Gauge Theory Bootstrap:

Pion amplitudes and low energy parameters

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based on 2309.12402 and 2403.10772 with Martin Kruczenski

Low energy physics of asymptotically free gauge theory

asymptotically free gauge theory $SU(N_c)$ with N_f massive quarks $m_q \ll \Lambda_{\rm QCD}$

chiral symmetry breaking & confinement

$$\mathcal{L} = i\sum_j^{N_f} \bar{q}_j \not{D} q_j - \sum_j^{N_f} m_q \bar{q}_j q_j - \frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu} + \text{gauge fixing} + \text{ghost}$$

gauge theory parameters: N_c N_f m_q $\Lambda_{\rm QCD}$

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gauge theory parameters: $N_c \quad N_f \quad m_q \quad \Lambda_{\mathrm{QCD}}$

What is the low energy physics?

Physics of Goldstone bosons

chiral symmetry breaking

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$$

pseudo-Goldstone bosons dominate the low energy physics

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pseudo-Goldstone bosons dominate the low energy physics

e.g. $N_f = 2$ pions $\pi_0 = \pi^3$ $\pi_\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2)$

effective Lagrangian: $\mathcal{L} = \frac{f_\pi^2}{4}\left\{\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + m_\pi^2 \mathrm{Tr}\left(U + U^\dagger\right)\right\}$ $U = e^{i\frac{\vec{\tau}\cdot\vec{\pi}}{f_\pi}}$

The EFT approach

non-renormalizable, add new terms with unknown coefficients:

e.g.

$$
\begin{aligned}
\mathcal{L}_4 \;=\; & \frac{l_1}{4}\left\{\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\right\}^2 + \frac{l_2}{4}\mathrm{Tr}[D_\mu U(D_\nu U)^\dagger]\mathrm{Tr}[D^\mu U(D^\nu U)^\dagger] \\
& + \frac{l_3}{16}[\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)]^2 + \frac{l_4}{4}\mathrm{Tr}[D_\mu U(D^\mu\chi)^\dagger + D_\mu\chi(D^\mu U)^\dagger] \\
& + l_5\left[\mathrm{Tr}(f^R_{\mu\nu}U f^{\mu\nu}_L U^\dagger) - \frac{1}{2}\mathrm{Tr}(f^L_{\mu\nu}f^{\mu\nu}_L + f^R_{\mu\nu}f^{\mu\nu}_R)\right] \\
& + i\frac{l_6}{2}\mathrm{Tr}[f^R_{\mu\nu}D^\mu U(D^\nu U)^\dagger + f^L_{\mu\nu}(D^\mu U)^\dagger D^\nu U] \\
& - \frac{l_7}{16}[\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)]^2
\end{aligned}
$$

χPT: unknown coefficients determined from fitting with experimental data

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in principle should be computed from UV gauge theory

Strongly coupled physics → Gauge Theory Bootstrap

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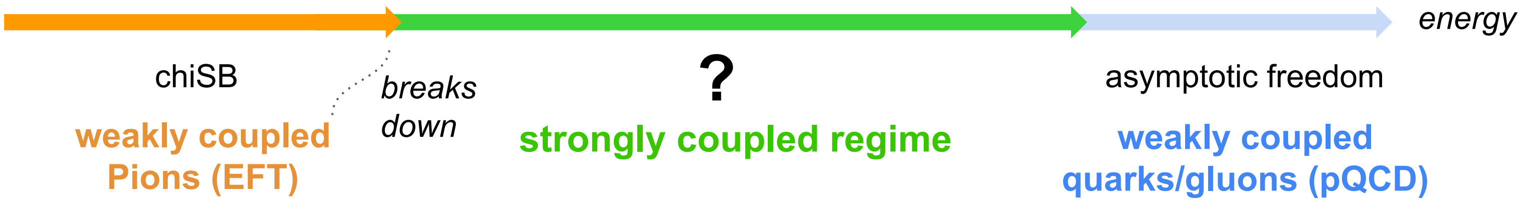
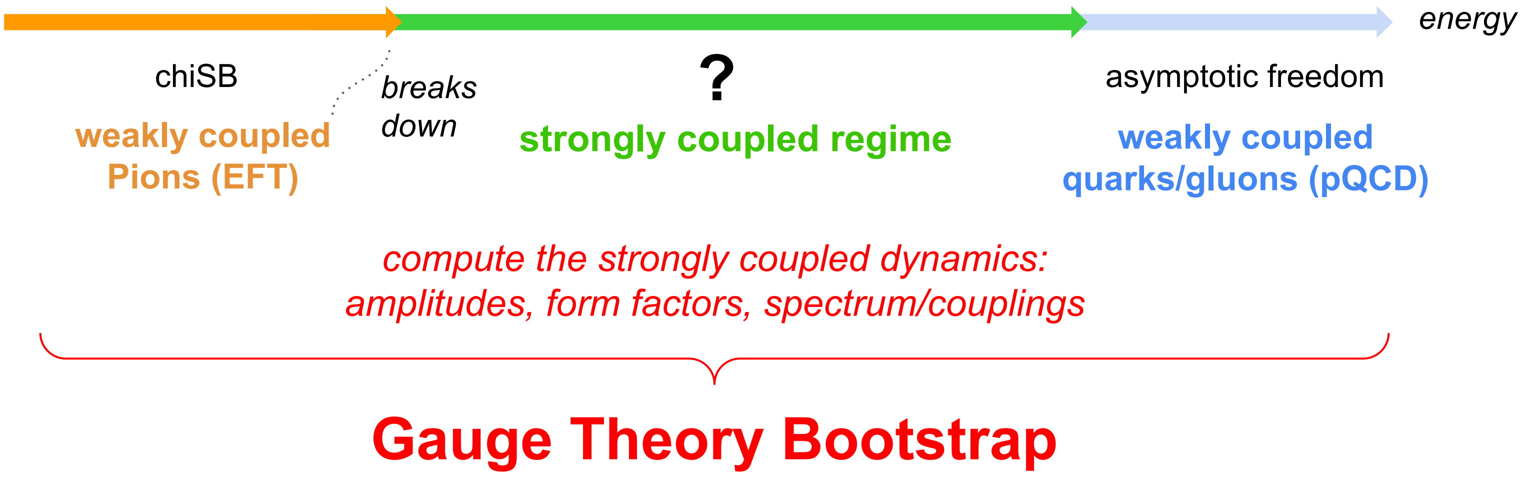


Strongly coupled physics → Gauge Theory Bootstrap



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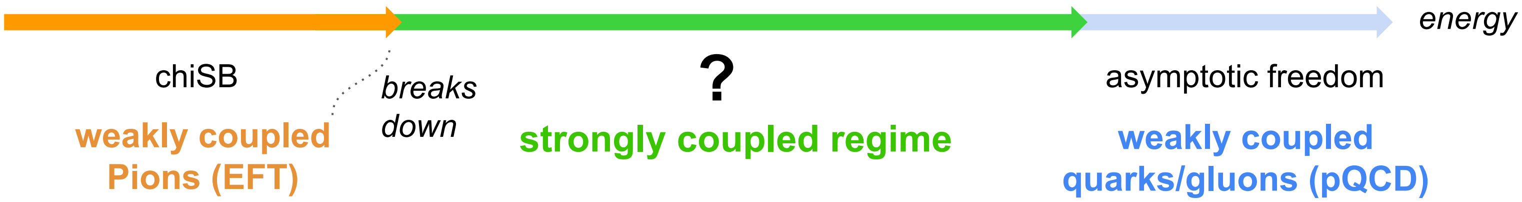


compute the strongly coupled dynamics:
amplitudes, form factors, spectrum/couplings

Gauge Theory Bootstrap

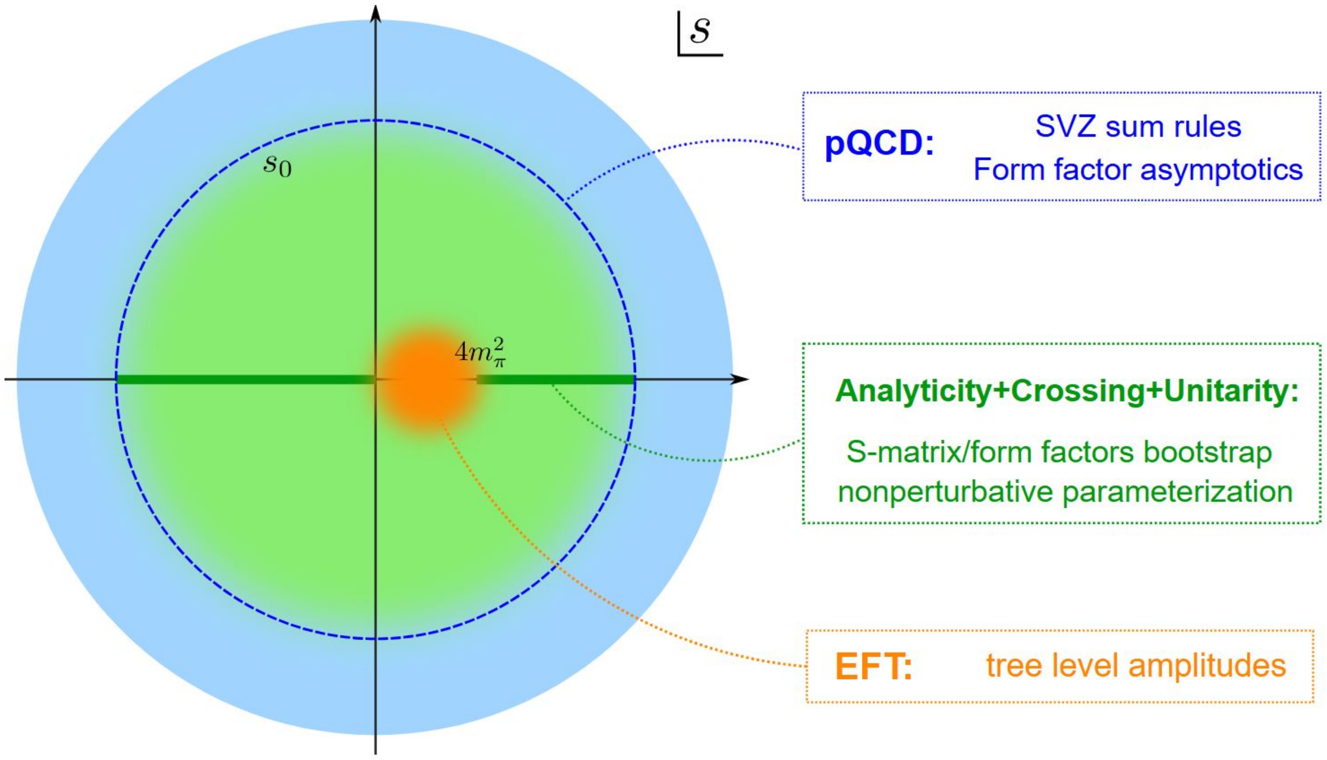
rules of the game:

assume — *chiral symmetry breaking & confinement*

input — $\underbrace{N_c \quad N_f \quad m_q \quad \alpha_s}_{\textit{defining gauge theory}} \qquad \underbrace{f_\pi \quad m_\pi}_{\textit{universal low energy parameters}}$

theoretical/numerical computation, not using experimental scattering data as input

Gauge Theory Bootstrap: summary



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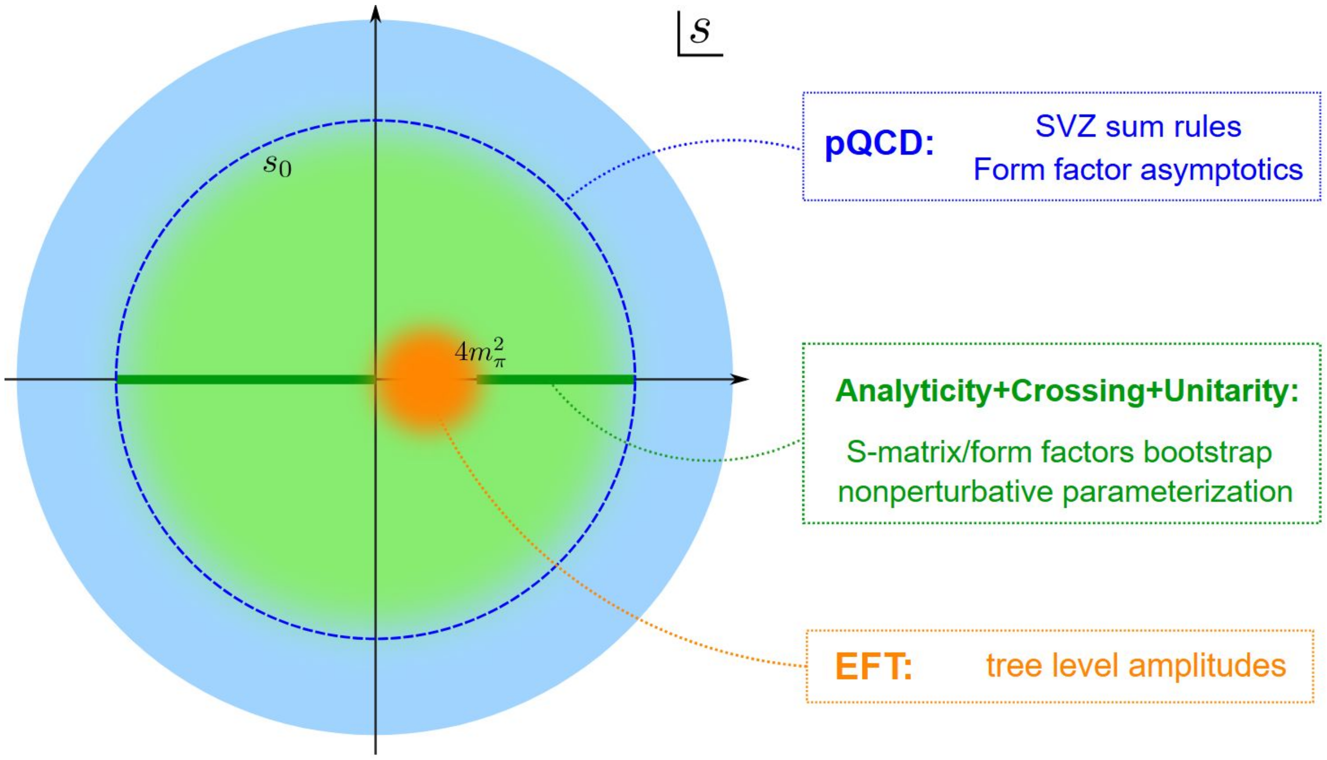

look for amplitudes/form factors that: 1, satisfy generic consistency conditions (analyticity, crossing, unitarity)
2, match low energy behavior (chiSB) and high energy (pQCD)

