Difference Equations and Integral Families for Witten Diagrams

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2406.04186

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Flat space Feynman integrals and amplitudes

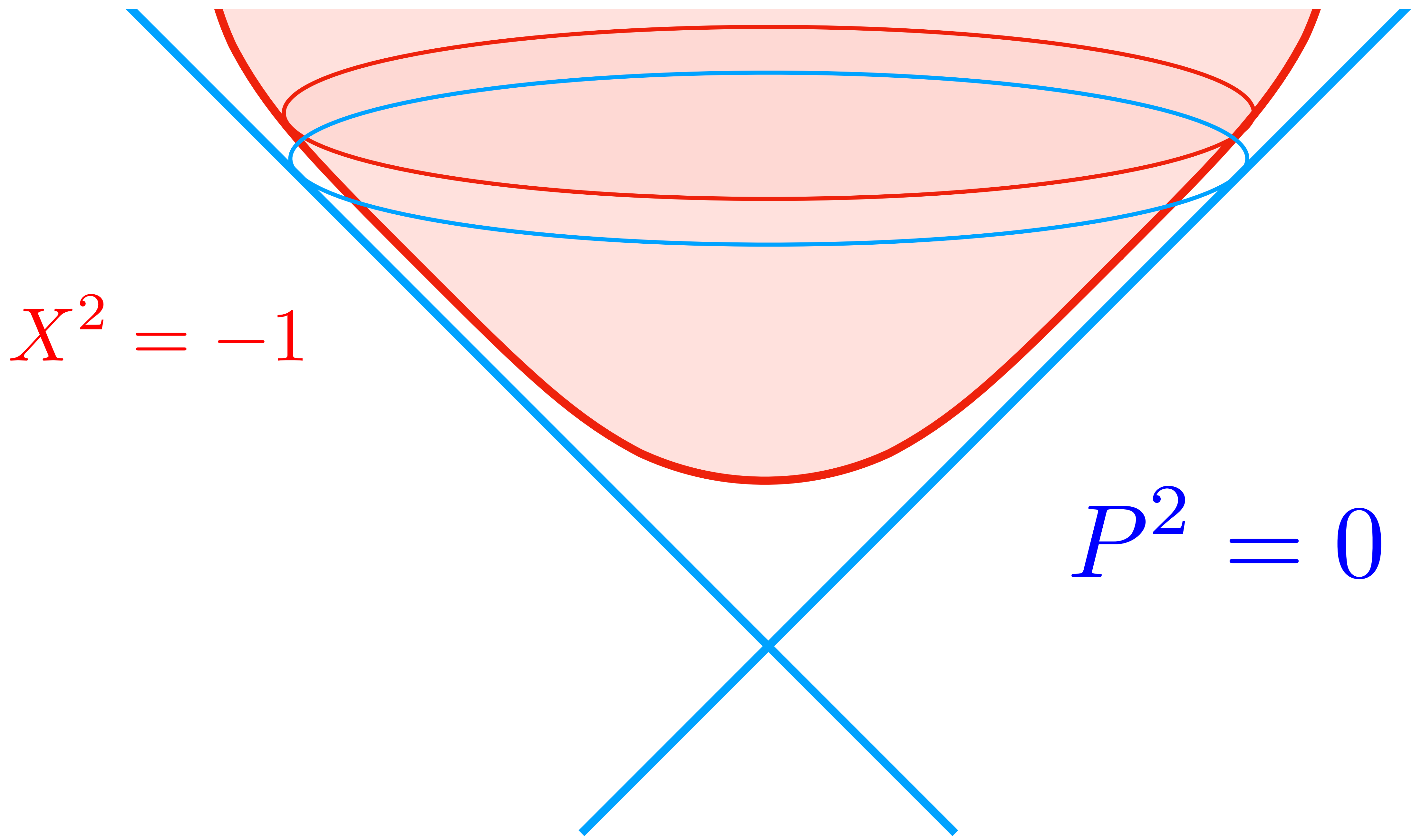
- Tree level amplitudes are rational functions
- Compute integrand using Feynman diagrams/recursion/Amplituhedron/etc
- Standard tools for integrating higher loop amplitudes are
  - Integration-by-parts
  - Differential equations

