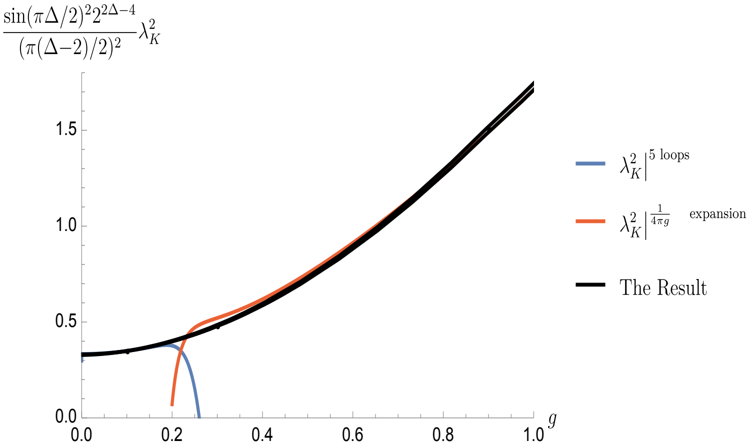
Rigorous Results for OPE

- Doing numerical bootstrap with the discussed set of dispersive sum rules gives us our final results for the OPE of Konishi: rigorous bound from weak to strong coupling.



Rigorous Results for OPE: Zoomed in

- Focusing close to the breaking point to see the validity of analytic expansions:



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- Saturating Lower bounds are only possible once we give additional data on the t'Hooft coupling to the bootstrap problem using the integrated sum rules.

Outline

Flat Space Limit: $g \gg 1$

- To go to very large coupling it is useful to think of the flat space limit of the dual AdS amplitude. Using Mellin space variables s, t, u:

$$4\pi g = \frac{R^2_{\text{AdS}}}{\alpha'} \to \infty, \quad s, t, u \to \infty \quad \text{with} \quad \frac{s}{4\pi g} = \text{fixed}$$

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- Leading order OPE coefficient at $g \gg 1$ comes from the flat space Virasoro-Shapiro amplitude:

$$2\sin^2(\pi\Delta_K/2)\lambda_K^2 = \frac{\pi^4 g^2}{2}\lambda_{\Delta_K}^{2,\text{free}}(1 + O(1/g))$$

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$$4\pi g = \frac{R^2_{\text{AdS}}}{\alpha'} \to \infty, \quad s, t, u \to \infty \quad \text{with} \quad \frac{s}{4\pi g} = \text{fixed}$$

- Leading order OPE coefficient at $g \gg 1$ comes from the flat space Virasoro-Shapiro amplitude:

$$2\sin^2(\pi\Delta_K/2)\lambda_K^2 = \frac{\pi^4 g^2}{2}\lambda_{\Delta_K}^{2,\text{free}}(1 + O(1/g))$$

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[Caron-Huot,Duong,2017]

- Thus at $g \gg 1$ our numerical bootstrap becomes equivalent to the flat space numerical bootstrap: well-understood and serves as a toy model.

Summary

- We find rigorous sandwiching lower bounds and upper bounds for OPE coefficient of the Konishi operator using numerical bootstrap, itegrated constraints and integrability for weak and strong couplings with error.

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[Alday, Hansen, Silva2022-2024: See Tobias's Talk]

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- We find bounds for the OPE coefficients of the other heavier single trace operators.

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- Apply these methods to other theories, For example ABJM in the planar limit?

Thank you!