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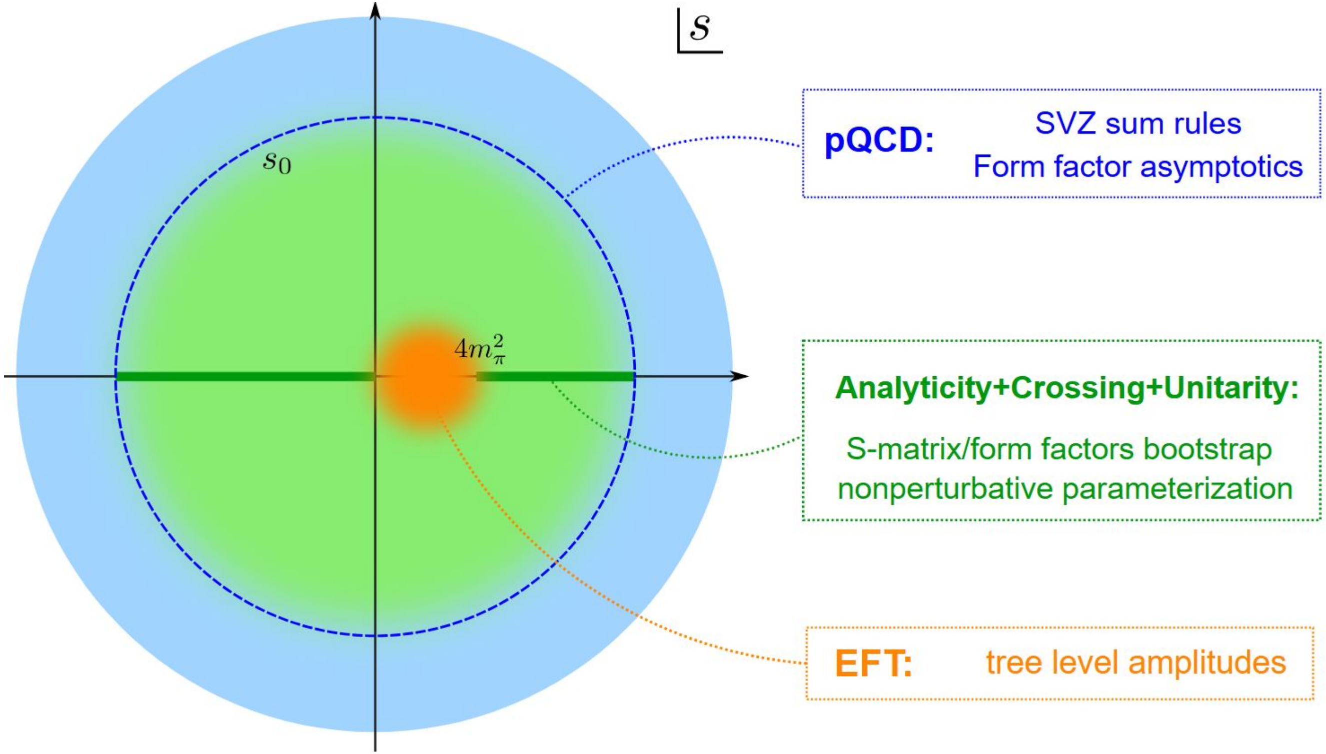


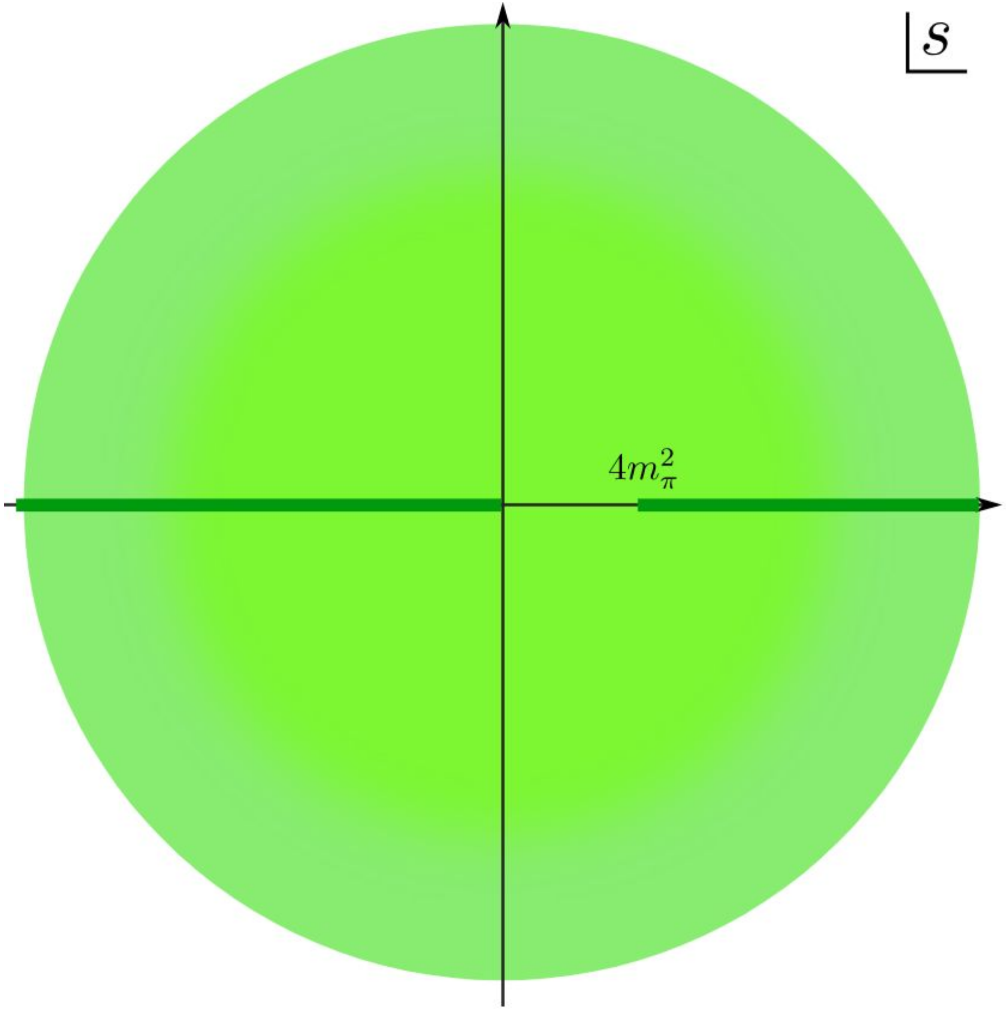
test with

$N_f = 2 \quad N_c = 3$

compare with experiments

changeable in the approach

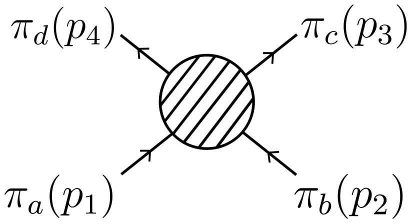
look for amplitudes/form factors that: 1, satisfy generic consistency conditions (analyticity, crossing, unitarity)
2, match low energy behavior (chiSB) and high energy (pQCD)



Analyticity+Crossing+Unitarity:

S-matrix bootstrap
nonperturbative parameterization

S-matrix bootstrap parameterization



modern S-matrix bootstrap: [Paulos, Penedones, Toledo, van Rees, Vieira, 2016&2017]

$$\langle p_1, a; p_2, b|\mathbf{T}|p_3, c; p_4, d\rangle = A(s,t,u)\delta_{ab}\delta_{cd} + A(t,s,u)\delta_{ac}\delta_{bd} + A(u,t,s)\delta_{ad}\delta_{bc}$$

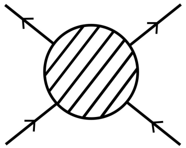
Crossing $A(s,t,u) = A(s,u,t)$

Analyticity cuts $s,t,u > 4$

$$m_\pi = 1$$

$$s + t + u = 4$$

S-matrix bootstrap parameterization



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Crossing $A(s,t,u) = A(s,u,t)$ ***Analyticity*** cuts $s,t,u > 4$

$$m_\pi = 1$$

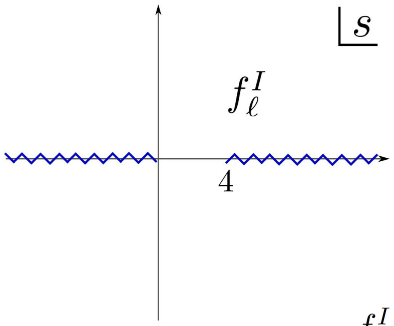
nonperturbative parameterization encoding **Analyticity** and **Crossing**:

$$s + t + u = 4$$

$$A(s,t,u) = \frac{1}{\pi^2}\int_4^\infty dx \int_4^\infty dy \left[\frac{\rho_1(x,y)}{(x-s)(y-t)} + \frac{\rho_1(x,y)}{(x-s)(y-u)} + \frac{\rho_2(x,y)}{(x-t)(y-u)}\right]$$ +subtraction terms

parameters: $\{\rho_{\alpha=1,2}(x,y), \dots\}$ numerics: discretize $\{\rho_{\alpha,ij}, \dots\}$ bootstrap variables

S-matrix bootstrap parameterization



analytic function of s

$$f_\ell^I(s) = \frac{1}{4}\int_{-1}^{1} d\cos\theta P_\ell(\cos\theta) T^I(s,t)$$

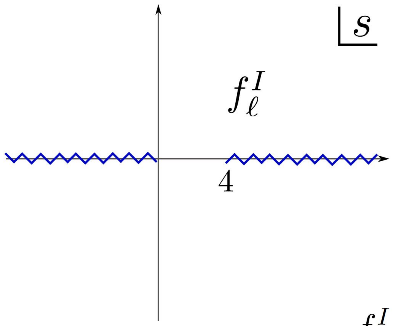
$$\begin{aligned} T^{I=0}(s,t,u) &= 3A(s,t,u) + A(t,s,u) + A(u,t,s) \\ T^{I=1}(s,t,u) &= A(t,s,u) - A(u,t,s) \\ T^{I=2}(s,t,u) &= A(t,s,u) + A(u,t,s) \end{aligned}$$

$\Downarrow$

$SU(2)_V$ isospin

Symmetry

S-matrix bootstrap parameterization



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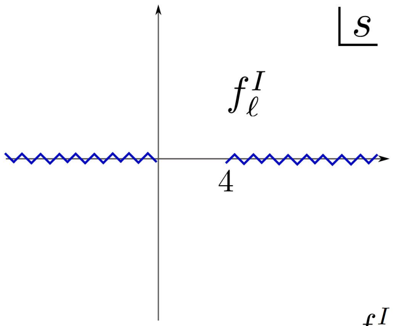
physical kinematic region $s > 4$

$$S_\ell^I(s) = 1 + i\pi\sqrt{\frac{s-4}{s}} f_\ell^I(s) = \eta_\ell^I(s)\, e^{2i\delta_\ell^I(s)}$$

inelasticity phase shift

unphysical region $f_\ell^I(0 < s < 4)$ *real linear functionals of bootstrap variables*

S-matrix bootstrap parameterization



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Unitarity

$$|S_\ell^I(s^+)| \leq 1, \; s > 4 \quad \forall \ell, I$$

$$\begin{pmatrix} 1 & S_\ell^I(s) \\ S_\ell^{I*}(s) & 1 \end{pmatrix} \succeq 0$$

positive semidefinite → convex space of amplitudes

convex optimization

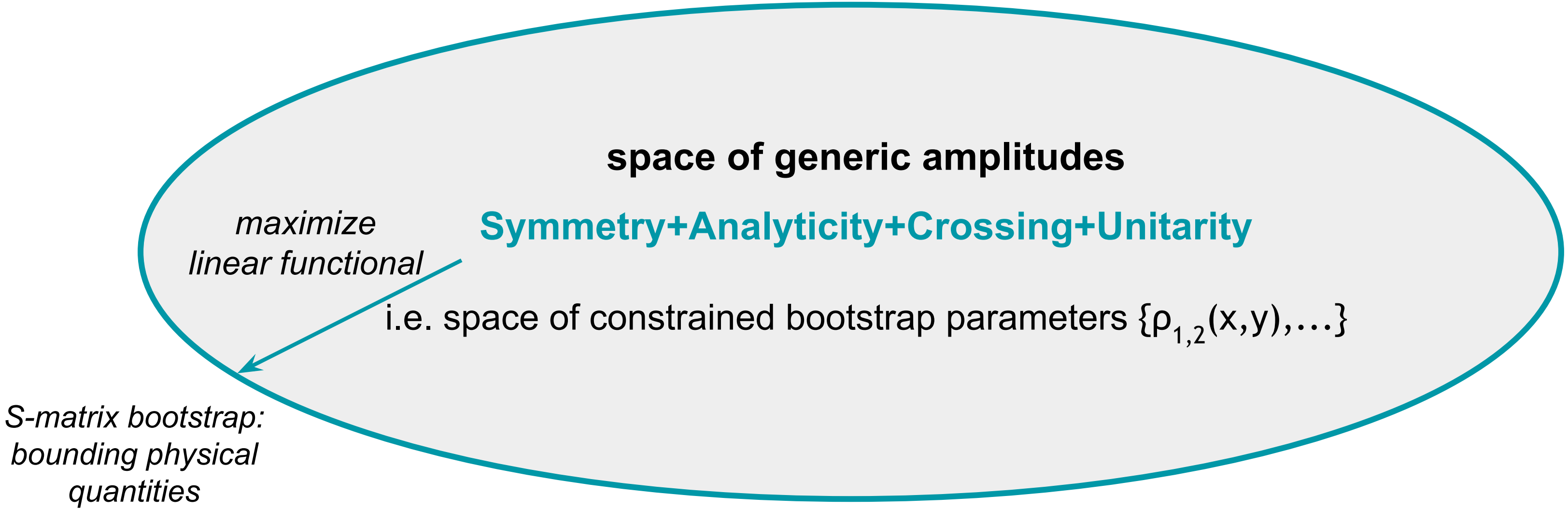
Bootstrap: maximization → nonperturbative computations

