Finite Feynman Integrals

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work with **Giulio Gambuti**, **Pavel Novichkov**, *and* Lorenzo Tancredi, [2311.16907]

and in progress with **Leonardo de la Cruz** *and* **Pavel Novichkov** [2407.nnnnn];
and **Marc Canay**;
and in progress with Yang Zhang, **Zhihou Wu**, *and* **Rourou Ma**

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Goal

- Recast amplitudes in terms of Feynman integrals selected for degree of divergence

$$I[\mathcal{N}(\ell_i)] = \int \prod_{i=1}^{L} \mathrm{d}^D\ell_i \, \frac{\mathcal{N}(\ell_i)}{\mathcal{D}_1 \cdots \mathcal{D}_E},$$

- Hope: lead to simpler and more transparent representations

- First step: classify and organize **finite** integrals

Henn, Peraro, Stahlhofen, & Wasser; von Manteuffel, Panzer, & Schabinger

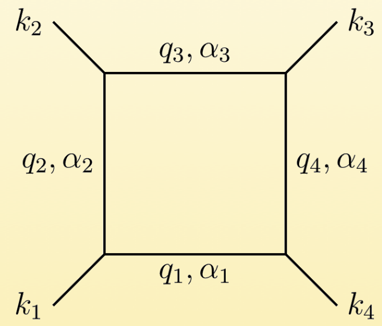
- Mostly gloss over fine print
  - look at locally IR-finite integrals (doable strictly in $D = 4$)
  - UV convergent by power counting ("strongly UV convergent")

One-Loop Example

- Canonical basis: box, triangle, bubble

⇒Tancredi's talk

- Box and Triangle have $\frac{1}{\epsilon^2}$ divergence (IR)
- Bubble has $\frac{1}{\epsilon}$ divergence (UV)



- Can trade $D = 4$ box ($\frac{1}{\epsilon^2}$ divergence) for $D = 6$ box (finite)

$$\text{Box}^{D=4} = c_0 \text{Box}^{D=6} + \sum c_i^{(3)}\ \text{Tri}_i$$

This isolates all IR divergences in triangles

How Do IR Singularities Arise?

- Look at

$$\int d^D\ell \frac{1}{\cdots(\ell-K_1)^2(\ell-K_2)^2(\ell-K_3)^2\cdots}$$

Singularities arise from regions where the denominator vanishes

- One denominator vanishing is integrable
- Two denominators vanishing in an invariant-independent way gives a $\frac{1}{\epsilon}$ singularity
- Three denominators vanishing in an invariant-independent way gives a $\frac{1}{\epsilon^2}$ singularity

How Do IR Singularities Arise?

- Genericallly,
  - two denominators vanishing must be adjacent propagators separated by a massless leg:
    $K_2 - K_1$ massless, singularity arises from $\ell \sim K_2 - K_1$
  - three denominators vanishing must be adjacent propagators separated by a pair of massless legs:
    $K_2 - K_1$and $K_3 - K_2$ massless, singularity arises when middle momentum is soft $\ell \sim K_2$

- Generalize this to higher loops

- Find numerators that vanish in those regions

Analytic Strategy

Gambuti, Novichkov, Tancredi, DAK

- Derivable
- Proceed topology by topology

- Solve Landau equations in mixed representation:
  - for all loops i, $\sum_{d=1}^{N} \alpha_d \frac{\partial}{\partial \ell_i} \mathrm{Den}_d = 0$
  - for all denominators d, $\alpha_d \mathrm{Den}_d = 0$
  - at least one α_d strictly positive, all nonnegative
  - subtleties for nonplanar integrals

$\Rightarrow$ McLeod's talk

- Each solution is a singular surface

Analytic Strategy

- Classify degree of divergence following Anastasiou & Sterman (based on Libby & Sterman)
  - planar: logarithmic soft & collinear singularities
  - nonplanar: soft singularities can collide to give power divergences
- Build finite numerators
  - start with all factors: $\ell_1^2, \ell_1 \cdot \ell_2, \ell_2^2; \ell_i \cdot k_{1,2,4}$
  - build all numerators of fixed degree, *e.g.*

$$c_1 \; \ell_1 \cdot \ell_2 + c_2 \; \ell_1 \cdot k_4 \, \ell_2 \cdot k_1 + c_3 \; (\ell_1 \cdot k_2)^2 + \cdots$$

  - for each singular surface, require coefficients of singular scaling terms to vanish

$\rightarrow$ Linear equation(s) for the c_i

Independent Numerators

- How many are there (cumulative)?