Beyond Flat Space

Beyond Flat Space
Anti-de Sitter

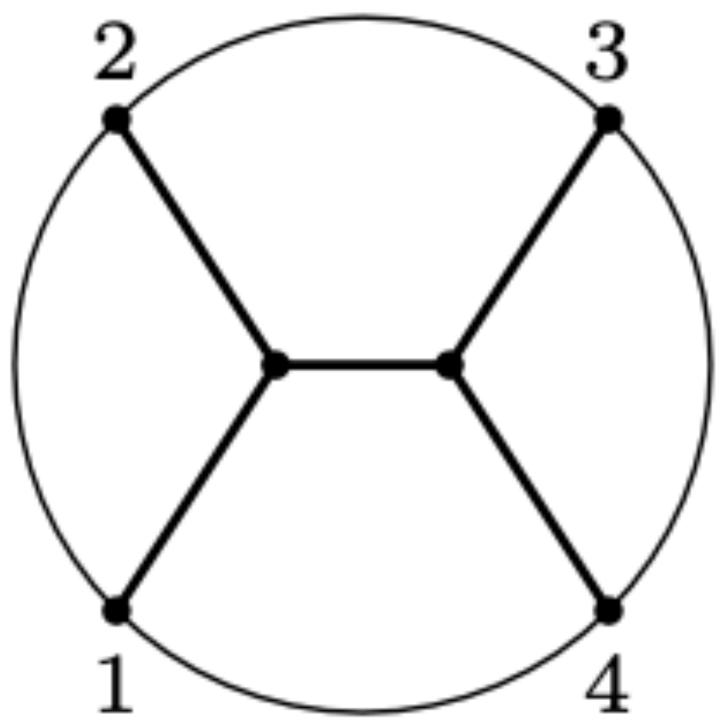
Boundary correlators in Anti-de Sitter

- Bulk Coordinate: X^A
- Boundary Coordinate: P^A
- Bulk-Boundary Propagator: $E_\Delta(X, P)$
- Bulk-Bulk Propagator: $G_\Delta(X, X')$
- Mass corresponds to $\Delta = \delta + h$



Witten diagrams

$$\mathcal{A}_{\text{s-channel}} =$$



$$= \int_{\text{AdS}} d^{d+1}X_1 d^{d+1}X_2 E(X_1, P_1) E(X_1, P_2)$$
$$\times G(X_1, X_2) E(X_2, P_3) E(X_2, P_4)$$

Problems

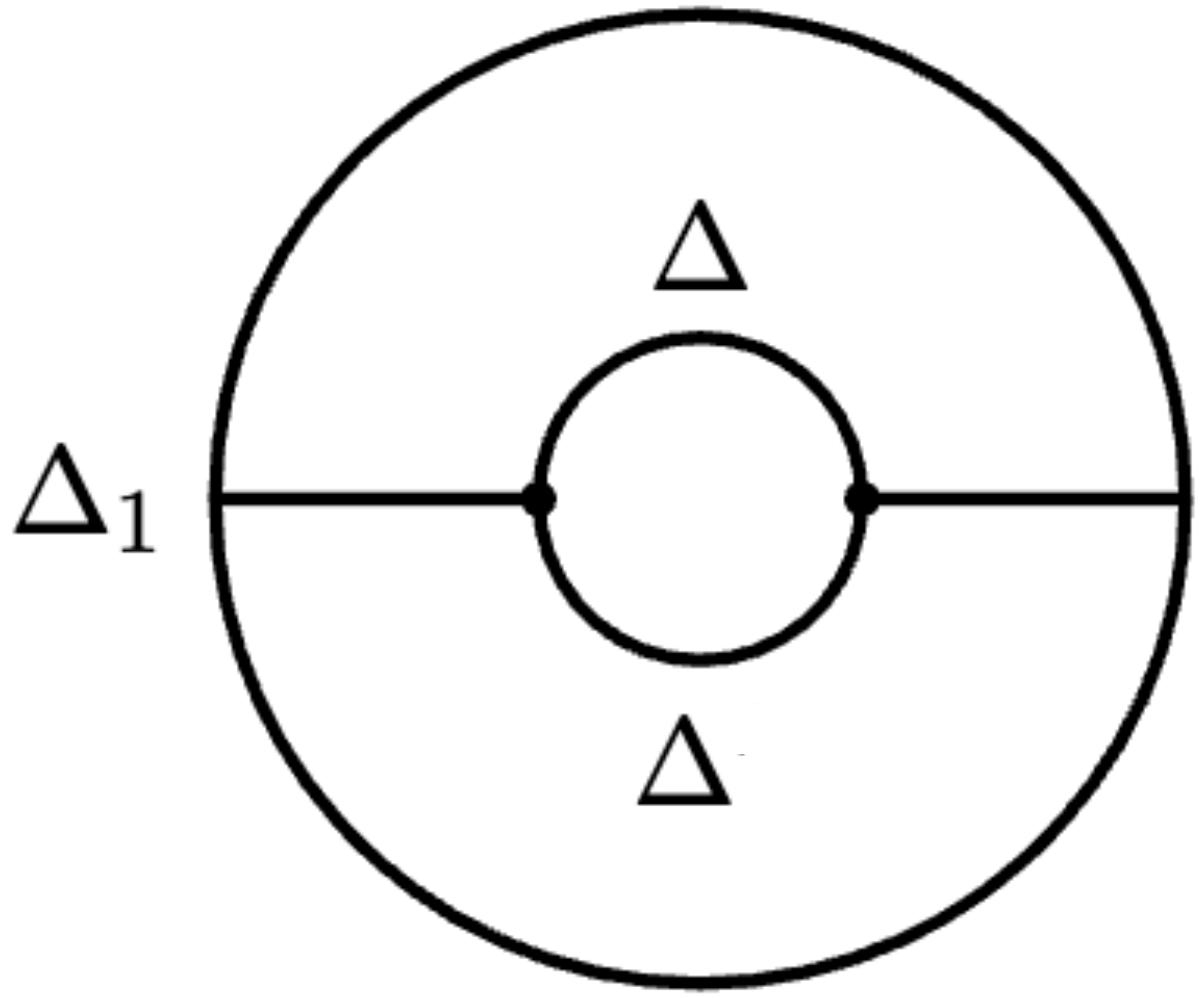
- Non-trivial integrals are required even at tree-level.
- Propagators are generically not nice functions!
- Results are generically in terms of digamma, hypergeometric functions of mass and dimension.
- One solution is to work in Mellin space.