space of generic amplitudes

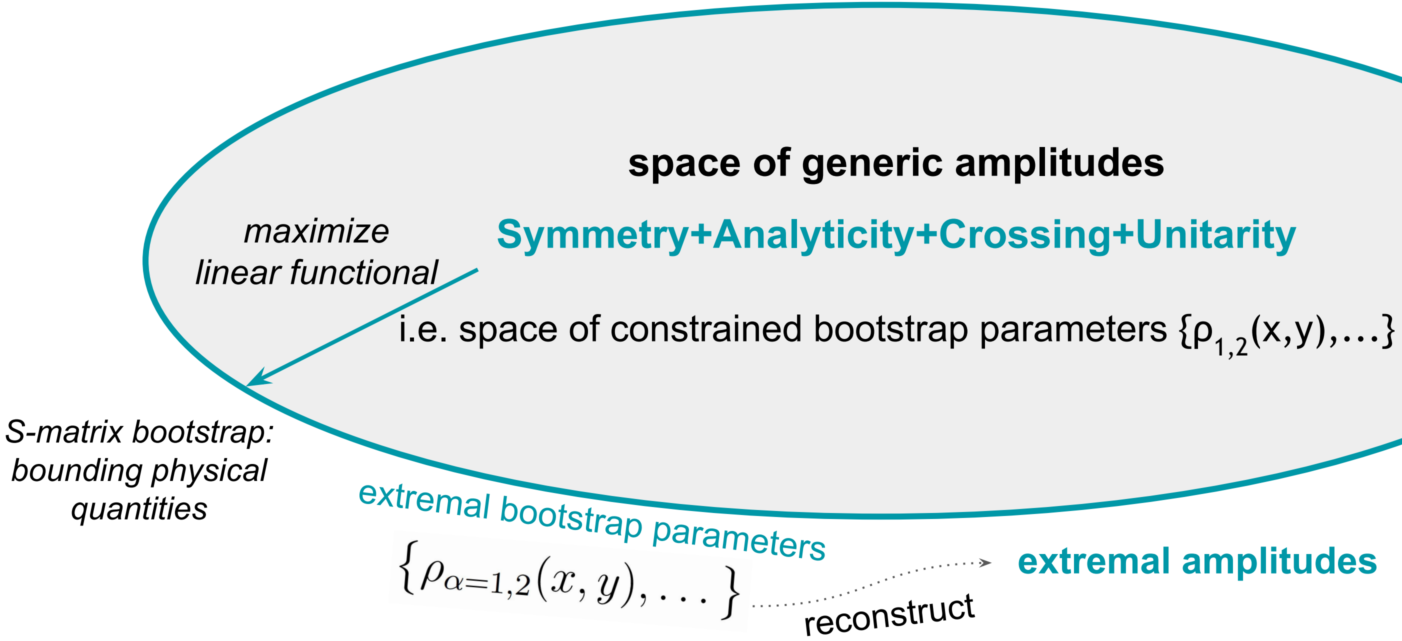
Symmetry+Analyticity+Crossing+Unitarity

i.e. space of constrained bootstrap parameters $\{\rho_{1,2}(x,y),\ldots\}$

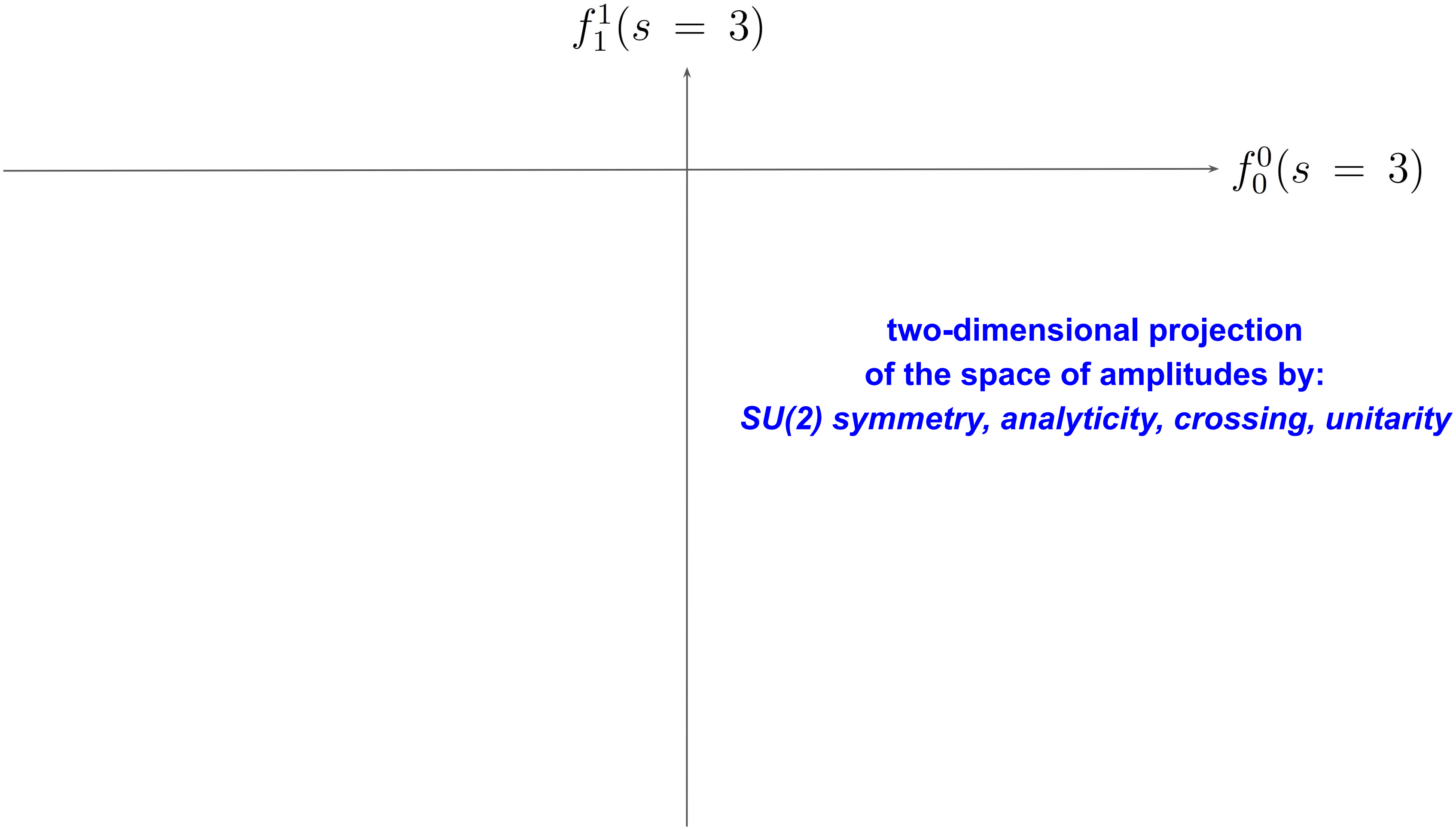
Bootstrap: maximization → nonperturbative computations