Exclusion Plot

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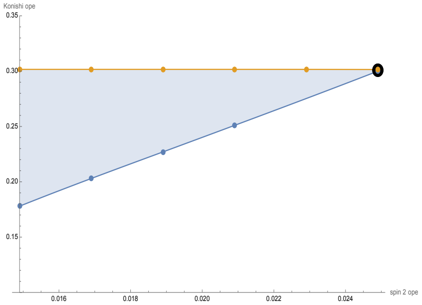
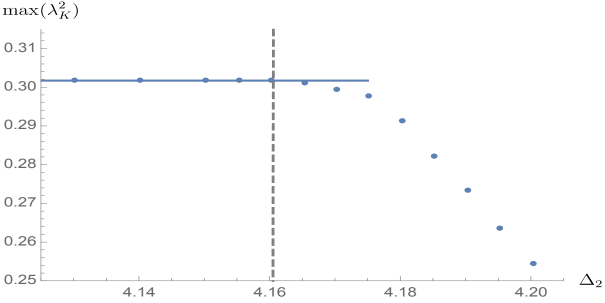
- Since integrated sum rules input additional information on coupling, we might think that inputting data for the coupling in other ways, would still gives us a lower bound for Konishi.
- Indeed, we can check this for $g = 0.1$, that we have perturbative data for the OPE coefficient of other operators. See the exclusion plot for OPE of Konishi in terms of OPE of next operator in the trajectory, the operator with spin 2.



Some Properties:

- The bound has a dependence on the leading trajectory single-trace data: for example if we impose positivity on the interval that includes points close to spin 2 operator but not the exact spin 2 operator of the theory, we can rule out the theory. See in the plot the bound only stabilizes after we include spin 2.



- This is not the case for operators on the subleading trajectories

$X_{u,v}$:

- $u \leftrightarrow v$ crossing symmetry (symmetry between s & t-channel) is ensured by choosing a crossing symmetric kernel:

$$\mathcal{P}^{N=4}_{\Delta,J}(u,v) = \mathcal{P}^{N=4}_{\Delta,J}(v,u)$$

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$$\mathcal{P}^{N=4}_{\Delta,J}(u,v) = \mathcal{P}^{N=4}_{\Delta,J}(v,u)$$

- However imposing crossing symmetry between s and u-channel gives us infinitely non-trivial constraints:

$$0 = \sum_{\substack{(\Delta,J) \\ \text{Long ST}}} \lambda^2_{\Delta,J} \underbrace{\left(\mathcal{P}^{N=4}_{\Delta,J}(u,v) - u^{-4}\mathcal{P}^{N=4}_{\Delta,J}(1/u, v/u)\right)}_{X_{u,v}[\Delta,J]}$$

Here, we also introduced the notation $X_{u,v}[\Delta, J]$ for the action of sum rules on different conformal blocks.

B_t & B_v Family of Sum Rules

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- This gives set of 2 continuous families of sum rules tailored for $\mathcal{N}=4$ SYM.

Projective Sum Rules

We want to define projection sum rules that solve the 1-loop problem analytically.

- At weak coupling dispersive sum rules delays the contribution of ST operators with twists $4+2m+O(g) \geq 4$ for $m \geq 0$ using dDisc to two loop (order g^2)

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- At weak coupling dispersive sum rules delays the contribution of ST operators with twists $4 + 2m + O(g) \geq 4$ for $m \geq 0$ using dDisc to two loop (order g^2)
- The only single trace operators that survive at one-loop are operators with twist $\tau = 2 + O(g)$. We want to build sum rules that singles out the contribution of a single spin at those twists. We do this by finding the kernel to integrate $\widehat{B}_t$ against:

$$W[\Delta, J] \equiv \int \frac{dt}{4\pi i} W[t] \widehat{B}_t[\Delta, J]$$

Projective Sum Rule: $\Phi_{\ell,\ell+2}$ & Ψ_ℓ

- The projective sum rules that we build are:

$$\lim_{\tau\to 2} \Psi_\ell[\tau+J,J] = \delta_{\ell,J}(1+(\tau-2)\beta_\ell) + O((\tau-2)^2)$$

$$\lim_{\tau\to 2} \Phi_{\ell,\ell+2}[\tau+J,J] = 0 + (\tau-2)\left(\delta_{\ell,J} - \frac{\Phi^\infty_\ell}{\Phi^\infty_{\ell+2}}\delta_{\ell+2,J}\right) + O((\tau-2)^2)$$

  Here β and Φ^∞_ℓ is known.
- Acting with Ψ_ℓ on the correlator then relates the anomalous ope coefficient and anomalous dimension for spin ℓ operator at one loop.
- Acting with $\Phi_{\ell,\ell+2}$ on the correlator then relates the anomalous dimension for spin ℓ operator to anomalous dimension of spin $\ell+2$ at one loop.