Discrete vs Continuum

- Spectrum in AdS is discrete, not continuous.
  - Example: energy spectrum of 2-body states
- One might expect derivatives and integrals to be replaced by difference equations and sums
  - $D_x^{\pm}\,[f(x)] \equiv \dfrac{f(x \pm 2) - f(x)}{2}$
  - $D_x^{\pm}[f(x)\,g(x)] = D_x^{\pm}[f(x)]\,g(x \pm 2) + f(x)\,D_x^{\pm}[g(x)]$

Reverse engineer a difference eq. for 2-point bubble in d=2

The result



$$\propto \frac{\psi(\Delta - \frac{\Delta_1}{2}) - \psi(\Delta + \frac{\Delta_1}{2} - 1)}{8\pi(\Delta_1 - 1)}$$

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Difference equations

- Digamma function obeys nice difference equation:

$$\psi(x+1) - \psi(x) = \frac{1}{x}$$

Difference equations

- Digamma function obeys nice difference equation:

$$\psi(x+1) - \psi(x) = \frac{1}{x}$$

- Can infer difference equation for bubble

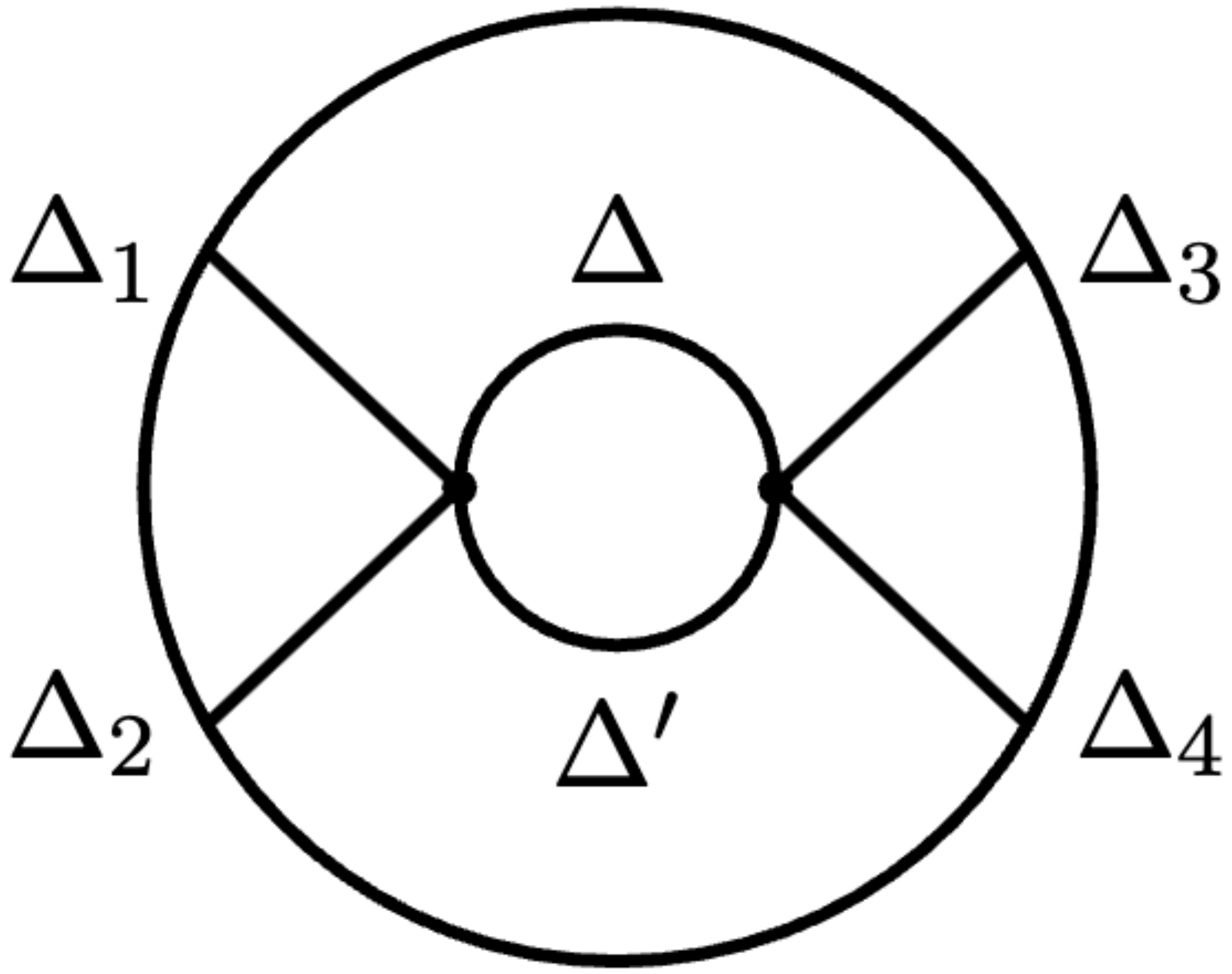
$$f(\Delta_1) = \frac{\psi(\Delta - \frac{\Delta_1}{2}) - \psi(\Delta + \frac{\Delta_1}{2} - 1)}{8\pi(\Delta_1 - 1)}$$