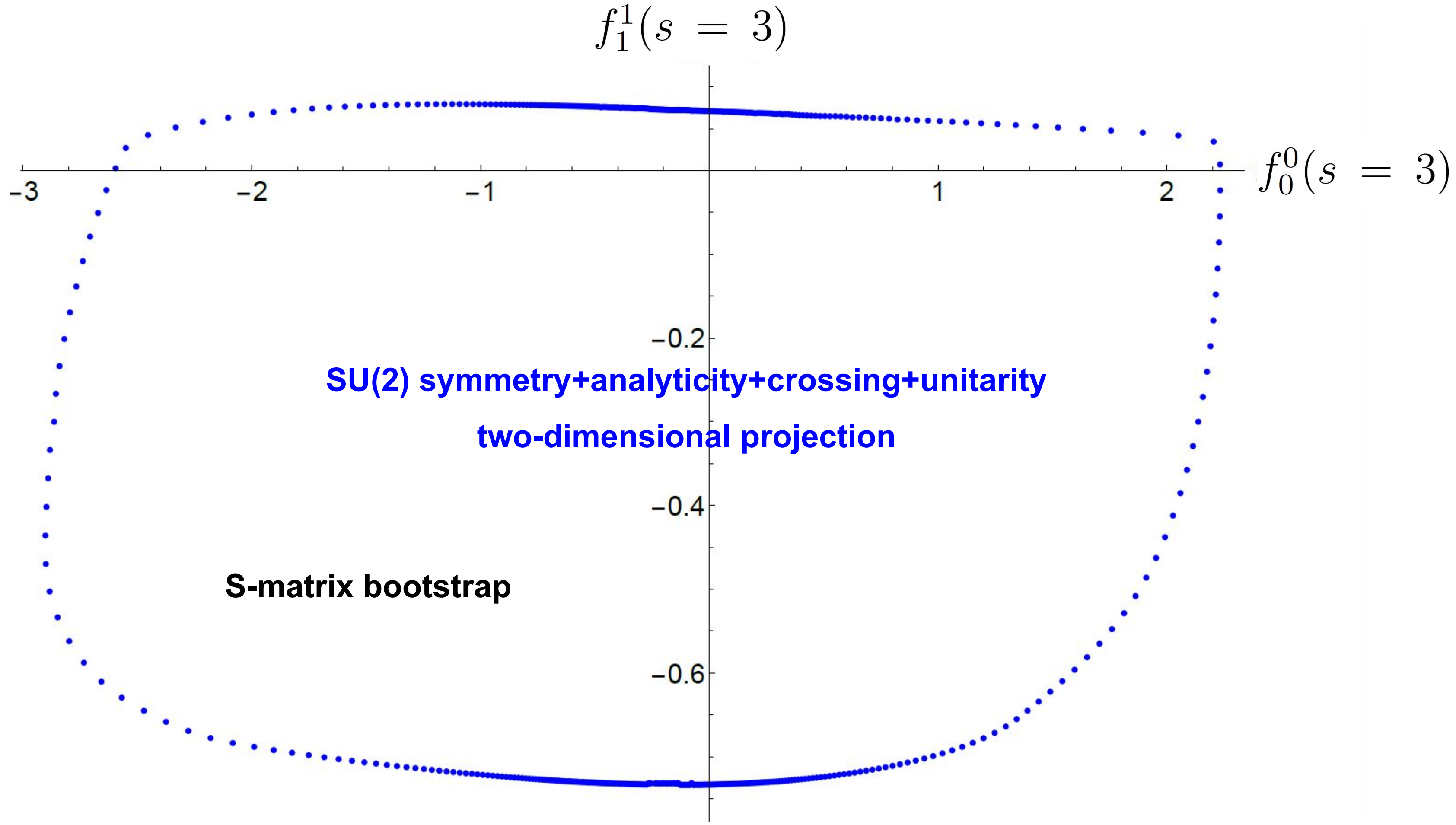


Bootstrap: maximization → nonperturbative computations

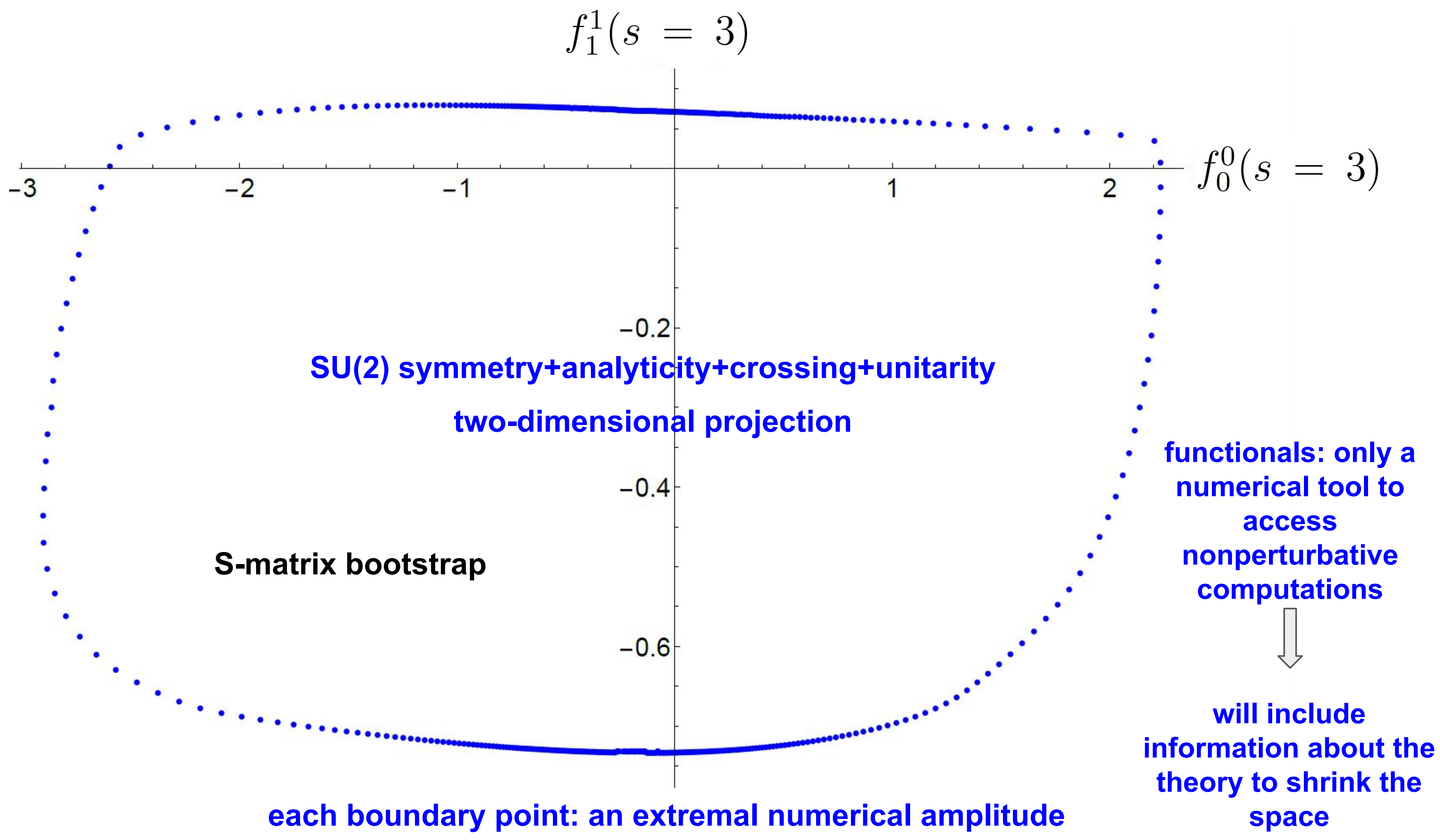


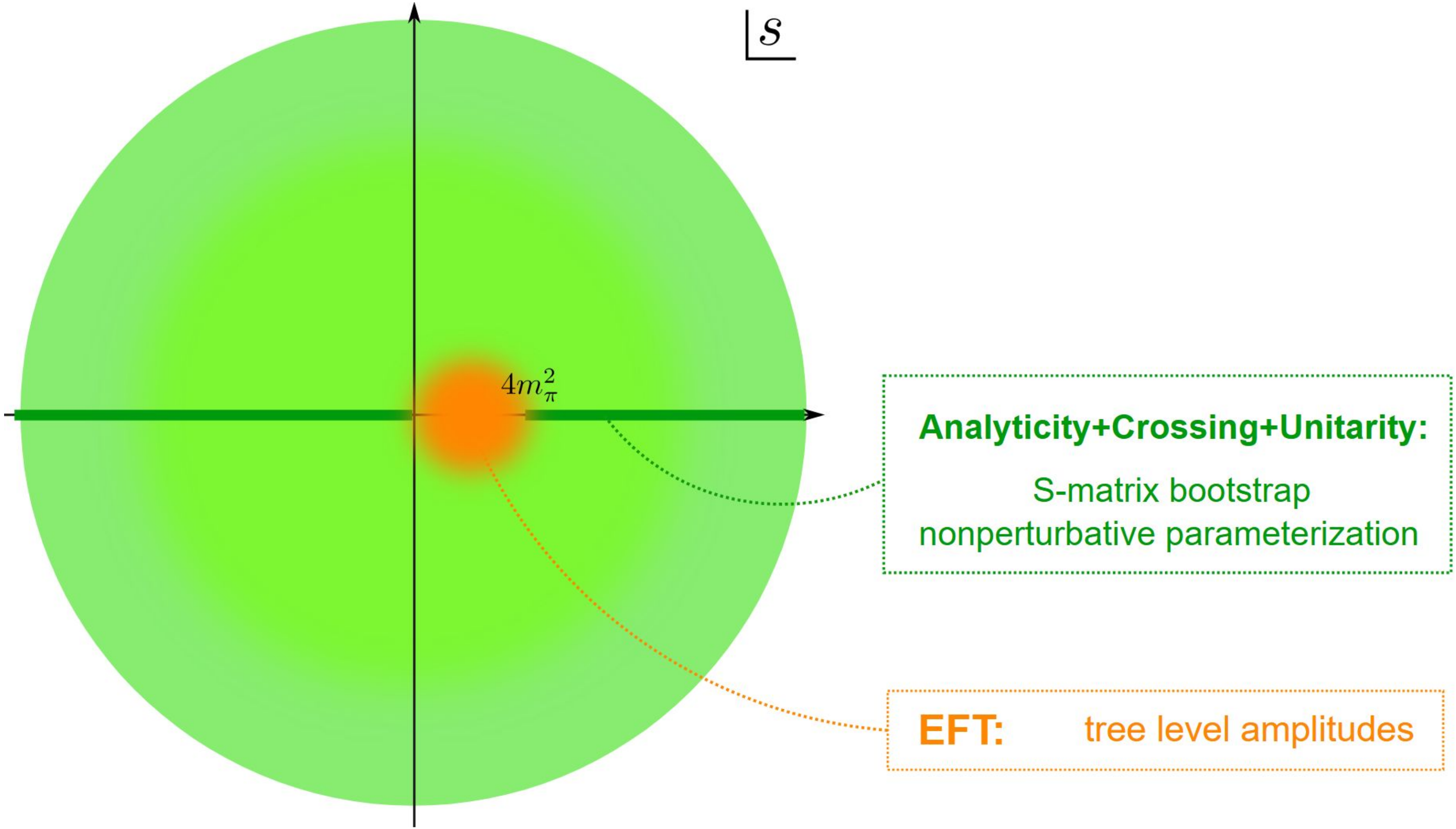
maximization → non-perturbative numerical computation of scattering amplitudes


$f_1^1(s = 3)$
$f_0^0(s = 3)$
two-dimensional projection
of the space of amplitudes by:
SU(2) symmetry, analyticity, crossing, unitarity



each boundary point: an extremal numerical amplitude


$f_1^1(s = 3)$
$f_0^0(s = 3)$
−3
−2
−1
1
2
−0.2
−0.4
−0.6
SU(2) symmetry+analyticity+crossing+unitarity
two-dimensional projection
S-matrix bootstrap
functionals: only a numerical tool to access nonperturbative computations
will include information about the theory to shrink the space
each boundary point: an extremal numerical amplitude


s
$4m_\pi^2$
Analyticity+Crossing+Unitarity:
S-matrix bootstrap
nonperturbative parameterization
EFT:
tree level amplitudes

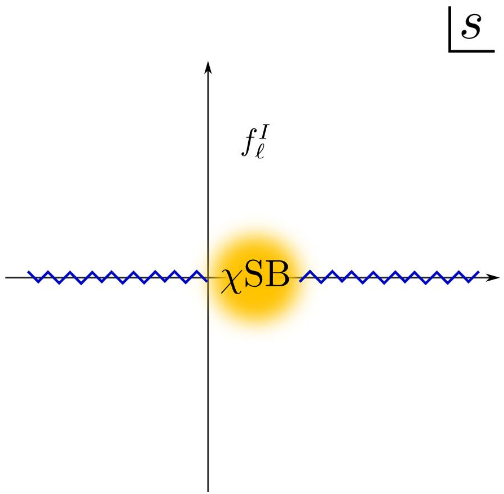
Chiral symmetry breaking (tree EFT) input

chiral symmetry breaking: weakly coupled Goldstone bosons at very low energy

tree-level amplitude: $A_{\text{tree}}(s,t,u) = \frac{4}{\pi}\frac{s - m_\pi^2}{32\pi f_\pi^2}$ *linear in s* [Weinberg, 1966]

S0: $f^0_{0,\text{tree}}(s) = \frac{2}{\pi}\frac{2s - m_\pi^2}{32\pi f_\pi^2}$ P1: $f^1_{1,\text{tree}}(s) = \frac{2}{\pi}\frac{s - 4m_\pi^2}{96\pi f_\pi^2}$ S2: $f^2_{0,\text{tree}}(s) = \frac{2}{\pi}\frac{2m_\pi^2 - s}{32\pi f_\pi^2}$

good in unphysical region (very low energy) $0 < s < 4m_\pi^2$



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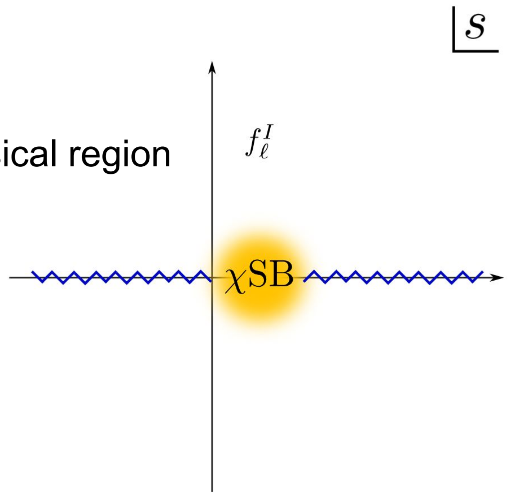
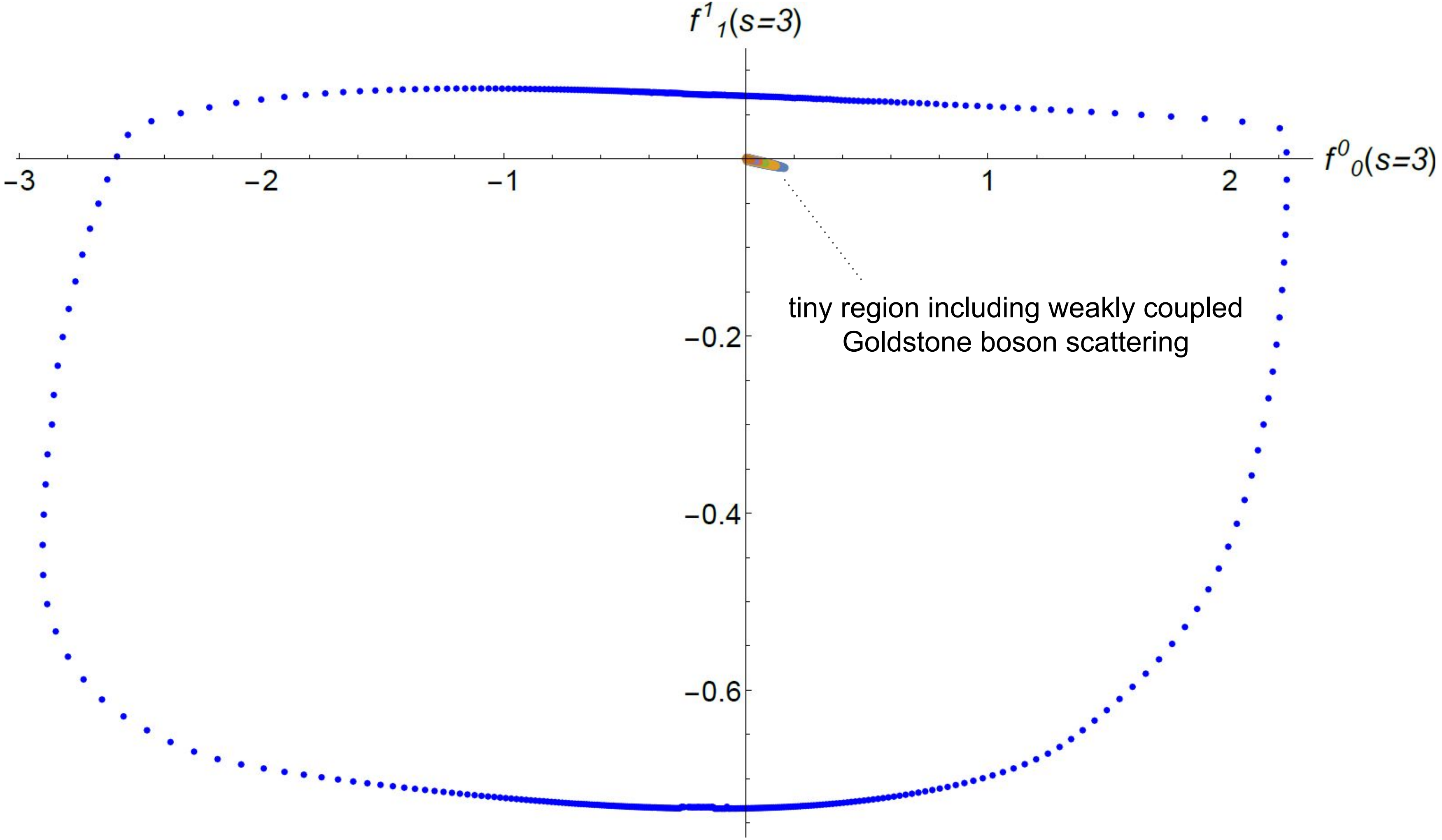
good in unphysical region (very low energy) $0 < s < 4m_\pi^2$