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- Acting with $\Phi_{\ell,\ell+2}$ on the correlator then relates the anomalous dimension for spin ℓ operator to anomalous dimension of spin $\ell+2$ at one loop.
- If we give the anomalous dimension of Konishi operator as input, these 2 functionals together solve the one-loop problem analytically!

[Caron-Huot, Coronado, Trinh, ZZ;2022]

Overview of the Optimization Problem

- Each dispersive sum rule discussed gives and infinite dimensional vector similar to the naive functional discussed previously.

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- Each dispersive sum rule discussed gives and infinite dimensional vector similar to the naive functional discussed previously.
- To setup a practical bootstrap problem, we consider a finite linear combinations of different types of sum rules:

$$0 = \sum_k \alpha_k (W_k^{\text{protected}} + \sum_{(\Delta,J)\text{ST}} \lambda^2_{\Delta,J} W_k[\Delta, J])$$

- Similar to the previous discussion, we separate the protected part and the target OPE coefficient we want to bound ($\lambda^2_{\text{Konishi}}$) and impose that all other terms are positive:

$$\sum_k \alpha_k W_k[\Delta', J'] \geq 0 \quad \forall \quad (\Delta', J') \text{ in single-trace spectrum,}$$

Side: Integrability will enter in determining the (Δ', J') that we need to impose positivity on.

Overview of the Optimization Problem

- The optimal bound is then found by solving a linear optimization problem on the α_k parameters:

$$\text{maximize} \qquad \sum_k \alpha_k W_k^{\text{protected}}$$

$$\text{such that} \qquad \sum_k \alpha_k W_k[\Delta_{\text{Konish}} i, 0] = \pm 1 \text{ and positivity holds.}$$

- We find similar looking equation to the original bootstrap attempt we had but now since we killed the double traces, it holds.

$$-\sum_k \alpha_k W_k^{\text{protected}} \pm \lambda^2_{\text{Konishi}} \geq 0\,.$$

Positivity on Single-Trace Spectrum

- With data available from combination of integrability methods such as Quantum Spectral Curve data, large spin, etc, we estimate the leading trajectory single-trace data, large spin gap and spin 0 gap.

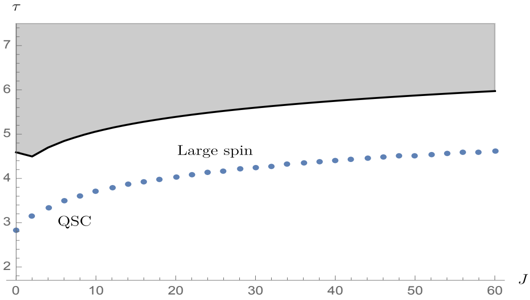

Figure: spectrum at g=0.3 as a reference

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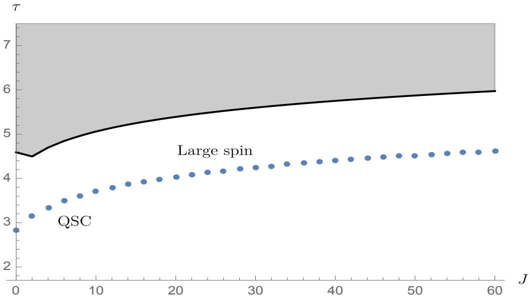

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- We then impose positivity on the leading Regge trajectory and everywhere above the gap. We assume the large spin gap all the way to spin 2.