$$D^{+}_{\Delta_1}[(\Delta_1 - 1)f(\Delta_1)] = \frac{1 - \Delta}{\pi\left(4(\Delta - 1)^2 - \Delta_1^2\right)}$$

Comments

- Somewhat trivial because we already knew the result before hand.
- Goal is to compute difference equations without knowing the solution.

Example of more non-trivial result



$$
\begin{aligned}
0 = {} & 512\pi\Gamma^2\left[6-\frac{s}{2}\right]\mathcal{M}(s-8) - 128\pi(4s^2-43s+140)\Gamma^2\left[5-\frac{s}{2}\right]\mathcal{M}(s-6) \\
& + 32\pi(6s^4-105s^3+738s^2-2408s+3040)\Gamma^2\left[4-\frac{s}{2}\right]\mathcal{M}(s-4) \\
& - 8\pi(4s^6-81s^5+700s^4-3268s^3+8608s^2-12032s+6912)\Gamma^2\left[3-\frac{s}{2}\right]\mathcal{M}(s-2) \\
& + 2\pi(s-4)^3(s-2)^2(s^2-3s+2)s\Gamma^2\left[2-\frac{s}{2}\right]\mathcal{M}(s) + 48(s-4)^3\Gamma^2\left[2-\frac{s}{2}\right]. \qquad (5.'
\end{aligned}
$$

$d = 2, \quad \Delta_i = \Delta = \Delta' = 2$

Derived new difference equation for 4-point bubble in Mellin space that agrees with known result for specific values.

How do you compute the difference equations without knowing the answer?

Derivation of the one-loop tadpole

The tadpole

$$A_{\text{tad}}(P_i) = \int_{\text{AdS}} dX \; G_{\Delta}(X,X) \prod_{i=1}^{p} E_{\Delta_i}(X,P_i)$$

$$= G_{\Delta}(X,X) \int_{\text{AdS}} dX \prod_{i=1}^{p} E_{\Delta_i}(X,P_i)$$

$$= G_{\Delta}(X,X) C_n$$

The tadpole

$$A_{\text{tad}}(P_i) = \int_{\text{AdS}} dX\; G_\Delta(X,X) \prod_{i=1}^{p} E_{\Delta_i}(X,P_i)$$

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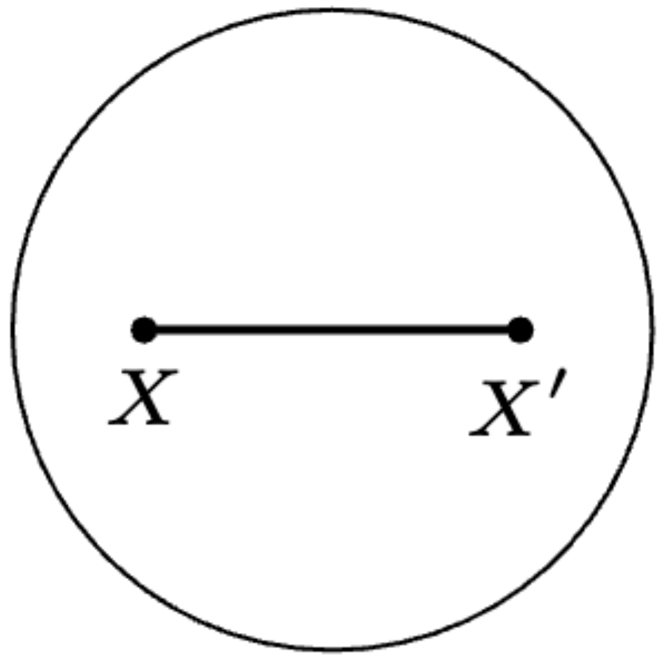
$$= G_\Delta(X,X)\, C_n$$