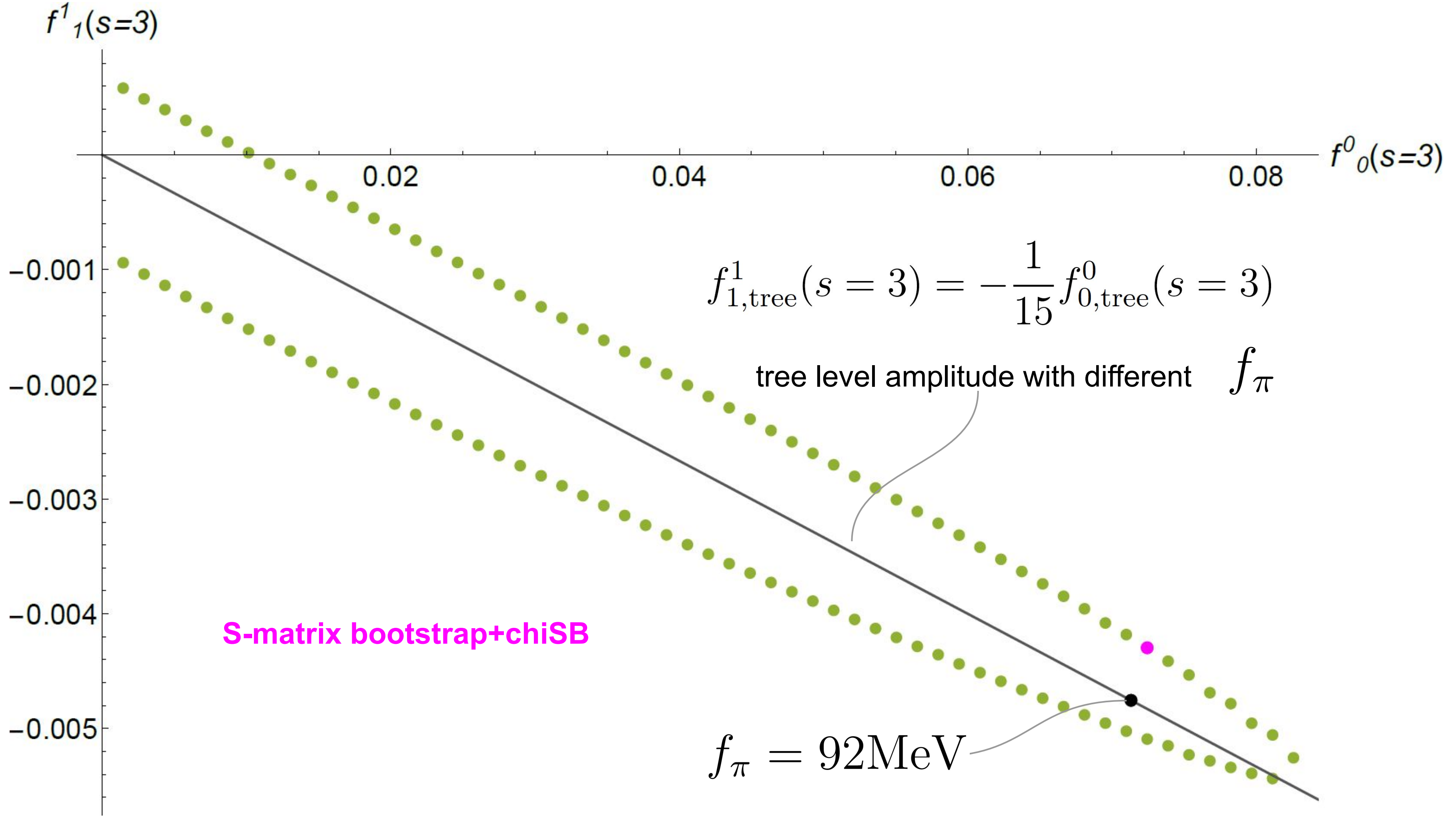
numerically requires p.w. in the bootstrap match the tree level p.w. in unphysical region

$$f^0_0(s) \simeq f^0_{0,\text{tree}}(s) \qquad f^1_1(s) \simeq f^1_{1,\text{tree}}(s) \qquad f^2_0(s) \simeq f^2_{0,\text{tree}}(s) \qquad 0 < s < 4m_\pi^2$$

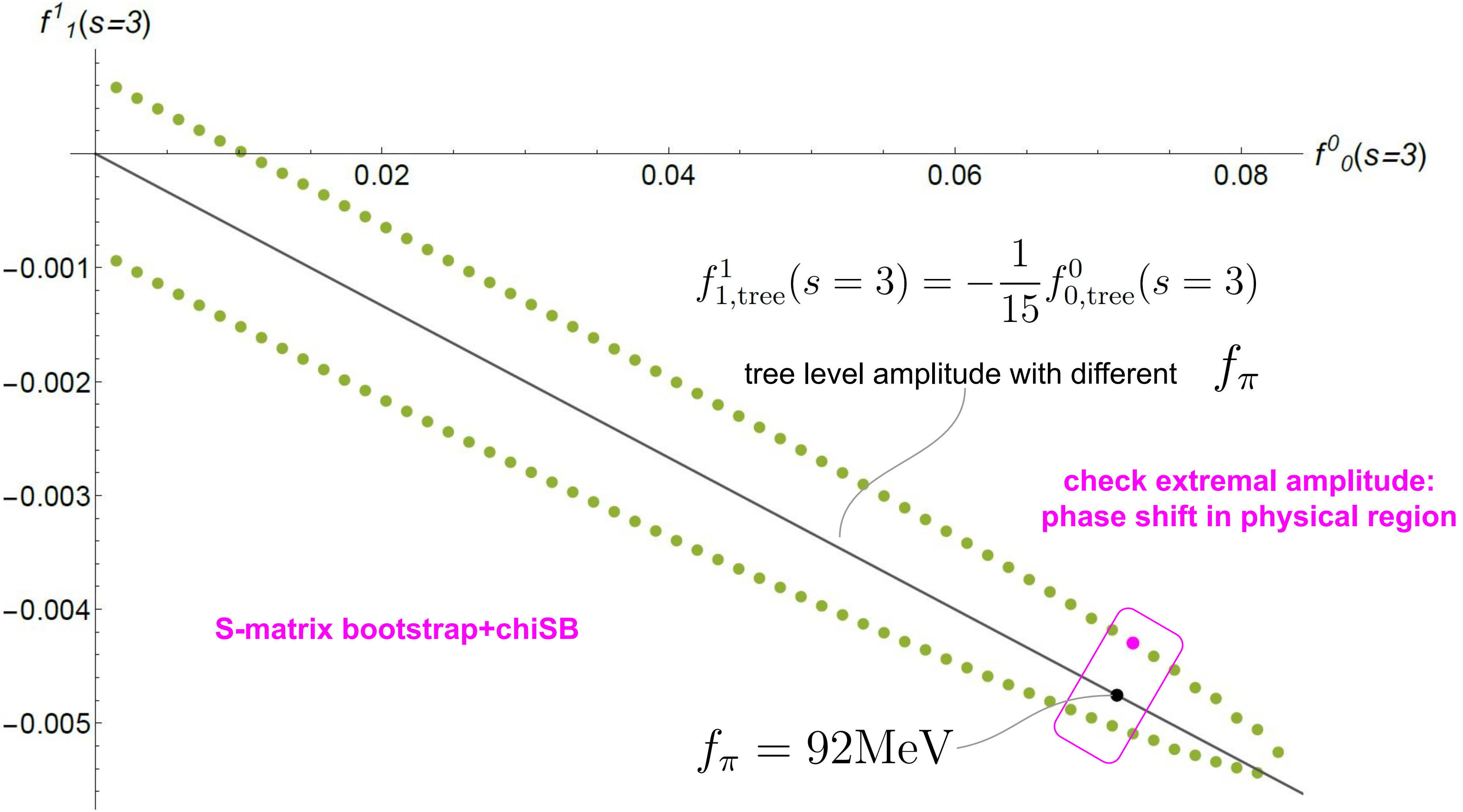




$f^1{}_1(s=3)$
$f^0{}_0(s=3)$
−3
−2
−1
1
2
−0.2
−0.4
−0.6
tiny region including weakly coupled
Goldstone boson scattering



$f^1_1(s=3)$
$f^0_0(s=3)$
0.02
0.04
0.06
0.08
−0.001
−0.002
−0.003
−0.004
−0.005
$f^1_{1,\text{tree}}(s=3) = -\frac{1}{15} f^0_{0,\text{tree}}(s=3)$
tree level amplitude with different f_π
S-matrix bootstrap+chiSB
$f_\pi = 92\text{MeV}$



$f^1_1(s=3)$
$f^0_0(s=3)$
0.02
0.04
0.06
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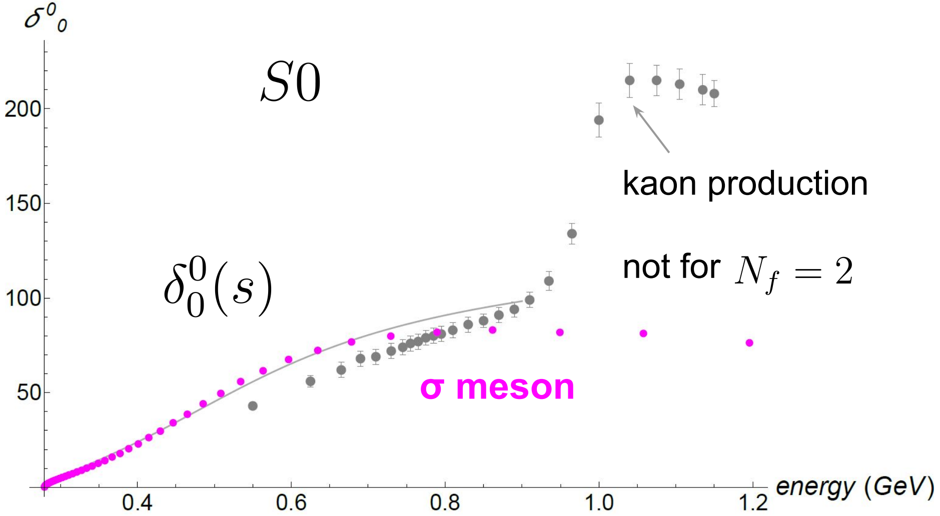
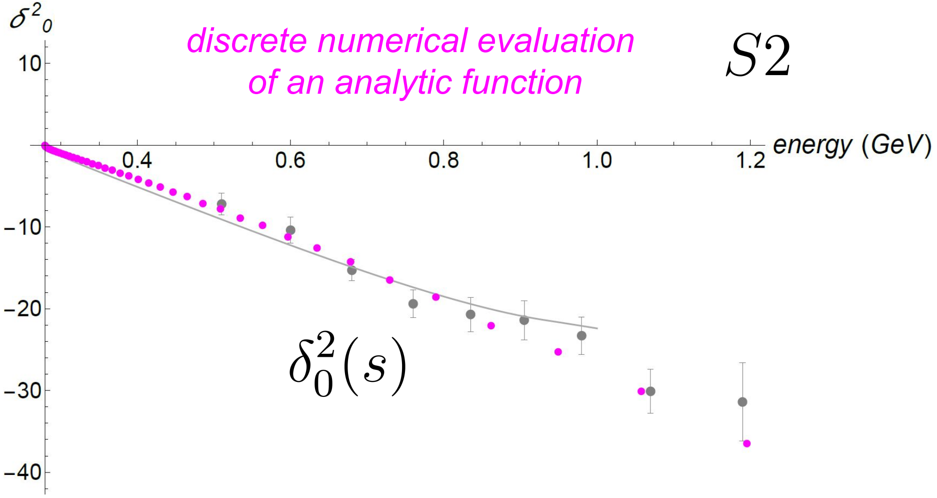
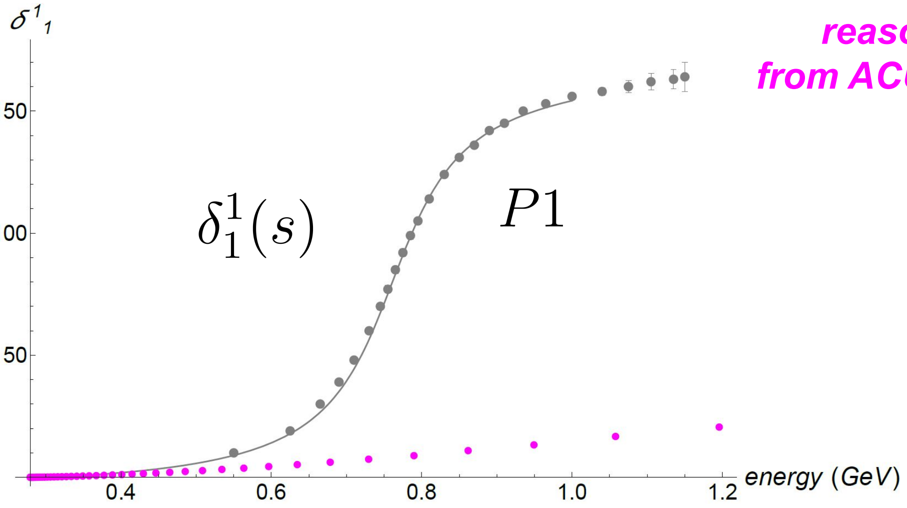
$f^1_{1,\text{tree}}(s=3) = -\frac{1}{15} f^0_{0,\text{tree}}(s=3)$
tree level amplitude with different f_π
check extremal amplitude:
phase shift in physical region
S-matrix bootstrap+chiSB
$f_\pi = 92\text{MeV}$



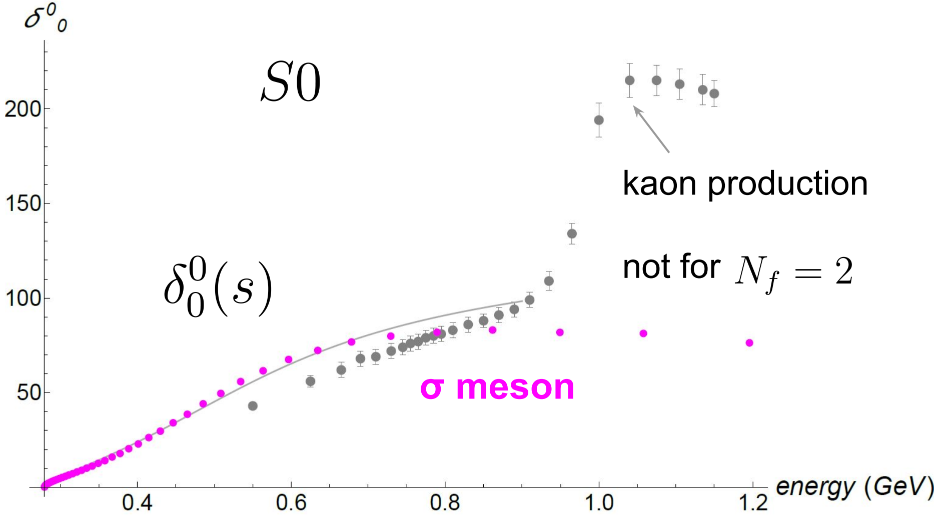


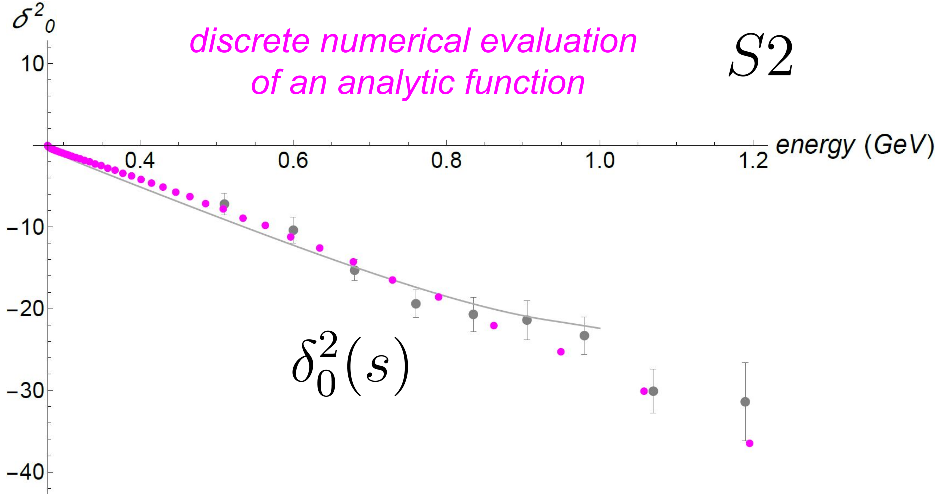
phase shifts with only chiSB (EFT) input

reasonable S0, S2 waves from ACU + weakly coupled EFT



experimental data (gray dots)
[Protopopescu et al, 1973]
[Losty et al, 1974]
pheno fit (gray line)
[Pelaez, Yndurain, 2005]