- 31 solutions to the Landau equations for planar double box

Max Order in ℓ	**1**	**2**	**3**	**4**	**5**
Finite	**0**	**2**	**18**	**89**	**247**

- Not all truly independent

  Poly(ℓ_i) (Finite numerator)

(subject to UV power-counting)

- Mathematical structure: ideal (before UV power-counting)
- "truncated ideal" (linear space) after

Independent Numerators

- Appropriate technology: Gröbner bases

- Compute Gröbner basis of order 2, retain independent remainders after dividing over the basis; iterate
- Or, just compute overall Gröbner basis all at once

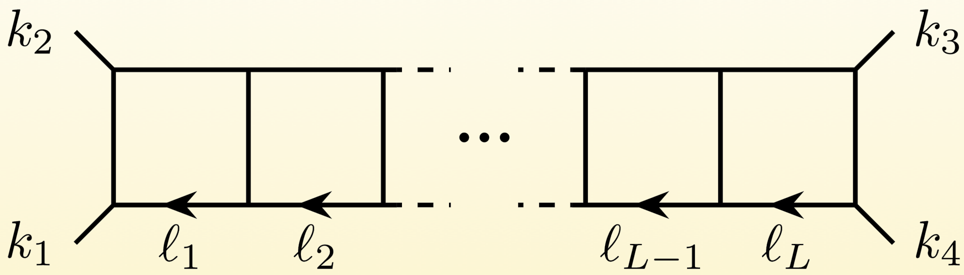
Max Order in ℓ	1	2	3	4	5
Finite	0	2	18	89	247
Independent New	0	2	4	4	0

- Define (van Neerven & Vermaseren)

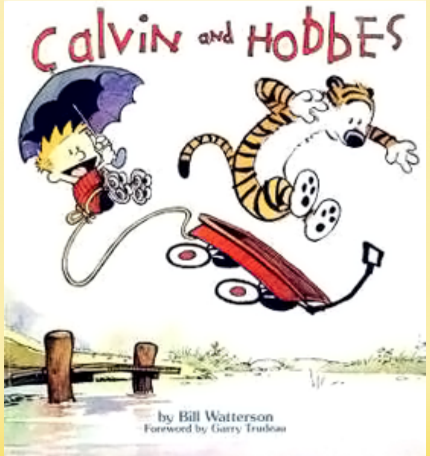
$$v_i^\mu \equiv \frac{G\begin{pmatrix} k_1 & \cdots & \mu & \cdots & k_R \\ k_1 & \cdots & k_i & \cdots & k_R \end{pmatrix}}{G\begin{pmatrix} k_1 & \cdots & k_R \end{pmatrix}}, \quad \nu_{ij} \equiv G\begin{pmatrix} l_i & k_1 & \cdots & k_R \\ l_j & k_1 & \cdots & k_R \end{pmatrix} \Big/ G\begin{pmatrix} k_1 & \cdots & k_R \end{pmatrix}$$

to get nice forms for generators

All-Loop Ladder Conjecture



Max Order in ℓ	1	2	3	4	5	6	7	8
Independent	0	2	$2L$	L^2	0	0	0	0
$\mathcal{O}(\epsilon)$ independent	0	0	0	$(3L^2-9L+8)/2$	$(L-2)(L^2-4L+5)$	$(L-2)\times$ $(L^3-9L+16)/8$	$(L-2)(L-3)\times$ $(L^2-5L+8)/4$	$(L-2)(L-3)\times$ $(L^2-5L+8)/8$



Geometric Strategy

De la Cruz, Novichkov, DAK

- Not yet derivable — lots of conjecture
- Parametric representation
- Focus on exponents of monomials

$$\alpha_1^{e_1}\alpha_2^{e_2}\cdots\alpha_n^{e_n}$$
$$\boldsymbol{e} \equiv (e_1, e_2, \dots, e_n)$$