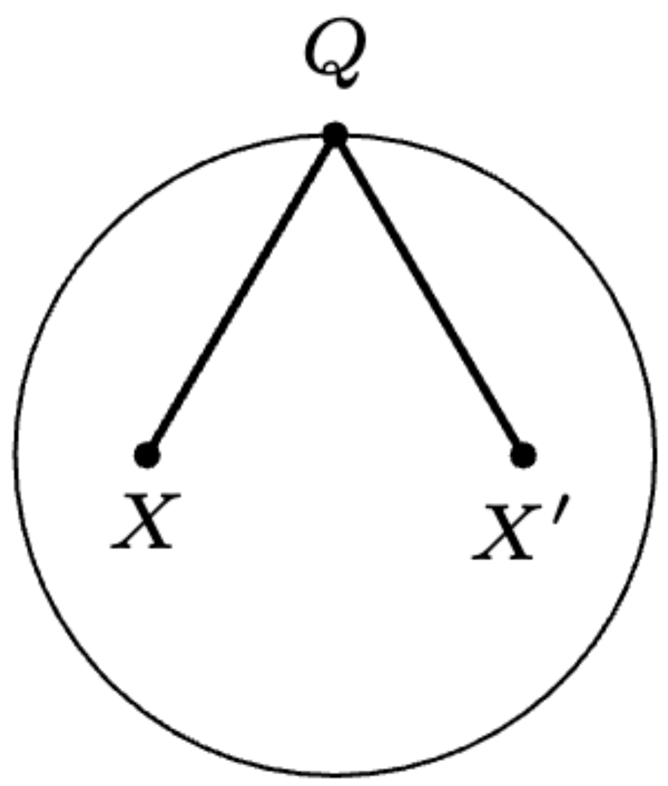
Spectral representation

$$G_\Delta(X_1, X_2) = \int \frac{dc}{2\pi i} \frac{1}{(\Delta - h)^2 - c^2} \Omega_c(X_1, X_2)$$

$$\Omega_c(X_1, X_2) = -\, 2c^2 \int_{\text{AdS}} dQ E_{d/2+c}(X_1, Q) E_{d/2-c}(X_2, Q)$$



$$\sim \int dc \int_{\partial \mathrm{AdS}} dQ \frac{1}{c^2 - d^2/4}$$



Spectral representation

$$G_\Delta(X,X) = -\frac{\Gamma(h)}{2\pi^h\Gamma(2h)}\int \frac{dc}{2\pi i}\frac{1}{c-\delta}W_{\text{tadpole}}(c)$$

$$W_{\text{tadpole}} = \frac{\Gamma(h+c)\Gamma(h-c)}{c\,\Gamma(-c)\Gamma(c)}$$

A family of integrals

AdS

$$I_n(\delta) = \int \frac{dc}{2\pi i} \Pi_n(c - \delta) W_{\text{tadpole}}(c)$$

$$\Pi_n(x) = \frac{1}{2^n (x)_n}$$

Flat Space

$$I_n(m^2) = \int d^d\ell \frac{1}{(\ell^2 + m^2)^n}$$

A family of integrals

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$$I_n(\delta) = \int \frac{dc}{2\pi i}\Pi_n(c-\delta)W_{\text{tadpole}}(c)$$

$$\Pi_n(x) = \frac{1}{2^n(x)_n}$$

$$D_x^+[\Pi_n(x)] = -n\,\Pi_{n+1}(x)$$

Flat Space

$$I_n(m^2) = \int d^d\ell \frac{1}{(\ell^2+m^2)^n}$$

$$\frac{\partial}{\partial x}\frac{1}{x^n} = \frac{-n}{x^{n+1}}$$

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$$D_x^+[\Pi_n(x)] = -\, n\, \Pi_{n+1}(x)$$

$$D_\delta^-\left[I_1\right] = -\, I_2$$

Flat Space

$$I_n(m^2) = \int d^d\ell \, \frac{1}{(\ell^2 + m^2)^n}$$

$$\frac{\partial}{\partial x}\frac{1}{x^n} = \frac{-n}{x^{n+1}}$$

$$\frac{\partial}{\partial m^2} I_1(m^2) = -\, I_2(m^2)$$

A family of integrals

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$$I_n(\delta) = \int \frac{dc}{2\pi i} \Pi_n(c-\delta) W_{\text{tadpole}}(c)$$

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$$D_\delta^-\left[I_1\right] = -I_2$$

Flat Space

$$I_n(m^2) = \int d^d\ell \frac{1}{(\ell^2+m^2)^n}$$

$$\frac{\partial}{\partial x}\frac{1}{x^n} = \frac{-n}{x^{n+1}}$$

$$\frac{\partial}{\partial m^2} I_1(m^2) = -I_2(m^2)$$

We need a second independent relation between I_1 and I_2

Summation-by-Parts

AdS

$$0 = \int \frac{dc}{2\pi i} D_c^+ \left[P'_c \, \Pi_n(c-\delta) \, W_{\text{tadpole}}(c) \right]$$