Functions in Mellin Space:

To introduce the rest of the functionals, we need to introduce the Mellin representation for identical-dimension operators:

$$\mathcal{H}(u,v) = \int\int \frac{dsdt}{(4\pi i)^2} u^{\frac{s}{2}-\Delta} v^{\frac{t}{2}-\Delta} \Gamma(\Delta - \frac{s}{2})^2 \Gamma(\Delta - \frac{t}{2})^2 \Gamma(\Delta - \frac{u}{2})^2 M_{s,t}$$

with $s + t + u = 4\Delta$ for $\Delta = 4$.

The poles of the Mellin amplitude, according to the s-channel OPE occur at descendants of primaries:

$$M(s,t) \sim \frac{\lambda^2_{\Delta,J} \mathcal{Q}^m_{\Delta+4,J}(t)}{s - (\Delta + 4 - J + 2m)}$$

Importantly $\mathcal{Q}^m_{\Delta+4,J}(t) \propto 2\sin^2\left(\frac{\tau-2\Delta}{2}\pi\right)$. The Mellin amplitude readily has double-zero on double traces.

Dispersion Relation for Mellin Amplitudes

- Let us now move towards defining the dispersive functionals by writing dispersion relation for $M(s,t)$:
- Using $\lim_{s'\to\infty} M(s',t') \sim s^{J_*-4}$, where $J_* < 2$, we can write the following fixed-u dispersion relations:

$$M(s,t) = \oint \frac{ds'}{2\pi i} \frac{M(s',t')}{s-s'}$$

$$M(s,t) = \oint \frac{ds'}{2\pi i} \frac{(s'-s_0)(t'-s_0)}{(s-s_0)(t-s_0)} \frac{M(s',t')}{s-s'}$$

More subtraction would cancel the zero in dDisc and makes us sensitive to double traces.

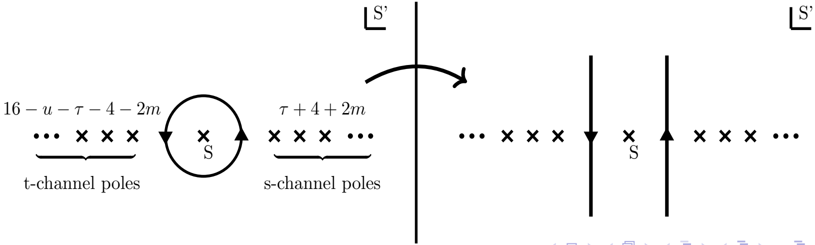
Mellin Amplitudes

Assuming the same spectrum in the s- and t- channel, we can deform the contour to pick up the poles and write their combined contribution a:

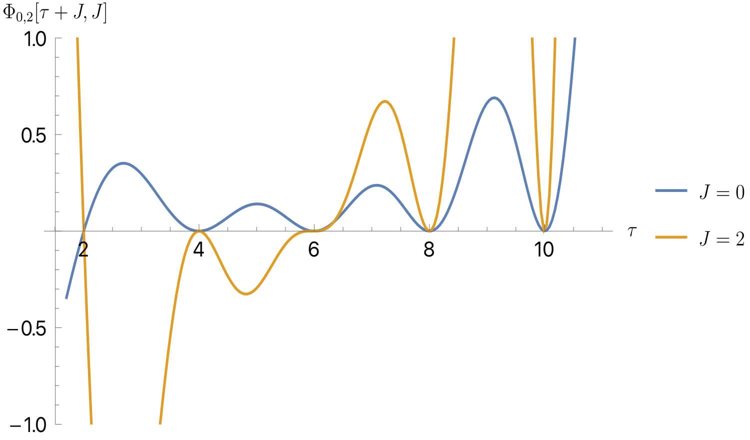
$$M(s,t) = M(s,t)^{\text{strong}} + \sum_{\substack{(\Delta,J) \\ \text{Long ST}}} \lambda^2_{\Delta,J} \widehat{\mathcal{P}}^{N=4}_{s,t}[\Delta, J]$$

with Polyakove-Regge block given as:

$$\widehat{\mathcal{P}}^{N=4}_{s,t}[\Delta, J] = \sum_{m=0}^{\infty} \mathcal{Q}^m_{\Delta+4,J}(16-s-t)\left[\frac{1}{s-(\tau+2m+4)} + \frac{1}{t-(\tau+2m+4)}\right]$$