- Build on theorem of Berkesch, Forsgård, & Passare on convergence of Euler–Mellin integrals

Geometric Strategy

- Newton polytope: convex hull of all positive-weight linear combinations of all exponent vectors in a given polynomial
- H-representation: region bounded by set of inequalities
- Relation to tropical geometry to be explored

- BFP instructs us to look at Newton polytope of Symanzik polynomials

$$\text{Newton}\left(\left[\mathcal{U}^{E-\frac{D}{2}(L+1)-r}\mathcal{F}^{\frac{DL}{2}-E}\right]^{-1}\right)$$

E propagators, D dimensions, L loops, rank r

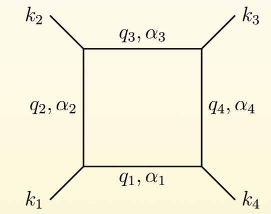
Geometric Strategy

- Reexpress polytope as weighted Minowski sum

$$\left(r - E + \frac{D}{2}(L+1)\right)\text{Newton}(\mathcal{U}) + (E - DL/2)\text{Newton}(\mathcal{F})$$

- BFP: integral of a Feynman-parameter monomial converges if
  - $\mathcal{U}$ and $\mathcal{F}$ have no zeros on faces of polytope (true for planar integrals)
  - the vector $\boldsymbol{e} + \mathbf{1}$ lies in the `relative interior' of the polytope
- Find interior with tools like NConvex, or via conjecture on generating function
- Conjecture: integral is finite iff each Feynman-parameter monomial is in the relative interior

Geometric Strategy: Example

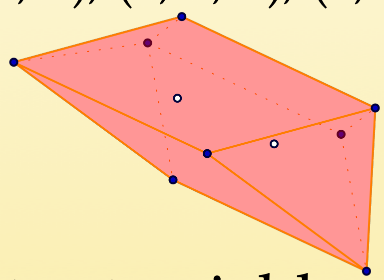


- Massless box

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \quad \mathcal{F} = s\alpha_1\alpha_3 + t\alpha_2\alpha_4$$

- Look at rank two: exponents w/lattice points in polytope

(0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 1, 0), (0, 1, 1), (0, 2, 0), (1, 0, 0), (1, 0, 1), (1, 1, 0), (2, 0, 0) + 18 others



- Relative-interior exponents

(1,0,1),(0,1,0)

- Require general loop-momentum numerator to yield only these
  - Require coefficients to be D-independent

N_1: $(s+t)\ell\cdot k_1 + t\ell\cdot k_2 - s\ell\cdot k_4 - (s+t)\,\ell^2$

N_2: $(t^2-s^2)(\ell\cdot k_1)^2 + 2t^2\ell\cdot k_1\,\ell\cdot k_2 + t^2(\ell\cdot k_2)^2 - 2s^2\ell\cdot k_1\,\ell\cdot k_4 - s^2(\ell\cdot k_4)^2 + st^2\ell^2$

N_3: $-(s+t)(\ell\cdot k_1)^2 - t\ell\cdot k_1\,\ell\cdot k_2 - (2s+t)\ell\cdot k_1\,\ell\cdot k_4 + t\ell\cdot k_2\,\ell\cdot k_4 - s(\ell\cdot k_4)^2 - \frac{1}{2}st\ell^2$

Comparison

- Do the results of the two approaches agree?

- Yes…

…but the comparison is subtle

  - Geometric numerators can vanish in parametric representation without being locally finite (special IBPs)
  - Need to consider locally finite IBPs too

Integration by Parts

Canay, Novichkov, Ma, Wu, Zhang, DAK

- Can avoid doubled propagators using generating vectors *aka* "syzygy method"

- Choose v in

$$\int d^D\ell_j \ \frac{\partial}{\partial \ell_i^\mu}\left[\frac{v^\mu\, N}{D_1 D_2 \cdots D_E}\right]$$