Flat Space

$$0 = \int d\ell \frac{\partial}{\partial \ell} \left[\dots \right]$$

Summation-by-Parts

AdS

$$0 = \int \frac{dc}{2\pi i} D_c^+ \left[P_c' \, \Pi_n(c-\delta) \, W_{\text{tadpole}}(c) \right]$$

$$I_{n \leqslant 0} = 0$$

Flat Space

$$0 = \int d\ell \, \frac{\partial}{\partial \ell} \left[\dots \right]$$

$$0 = \int d^d\ell \, \frac{1}{(\ell^2 + m^2)^{n \leq 0}}$$

Summation-by-Parts

AdS

$$0 = \int \frac{dc}{2\pi i} D_c^+ \left[P'_c \, \Pi_n(c-\delta) \, W_{\text{tadpole}}(c) \right]$$

$$I_{n \leqslant 0} = 0$$

⬇

$$I_2(\delta) = \frac{(\delta-1)(2h-1)}{(\delta+h-2)(\delta+h-1)} I_1(\delta)$$

Flat Space

$$0 = \int d\ell \frac{\partial}{\partial \ell} \left[\ldots \right]$$

$$0 = \int d^d\ell \frac{1}{(\ell^2+m^2)^{n \leq 0}}$$

Summation-by-Parts

AdS

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Flat Space

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$$0 = \int d^d\ell \frac{1}{(\ell^2+m^2)^{n\leq 0}}$$

P' is chosen to cancel undesirable pole terms from $W_{\text{tadpole}}(c+2)$

Difference equation

$$D_\delta^-\left[I_1\right] = -I_2$$

$$I_2(\delta) = \frac{(\delta-1)(2h-1)}{(\delta+h-2)(\delta+h-1)} I_1(\delta)$$

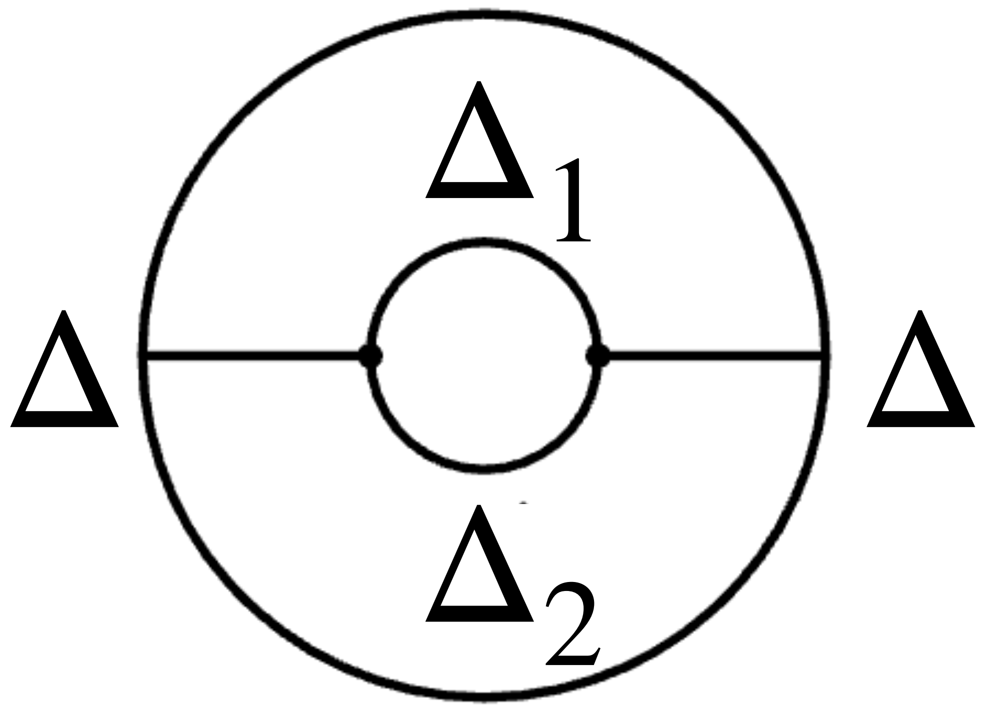
⬇

$$D_\delta^-\left[I_1(\delta)\right] = -\frac{(\delta-1)(2h-1)}{(\delta+h-2)(\delta+h-1)} I_1(\delta)$$

Result agrees with previous derivation in d=2!

Beyond the tadpole

Two-point bubble computation



$$A_{\text{2-bub}} \propto A_{\text{2-pt}} \int \prod_{a=1}^{2} \left[\frac{dc_a}{2\pi i}\right] \frac{1}{c_1 - \delta_1} \frac{1}{c_2 - \delta_2} W(c_1, c_2, \Delta)$$

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Summation-by-parts

$$0 = \int \frac{dc_1 dc_2}{(2\pi i)^2} D^+_{c_1} \left[P' \Pi_{n_1}(c_1 - \delta_1) \Pi_{n_2}(c_2 - \delta_2) W(c_1, c_2, \Delta) \right]$$

$$= \int \frac{dc_1 dc_2}{(2\pi i)^2} [\text{polynomial in } c_1, c_2] \Pi_{n_1}(c_1 - \delta_1) \Pi_{n_2}(c_2 - \delta_2) W(c_1, c_2, \Delta)$$

$$= \sum_{n_1', n_2'} a_{n_1', n_2'} \int \frac{dc_1 dc_2}{(2\pi i)^2} \Pi_{n_1'}(c_1 - \delta_1) \Pi_{n_2'}(c_2 - \delta_2) W(c_1, c_2, \Delta)$$