$\Phi_{0,2}$ Functional



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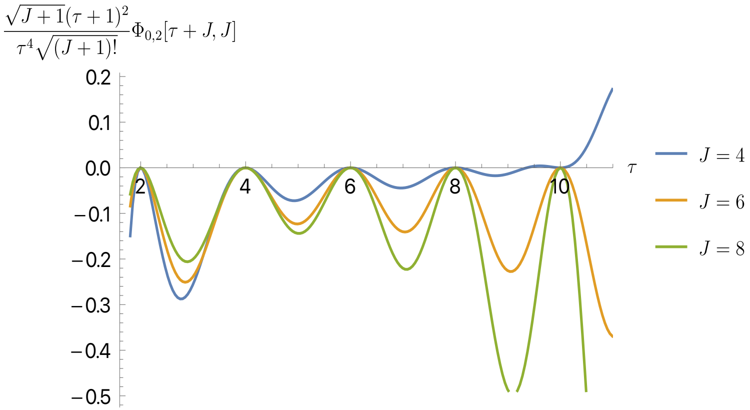
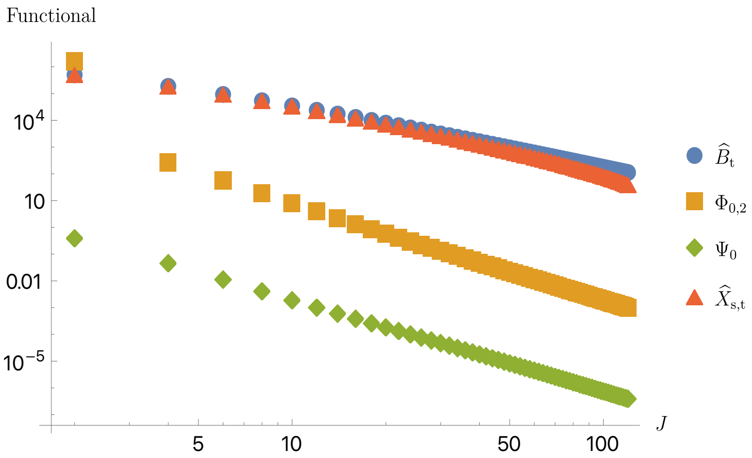
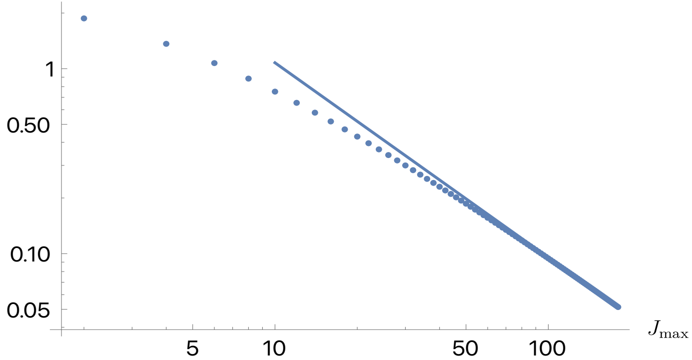



Functional
10^4
10
0.01
10^{-5}
5
10
50
100
J
$\widehat{B}_{\rm t}$
$\Phi_{0,2}$
Ψ_0
$\widehat{X}_{\rm s,t}$

$$8 - \sum_{J=0}^{J_{\max}} \lambda^2_{2+J,J} \widehat{B}_5[2+J, J]$$



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Necessary criteria for numerical bootstrap