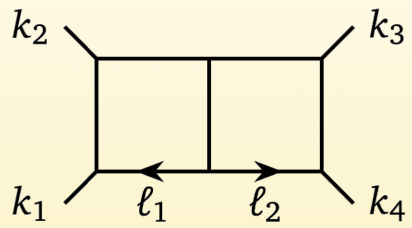
Such that $v \cdot \frac{\partial}{\partial \ell_i} D_j \propto D_j$ for all D_j

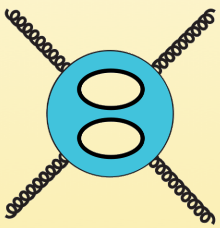
Find only IBPs for finite numerators by also requiring

$$v \cdot \frac{\partial}{\partial \ell_i} N_j = c_m N_m$$

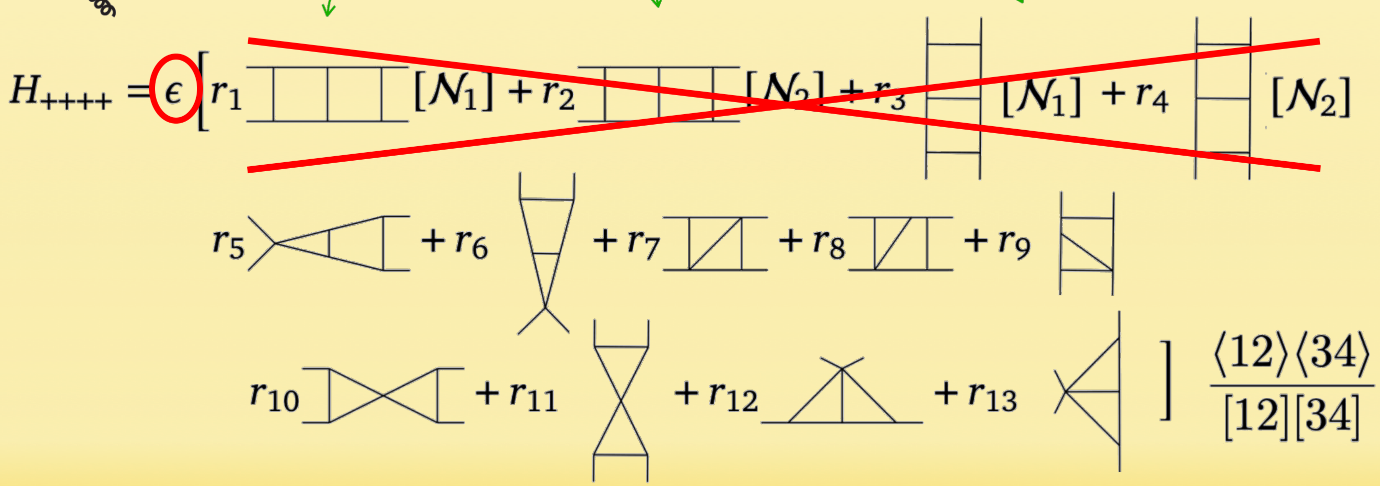
Use of New Integrals

Look at two-loop $A_4(++++)$





$$\mathcal{N}_1 = G\begin{pmatrix}\ell_1 & 1 & 2\\ 1 & 2 & 4\end{pmatrix} G\begin{pmatrix}\ell_2 & 3 & 4\\ 1 & 2 & 4\end{pmatrix}, \quad \mathcal{N}_2 = G\begin{pmatrix}\ell_1 & 1 & 2\\ \ell_2 & 3 & 4\end{pmatrix}$$



Coefficients simpler too

Integrating

Canay, Novichkov, DAK

- New opportunity to use existing tool: HyperInt

Panzer

- Top-level topology has complicated functions
- Rank-two finite pentabox
- All integrations but one are linear
- Separate $\sqrt{\Delta}$ by hand: $\alpha_6^2 - \Delta$

Summary

With numerators
Feynman integrals settle
happily bounded

- Finite integrals are a first step to exploring a new organization of scattering amplitudes
- Two approaches
  - Analytic approach: Landau equations + Anastasiou–Sterman scaling
  - Geometric approach: Newton polytopes + BFP theorem
- Applications
  - Amplitude structure
  - Integrating