P' is chosen to cancel undesirable pole terms from $W(c_1 + 2, c_2, \Delta)$

Collapsing propagators

$$\mathcal{I}^{\text{2-bub}}_{-1,n_2} = \frac{\Gamma(d-\boldsymbol{\Delta})\Gamma(\boldsymbol{\Delta})\Gamma(h)^2}{\Gamma(2h)}\mathcal{I}^{\text{tad}}_{n_2}(\delta_2)\,,$$

Remove poles

$$\mathcal{J} = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}c_1}{2\pi i}\frac{\mathrm{d}c_2}{2\pi i}\,\Pi_1(c_1-\delta_1)\Pi_2(c_2-\delta_2)\widetilde{P}(c_1,c_2,\boldsymbol{\Delta})W_{2\text{-bub}}(c_1,c_2,\boldsymbol{\Delta})$$

$$\begin{aligned} D^{+}_{\boldsymbol{\Delta}}[\mathcal{J}] &= \int_{-i\infty}^{i\infty} \frac{\mathrm{d}c_1}{2\pi i}\frac{\mathrm{d}c_2}{2\pi i}\,\Pi_1(c_1-\delta_2)\Pi_2(c_2-\delta_2)D^{+}_{\boldsymbol{\Delta}}\Big[\widetilde{P}(c_1,c_2,\boldsymbol{\Delta})W_{2\text{-bub}}(c_1,c_2,\boldsymbol{\Delta})\Big] \\ &= 4h(\boldsymbol{\Delta}-h)(2h^2-2h\boldsymbol{\Delta}+\boldsymbol{\Delta}^2-\delta_1^2-\delta_2^2)\mathcal{I}^{2\text{-bub}}_{1,1}(\boldsymbol{\Delta}) \\ &\quad -\frac{4h(\boldsymbol{\Delta}-h)\Gamma(2h-\boldsymbol{\Delta})\Gamma(\boldsymbol{\Delta})\Gamma(h)^2}{\Gamma(2h)}\Big[\mathcal{I}^{\text{tad}}_1(\delta_1)+\mathcal{I}^{\text{tad}}_1(\delta_2)\Big]\,. \end{aligned}$$

$\tilde{P}$ is necessary to cancels poles that appear when shifting Δ

Final difference equation

$$\mathcal{I}_{1,1}^{\text{2-bub}}(\mathbf{\Delta}+2) = \mathcal{I}_{1,1}^{\text{2-bub}}(\mathbf{\Delta}) \prod_{\sigma_1,\sigma_2=\pm 1} \frac{\mathbf{\Delta}+\sigma_1\delta_1+\sigma_2\delta_2}{\mathbf{\Delta}-2h+2+\sigma_1\delta_1+\sigma_2\delta_2}$$
$$- \frac{2(2h-1)(\mathbf{\Delta}-h+1)\Gamma(2h-\mathbf{\Delta}-2)\Gamma(\mathbf{\Delta})\Gamma(h)^2}{\Gamma(2h)\prod_{\sigma_1,\sigma_2=\pm 1}(\mathbf{\Delta}-2h+2+\sigma_1\delta_1+\sigma_2\delta_2)}$$
$$\times \left[(\mathbf{\Delta}^2-2h\mathbf{\Delta}+2\mathbf{\Delta}+\delta_1^2-\delta_2^2)\mathcal{I}_1^{\text{tad}}(\delta_2)+(\mathbf{\Delta}^2-2h\mathbf{\Delta}+2\mathbf{\Delta}+\delta_2^2-\delta_1^2)\mathcal{I}_1^{\text{tad}}(\delta_1)\right]$$

Similar result for internal mass.
Simplifies dramatically in d=2.

Results agree

Simplifications

$$\mathcal{I}_{1,1}^{\text{2-bub}}(\boldsymbol{\Delta}+2) = \mathcal{I}_{1,1}^{\text{2-bub}}(\boldsymbol{\Delta}) \prod_{\sigma_1,\sigma_2=\pm 1} \frac{\boldsymbol{\Delta}+\sigma_1\delta_1+\sigma_2\delta_2}{\boldsymbol{\Delta}-2h+2+\sigma_1\delta_1+\sigma_2\delta_2}$$

$$- \frac{2(2h-1)(\boldsymbol{\Delta}-h+1)\Gamma(2h-\boldsymbol{\Delta}-2)\Gamma(\boldsymbol{\Delta})\Gamma(h)^2}{\Gamma(2h)\prod_{\sigma_1,\sigma_2=\pm 1}(\boldsymbol{\Delta}-2h+2+\sigma_1\delta_1+\sigma_2\delta_2)} \quad (5$$

$$\times \left[(\boldsymbol{\Delta}^2-2h\boldsymbol{\Delta}+2\boldsymbol{\Delta}+\delta_1^2-\delta_2^2)\mathcal{I}_1^{\text{tad}}(\delta_2)+(\boldsymbol{\Delta}^2-2h\boldsymbol{\Delta}+2\boldsymbol{\Delta}+\delta_2^2-\delta_1^2)\mathcal{I}_1^{\text{tad}}(\delta_1)\right]$$

Agrees with known results.