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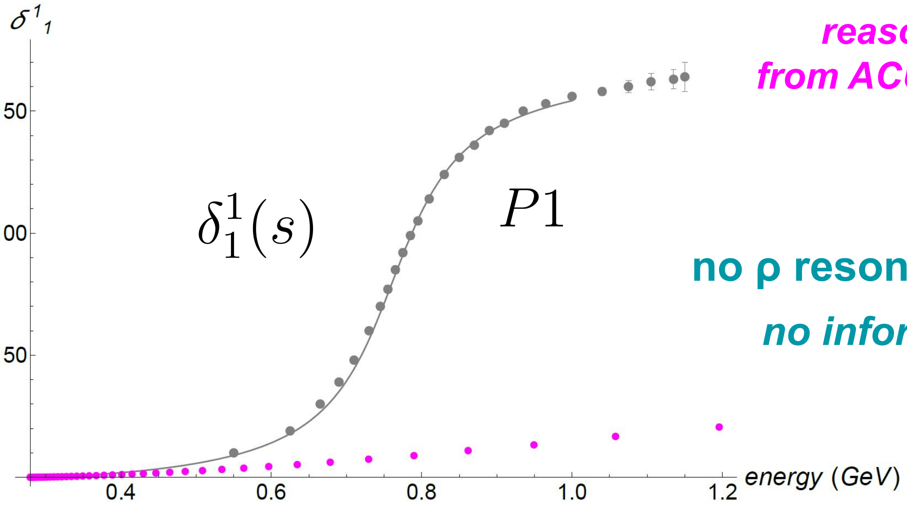

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[Protopopescu et al, 1973]
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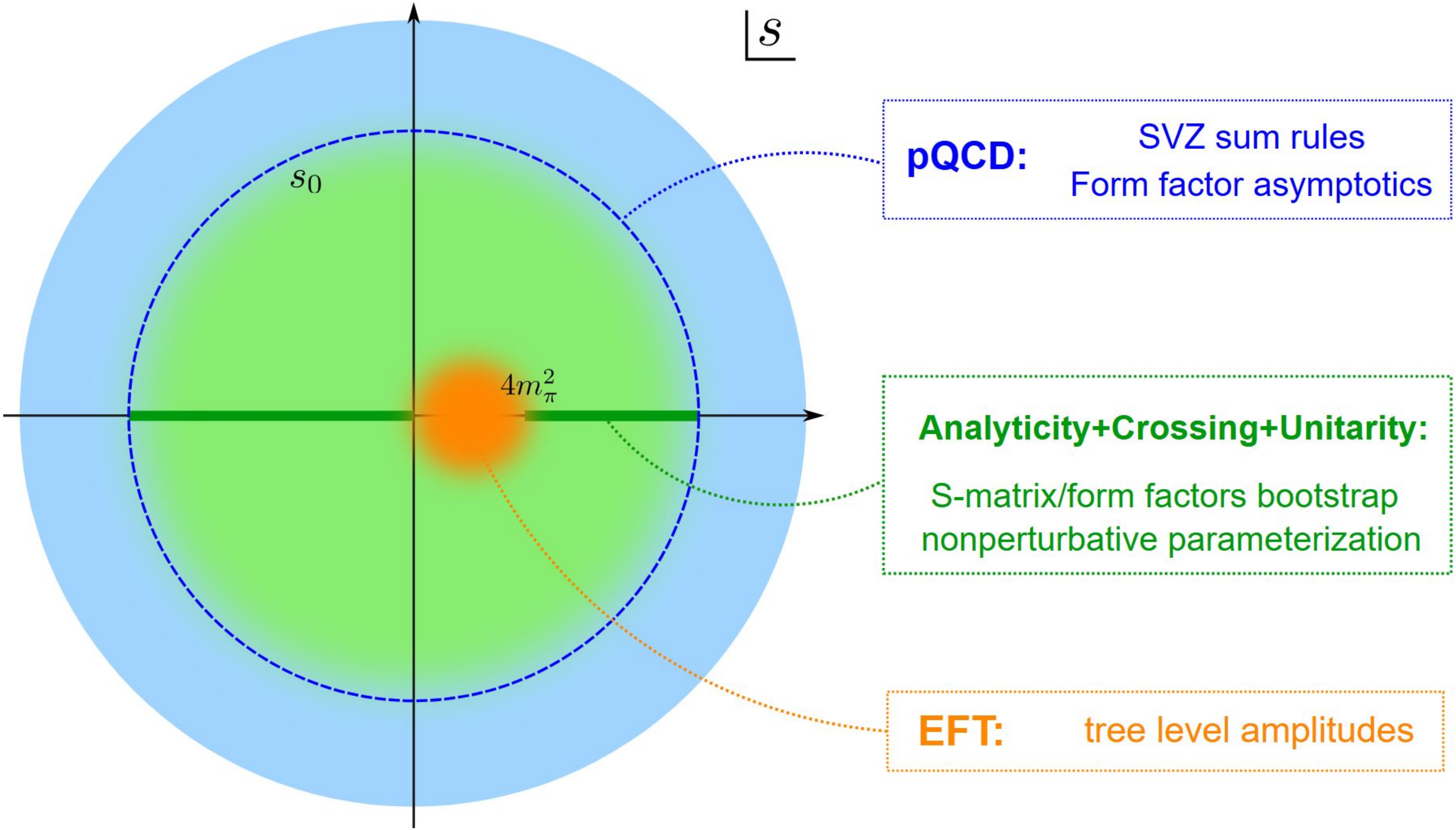
no ρ resonance without UV info

no information on QCD yet



s
s_0
4m_π^2
pQCD:
SVZ sum rules
Form factor asymptotics
Analyticity+Crossing+Unitarity:
S-matrix/form factors bootstrap
nonperturbative parameterization
EFT:
tree level amplitudes

S-matrix/form factor bootstrap

[Karateev, Kuhn, Penedones, 2019]

$$|\psi_1\rangle = |p_1, p_2\rangle_{in} \,, \qquad |\psi_2\rangle = |p_1, p_2\rangle_{out} \,, \qquad |\psi_3\rangle = \int dx e^{-i(p_1+p_2)\cdot x} \mathcal{O}(x)|0\rangle$$

state created by UV local operator

positive semidefinite matrix $\langle \psi_a | \psi_b \rangle = \begin{pmatrix} 1 & S & \mathcal{F} \\ S^* & 1 & \mathcal{F}^* \\ \mathcal{F}^* & \mathcal{F} & \rho \end{pmatrix} \succeq 0$

S-matrix/form factor bootstrap

[Karateev, Kuhn, Penedones, 2019]

$$|\psi_1\rangle = |p_1, p_2\rangle_{in}\,, \qquad |\psi_2\rangle = |p_1, p_2\rangle_{out}\,, \qquad |\psi_3\rangle = \int dx e^{-i(p_1+p_2)\cdot x}\mathcal{O}(x)|0\rangle$$

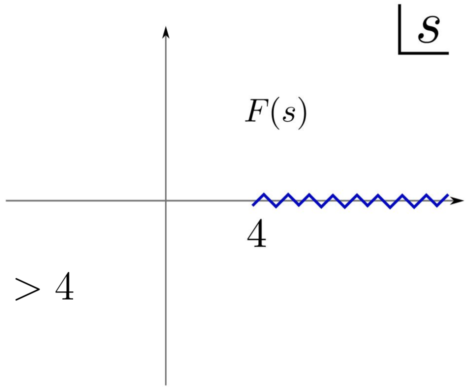
state created by UV local operator

positive semidefinite matrix $\langle\psi_a|\psi_b\rangle = \begin{pmatrix} 1 & S & \mathcal{F} \\ S^* & 1 & \mathcal{F}^* \\ \mathcal{F}^* & \mathcal{F} & \rho \end{pmatrix} \succeq 0$

2-particle form factor: $_{\text{out}}\langle p_1, p_2|\mathcal{O}(0)|0\rangle = F(s)$ *analytic function of s*

$$F(s) = \frac{1}{\pi}\int_4^\infty dx \frac{\text{Im}F(x)}{x-s}$$ +subtractions

spectral density: $\int \frac{d^4x}{(2\pi)^4} e^{iPx}\langle 0|\mathcal{O}^\dagger(x)\mathcal{O}(0)|0\rangle = \rho(s)$ *supported at* $s > 4$



S-matrix/form factor bootstrap

[Karateev, Kuhn, Penedones, 2019]

$$|\psi_1\rangle = |p_1, p_2\rangle_{in}\,, \qquad |\psi_2\rangle = |p_1, p_2\rangle_{out}\,, \qquad |\psi_3\rangle = \int dx e^{-i(p_1+p_2)\cdot x}\mathcal{O}(x)|0\rangle$$

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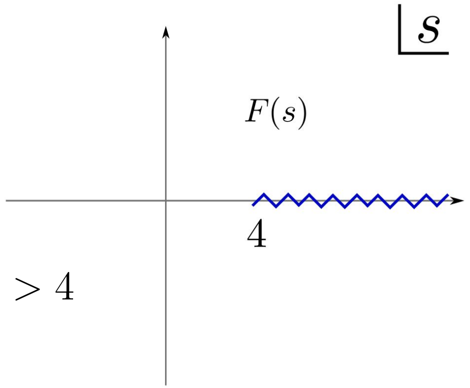


spectral density: $\int \frac{d^4x}{(2\pi)^4} e^{iPx}\langle 0|\mathcal{O}^\dagger(x)\mathcal{O}(0)|0\rangle = \rho(s)$ *supported at* $s > 4$

extended bootstrap variables: $\{\rho_{1,2}(x,y), \ldots, \text{Im}F(x), \rho(x)\}$

allow connection with UV theory

Current correlators from the UV gauge theory

to connect with UV gauge theory

$$
\begin{array}{c} \\ \langle \text{in}|_{P',I,\ell} \\ \langle \text{out}|_{P',I,\ell} \\ \langle 0|\mathcal{O}^\dagger_{P',I,\ell} \end{array}
\begin{array}{c} \begin{array}{ccc} |\text{in}\rangle_{P,I,\ell} & |\text{out}\rangle_{P,I,\ell} & \mathcal{O}_{P,I,\ell}|0\rangle \end{array} \\ \begin{pmatrix} 1 & S_\ell^I(s) & \mathcal{F}_\ell^I \\ S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*} \\ \mathcal{F}_\ell^{I*} & \mathcal{F}_\ell^I & \rho_\ell^I(s) \end{pmatrix} \end{array} \succeq 0 \qquad s > 4 \quad \forall \ell, I
$$

construct operators from gauge theory with desired quantum numbers

Current correlators from the UV gauge theory

to connect with UV gauge theory

$$\begin{array}{c|ccc} & |\text{in}\rangle_{P,I,\ell} & |\text{out}\rangle_{P,I,\ell} & \mathcal{O}_{P,I,\ell}|0\rangle \\ \langle\text{in}|_{P',I,\ell} & 1 & S_\ell^I(s) & \mathcal{F}_\ell^I \\ \langle\text{out}|_{P',I,\ell} & S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*} \\ \langle 0|\mathcal{O}^\dagger_{P',I,\ell} & \mathcal{F}_\ell^{I*} & \mathcal{F}_\ell^I & \rho_\ell^I(s) \end{array} \succeq 0 \qquad s > 4 \quad \forall \ell, I$$

construct operators from gauge theory with desired quantum numbers

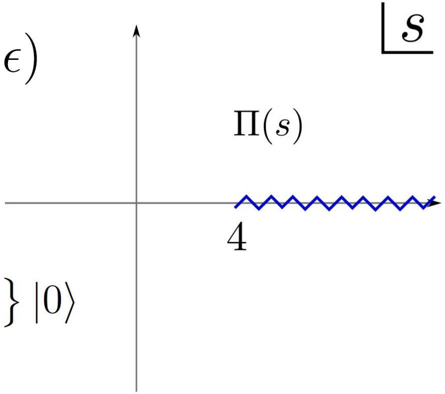
$$\rho_\ell^I(s) = 2\,\mathrm{Im}\Pi_\ell^I(x + i\epsilon)$$

e.g. vector (electromagnetic) current

$$P1 \quad : \quad j_V^\mu(x) = \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)$$

$\vdots$

$$\Pi_1^1(s) \quad = \quad i \int \frac{d^4x}{(2\pi)^4} e^{iPx} \langle 0|\hat{T}\left\{ j_\sigma^\dagger(x) j_\sigma(0) \right\}|0\rangle$$