Our functionals need to satisfy the following properties to be suitable for numerical bootstrap:

- swappability
- Asymptotic positivity of finite linear combination
- Completeness

Weak Coupling ($g = 0.1$): Finding the Spectrum

Next we discuss a nice feature about our optimized functional for this value of coupling:

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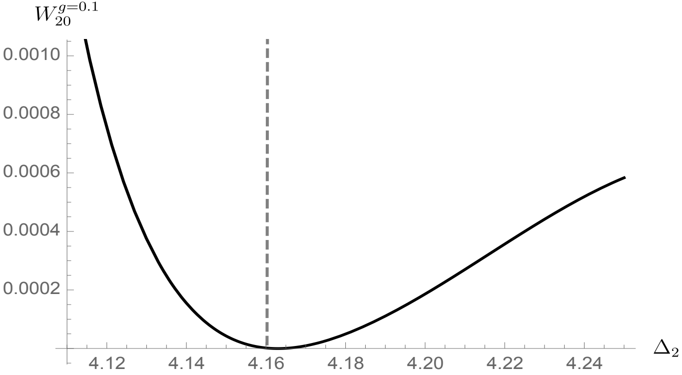
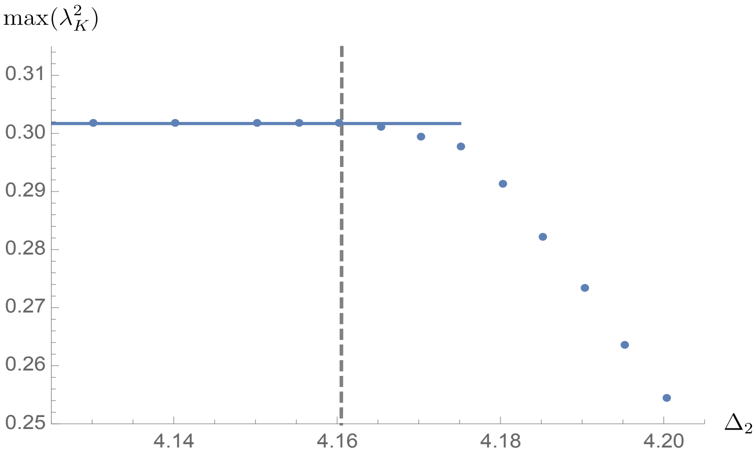
Next we discuss a nice feature about our optimized functional for this value of coupling:

- It has double zeroes close to the single trace operator for the first few spin! We illustrate this double zero for spin 2:



Weak Coupling: Finding the Spectrum

- This means that if we look at our upper bound as a function of $\Delta_2$we see a kink as a function of upper end or lower end of the window. If we vary the lower end:



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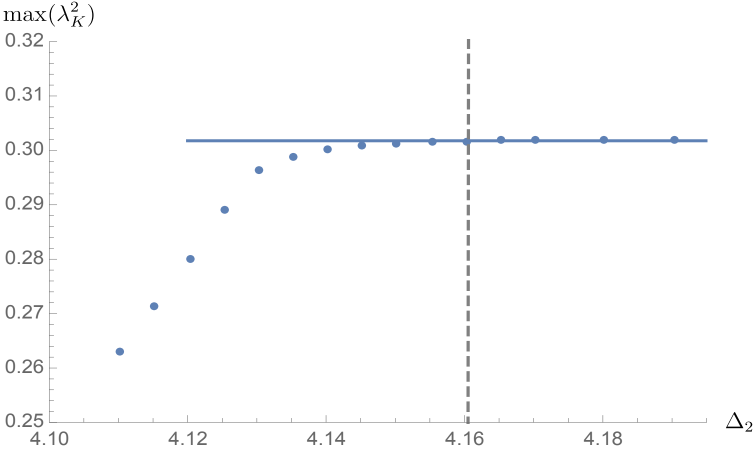

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- Asymptotically in spin for low twist we see that our optimized functional is as power low.