4-point bubble in Mellin space

- Need to introduce auxiliary variable so W is product of gamma functions
- Three integrals/sums over c_0, c_1 and c_2

$$\mathcal{I}^{\text{4-bub}}_{n,n_1,n_2} = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}c}{2\pi i}\frac{\mathrm{d}c_1}{2\pi i}\frac{\mathrm{d}c_2}{2\pi i}\Pi_n(c-\delta)\Pi_{n_1}(c_1-\delta_1)\Pi_{n_2}(c_2-\delta_2)W_{\text{4-bub}}(c,c_1,c_2,s)$$

- Care about $n = -1,\ n_1 = 1,\ n_2 = 1$
- "Scalelessness" only in n_1, n_2

```
-((-1 + 2*h)*(s - Δ[1, 2])*(s - Δ[3, 4])*
   (2*h - Δ[1, 2] - Δ[3, 4])*
   (-4 - 2*h*(-2 + s) + δ[1]^2 - δ[2]^2 -
    2*Δ[1, 2] - 2*Δ[3, 4] -
    Δ[1, 2]*Δ[3, 4] +
    s*(2 + Δ[1, 2] + Δ[3, 4]))*
   I4bub[-1, -1, 1][δ[1], δ[2], s]) -
 (-1 + 2*h)*(s - Δ[1, 2])*(s - Δ[3, 4])*
  (2*h - Δ[1, 2] - Δ[3, 4])*
  (-4 - 2*h*(-2 + s) - δ[1]^2 + δ[2]^2 -
   2*Δ[1, 2] - 2*Δ[3, 4] -
   Δ[1, 2]*Δ[3, 4] +
   s*(2 + Δ[1, 2] + Δ[3, 4]))*
  I4bub[-1, 1, -1][δ[1], δ[2], s] +
 256*I4bub[-1, 1, 1][δ[1], δ[2], -8 + s] +
 64*(-58 + 8*h*s - 4*s^2 + 2*δ[1]^2 + 2*δ[2]^2 -
   3*Δ[1, 2] - 3*Δ[3, 4] -
   Δ[1, 2]*Δ[3, 4] -
   2*h*(15 + Δ[1, 2] + Δ[3, 4]) +
   s*(27 + Δ[1, 2] + Δ[3, 4]))*
  I4bub[-1, 1, 1][δ[1], δ[2], -6 + s] +
 16*(376 + 6*s^4 - 44*δ[1]^2 + δ[1]^4 - 44*δ[2]^2 -
   2*δ[1]^2*δ[2]^2 + δ[2]^4 + 60*Δ[1, 2] -
   6*δ[1]^2*Δ[1, 2] - 6*δ[2]^2*
    Δ[1, 2] + 60*Δ[3, 4] -
   6*δ[1]^2*Δ[3, 4] - 6*δ[2]^2*
    Δ[3, 4] + 22*Δ[1, 2]*Δ[3, 4] -
   2*δ[1]^2*Δ[1, 2]*Δ[3, 4] -
   2*δ[2]^2*Δ[1, 2]*Δ[3, 4] -
   3*s^3*(19 + Δ[1, 2] + Δ[3, 4]) +
   3*s^2*(78 - 2*δ[1]^2 - 2*δ[2]^2 + 7*Δ[1, 2] +
     7*Δ[3, 4] + Δ[1, 2]*Δ[3, 4]) +
   4*h^2*(48 + 6*s^2 + 9*Δ[3, 4] + Δ[1, 2]*
       (9 + Δ[3, 4]) - 3*s*(11 + Δ[1, 2] +
        Δ[3, 4])) + s*(-468 - 58*Δ[1, 2] -
     58*Δ[3, 4] - 15*Δ[1, 2]*
```

$$\mathcal{I}^{\text{4-bub}}_{-1,1,1}(\delta_1-2,s)=\Big[1-2\lambda_{-1,1,1}(0)\Big]\mathcal{I}^{\text{4-bub}}_{-1,1,1}(\delta_1,s)-2\sum_{n=1}^{3}\lambda_{-1,1,1}(n)\mathcal{I}^{\text{4-bub}}_{-1,1,1}(\delta_1,s-2n)$$
$$-2\lambda_{-1,1,-1}\mathcal{I}^{\text{4-bub}}_{-1,1,-1}(\delta_1,s)-2\lambda_{-1,-1,1}\mathcal{I}^{\text{4-bub}}_{-1,-1,1}(\delta_1,s)\,.$$

$$0=\mathcal{R}_3(s)\equiv\sum_{n=0}^{4}\rho_{-1,1,1}(n)\mathcal{I}^{\text{4-bub}}_{-1,1,1}(s-2n)+\rho_{-1,1,-1}\mathcal{I}^{\text{4-bub}}_{-1,1,-1}(s)+\rho_{-1,-1,1}\mathcal{I}^{\text{4-bub}}_{-1,-1,1}(s)$$

Future Work

- What physical conditions fix the periodic function ambiguity.
- Similar techniques could be applied in de Sitter.
- Triangle computation
  - Hard because you need auxiliary integrals. We expect the same method should work.

Funded by Simons Foundation