Current correlators from the UV gauge theory

to connect with UV gauge theory

$$
\begin{array}{c|ccc}
 & |\mathrm{in}\rangle_{P,I,\ell} & |\mathrm{out}\rangle_{P,I,\ell} & \mathcal{O}_{P,I,\ell}|0\rangle \\
\hline
\langle \mathrm{in}|_{P',I,\ell} & 1 & S_\ell^I(s) & \mathcal{F}_\ell^I \\
\langle \mathrm{out}|_{P',I,\ell} & S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*} \\
\langle 0|\mathcal{O}^\dagger_{P',I,\ell} & \mathcal{F}_\ell^{I*} & \mathcal{F}_\ell^I & \rho_\ell^I(s)
\end{array} \succeq 0 \qquad s > 4 \quad \forall \ell, I
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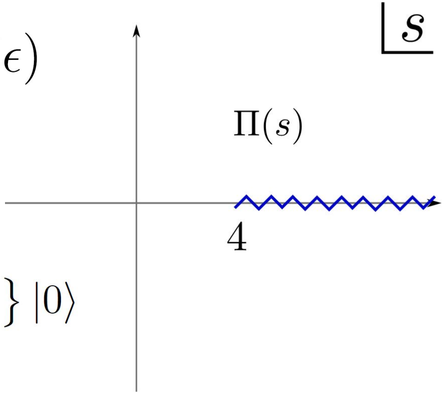

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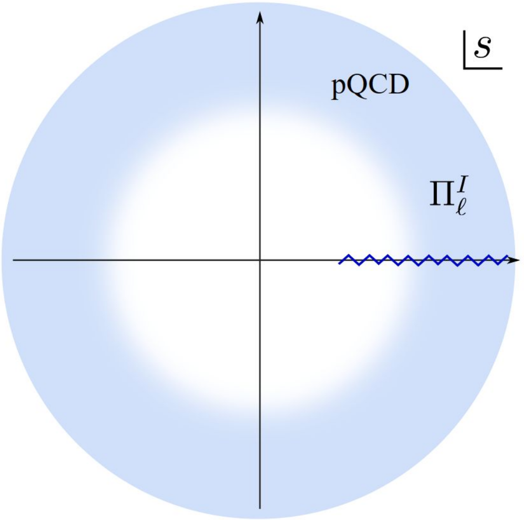
large spacelike momenta — asymptotic free region with pQCD computation

SVZ expansion

[Shifman, Vainshtein, Zakharov, 1979]

OPE: $$T\{j(x)j(0)\} = C_{\mathbb{1}}(x)\ \mathbb{1} + \sum_{\mathcal{O}} C_{\mathcal{O}}(x)\ \mathcal{O}(0)$$

$$\langle 0|T\{j(x)j(0)\}|0\rangle = C_{\mathbb{1}}(x) + C_{\bar{q}q}(x)\ \langle 0|m_q\bar{q}q|0\rangle + C_{G^2}(x)\ \langle 0|\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^{a\,\mu\nu}|0\rangle + \ldots$$



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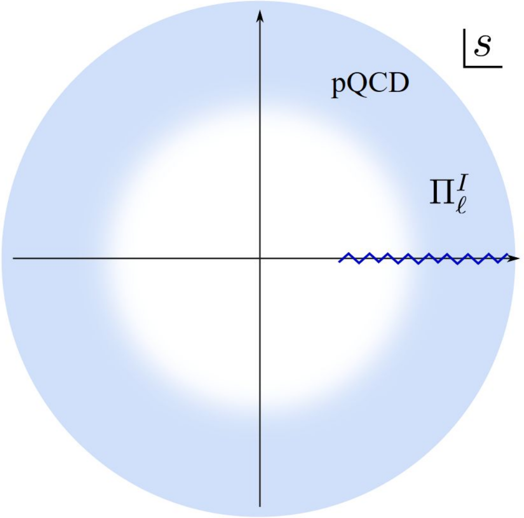
SB vacuum

quark condensate

gluon condensate

pQCD computation



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SB vacuum

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gluon condensate

pQCD computation

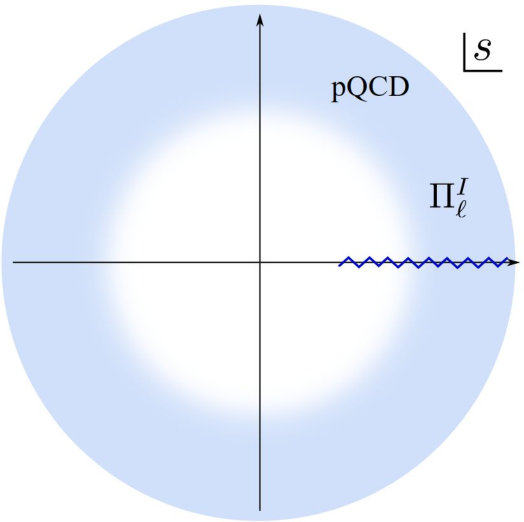
Fourier transform

large s expansion of vacuum polarization: e.g. vector current

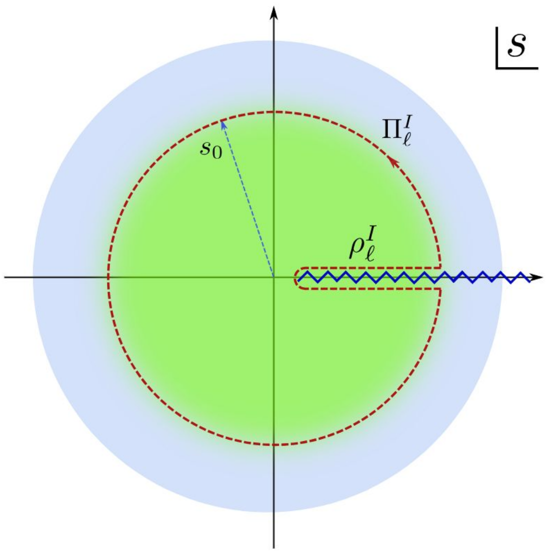
$$\Pi_1^1(s) = \frac{1}{2}\frac{1}{(2\pi)^4}\left\{-\frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)s\ln(-\frac{s}{\mu^2}) + \frac{1}{12s}\langle\frac{\alpha_s}{\pi}G^2\rangle + \frac{1}{s}\langle m_q\bar{q}q\rangle + \dots\right\}$$

$\vdots$

$N_c = 3$



Finite energy sum rule



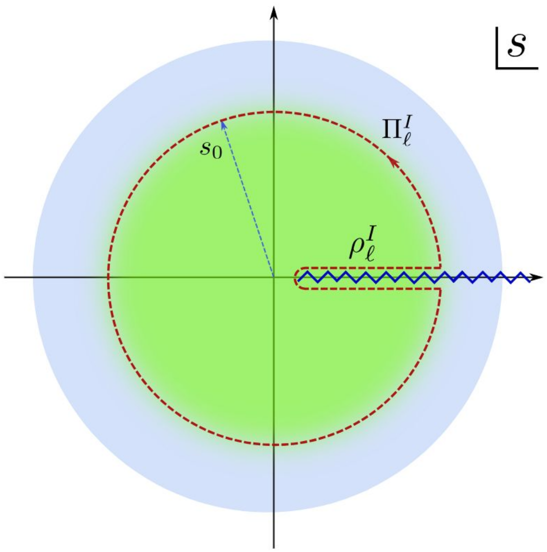
connect pQCD with bootstrap at s_0

contour integral $s^n\Pi(s)$ vanishes

SVZ

$$\int_4^{s_0} \rho(x) x^n dx = -s_0^{n+1} \int_0^{2\pi} e^{i(n+1)\varphi} \Pi(s_0 e^{i\varphi}) d\varphi$$

Finite energy sum rule



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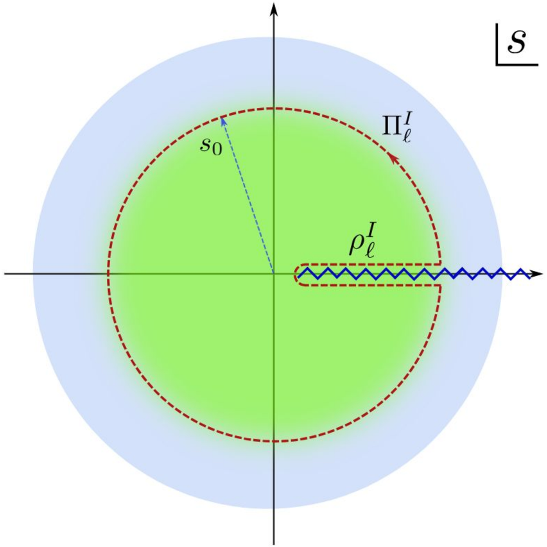
bootstrap variables ← **gauge theory information**

linear constraints

Finite energy sum rule



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SVZ

$$\int_4^{s_0} \rho(x) x^n dx = -s_0^{n+1} \int_0^{2\pi} e^{i(n+1)\varphi} \Pi(s_0 e^{i\varphi}) d\varphi$$

bootstrap variables ← **gauge theory information**

linear constraints

$$P1 \; : \; \frac{1}{s_0^{n+2}} \int_4^{s_0} \rho_1^1(x) x^n dx = \frac{1}{2(2\pi)^4} \left\{ \frac{1}{2\pi(n+2)} \left(1 + \frac{\alpha_s}{\pi}\right) - \frac{\delta_n \pi}{6 s_0^2} \langle \frac{\alpha_s}{\pi} G^2 \rangle - \frac{\delta_n 2\pi}{s_0^2} \langle m_q \bar{q} q \rangle + \ldots \right\}, \; n \geq -1$$

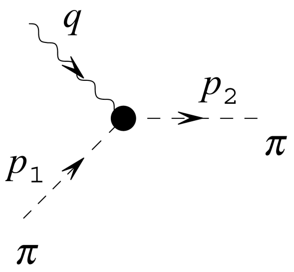
condensates suppressed at large s_0, not used as input

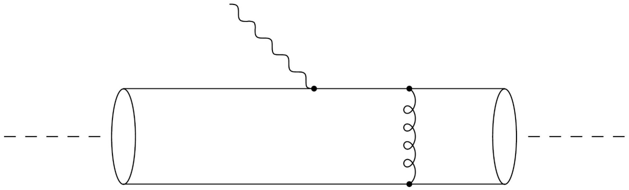
Asymptotic behavior of form factor from pQCD

perturbative QCD also controls asymptotic behavior of form factors

e.g. electromagnetic FF $\langle \pi(p_2)| J^\mu_{\text{em}}(0) |\pi(p_1)\rangle = (p_1^\mu + p_2^\mu) F_\pi(q^2)$

$$q = (p_2 - p_1)$$





$$F_\pi(s) \simeq -\frac{16\pi\alpha_s(s) f_\pi^2}{s}$$

[Lepage, Brodsky, 1979]

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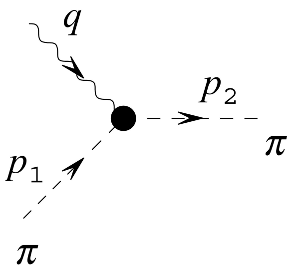

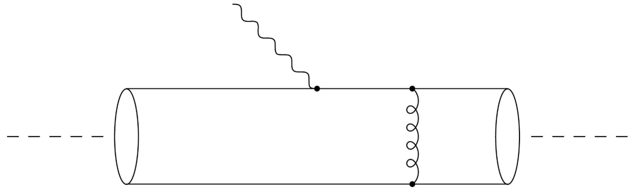

$$F_\pi(s) \simeq -\frac{16\pi\alpha_s(s) f_\pi^2}{s}$$

[Lepage, Brodsky, 1979]

evaluate to estimate

in practical numerical implementation
suffices to require smallness above $s = s_0$

$$||\mathcal{F}(s > s_0)|| \lesssim \epsilon$$

Test GTB with $N_f = 2$ $N_c = 3$: numerical input

$$s_0 = (1.2\,\mathrm{GeV})^2, \quad \alpha_s \simeq 0.41, \quad m_u \simeq 4\,\mathrm{MeV} \quad m_d \simeq 7.3\,\mathrm{MeV}$$

$$s_0 = (2\,\mathrm{GeV})^2, \quad \alpha_s \simeq 0.31, \quad m_u \simeq 3.6\,\mathrm{MeV} \quad m_d \simeq 6.5\,\mathrm{MeV}$$

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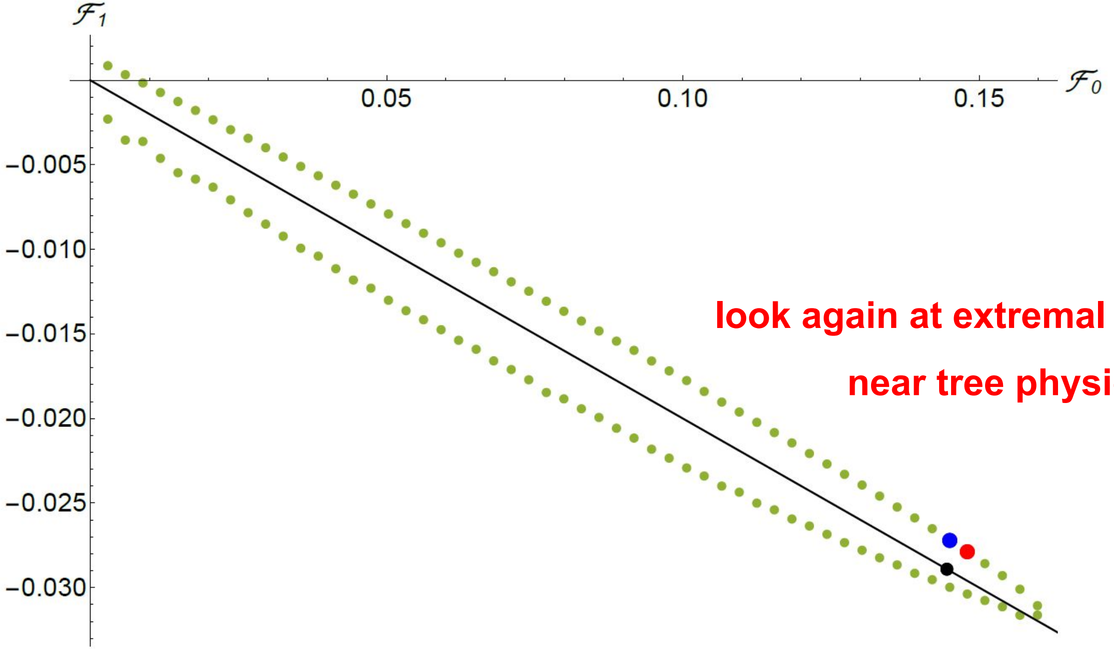
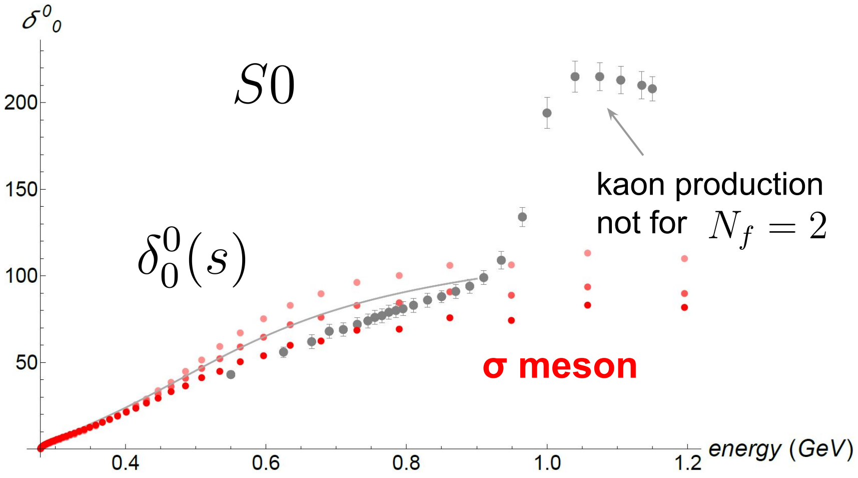
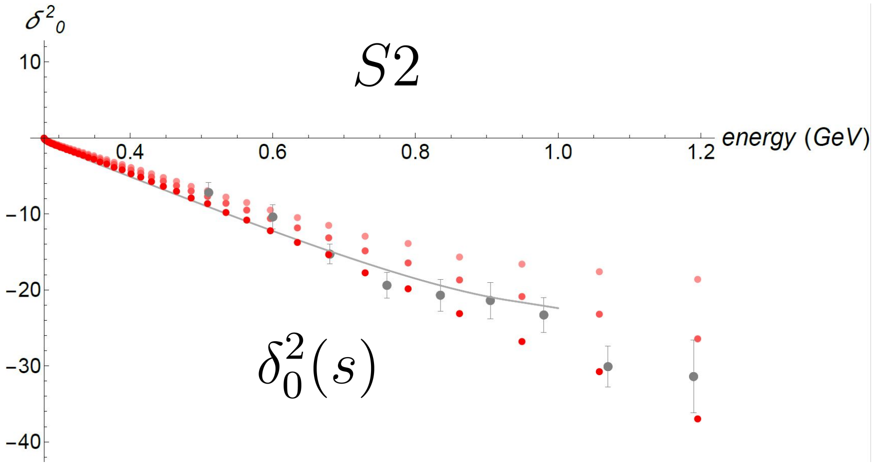


look again at extremal numerical amplitude:

near tree physical f_π point

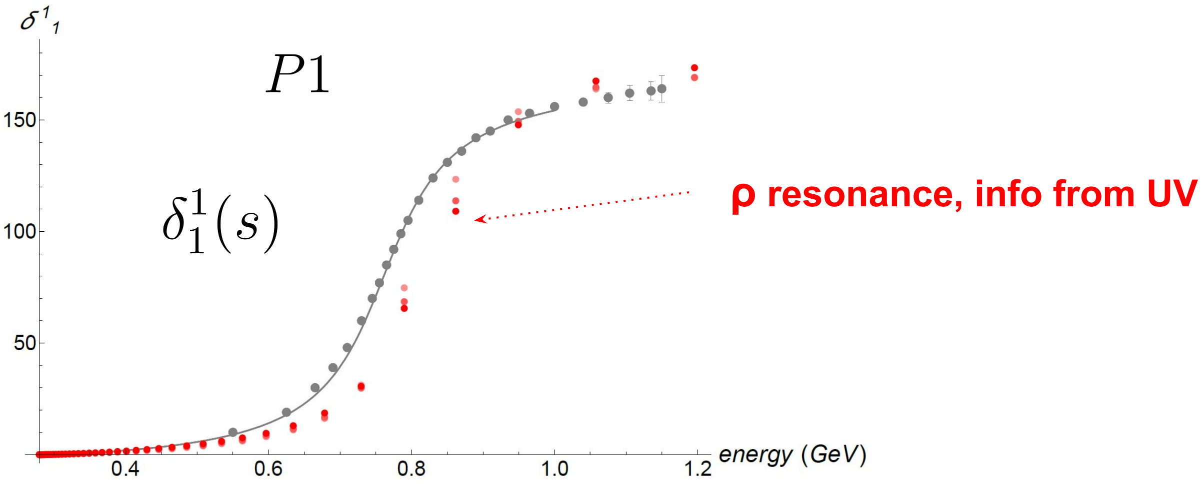




Gauge Theory Bootstrap

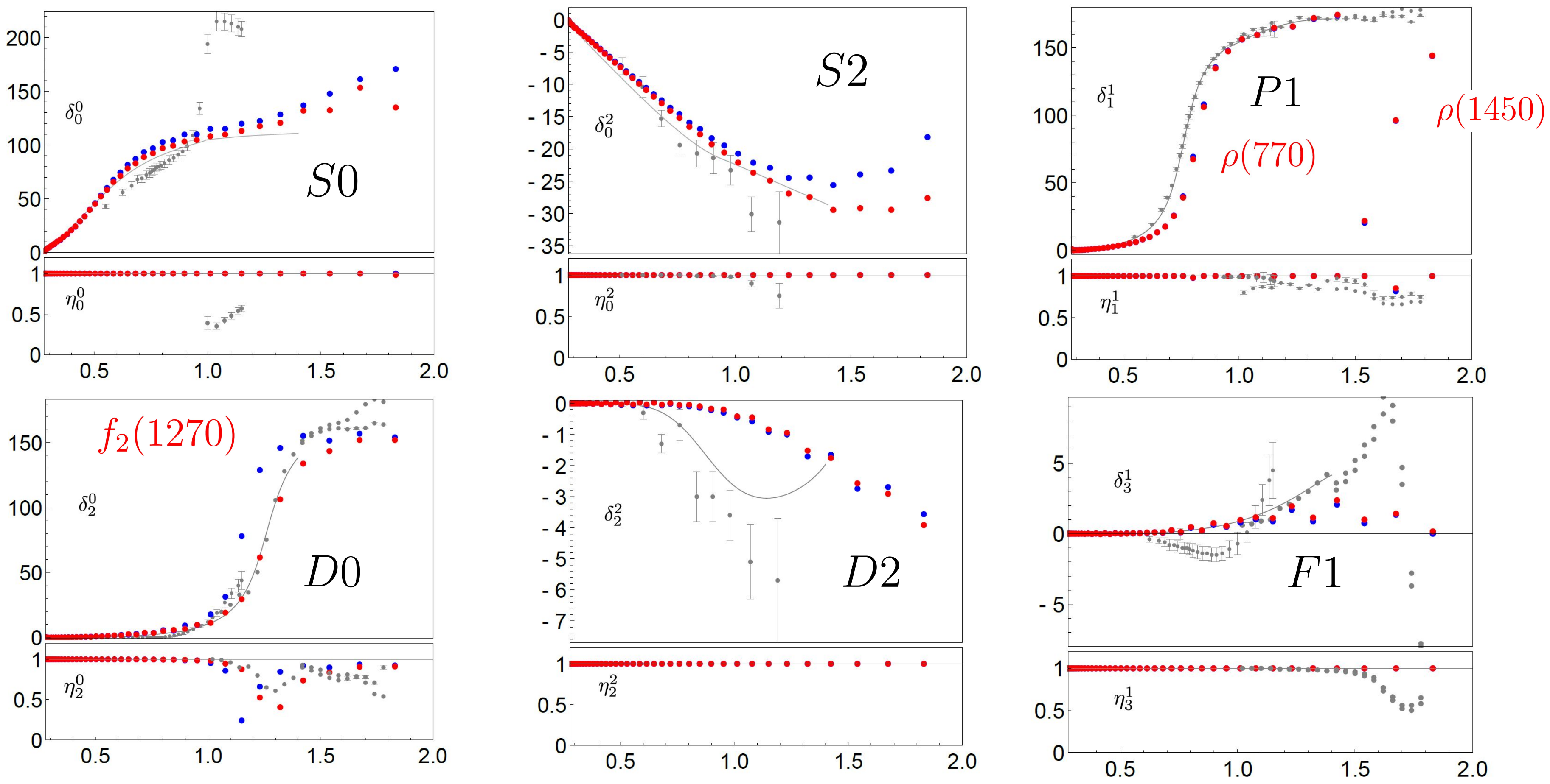
phase shifts up to 1.2 GeV

experimental data (gray dots)
[Protopopescu et al, 1973]
[Losty et al, 1974]
pheno fit (gray line)
[Pelaez, Yndurain, 2005]



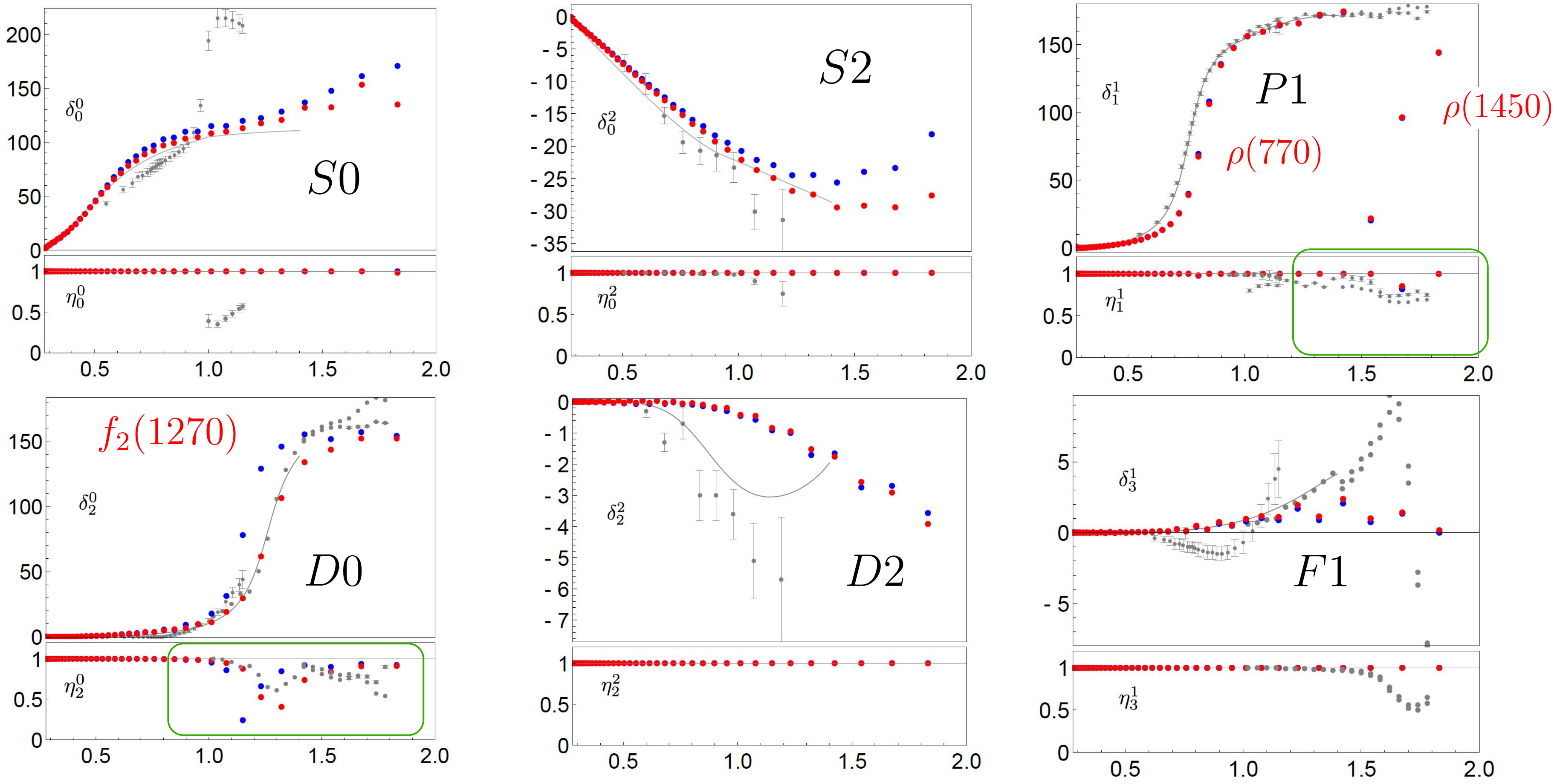
Gauge Theory Bootstrap

phase shifts up to 2 GeV



Gauge Theory Bootstrap

phase shifts up to 2 GeV



Low energy parameters: threshold expansion

scattering lengths and effective range parameters

$$\mathrm{Re} f_\ell^I(s) \overset{k\to 0}{\simeq} \frac{2m_\pi}{\pi} k^{2\ell}\left(a_\ell^I + b_\ell^I k^2 + \dots\right) \qquad k = \frac{\sqrt{s-4m_\pi^2}}{2}$$

	W	**GTB**	CGL	PY
$a_0^{(0)}$	0.16	0.178, 0.182	0.220 ± 0.005	0.230 ± 0.010
$a_0^{(2)}$	-0.046	-0.0369, -0.0378	-0.0444 ± 0.0010	-0.0422 ± 0.0022
$b_0^{(0)}$	0.18	0.287, 0.290	0.280 ± 0.001	0.268 ± 0.010
$b_0^{(2)}$	-0.092	-0.064, -0.066	-0.080 ± 0.001	-0.071 ± 0.004
$a_1^{(1)}$	31	28.0, 28.4	37.0 ± 0.13	$38.1 \pm 1.4\ (\times 10^{-3})$
$b_1^{(1)}$	0	2.86, 3.37	5.67 ± 0.13	$4.75 \pm 0.16\ (\times 10^{-3})$
$a_2^{(0)}$	0	12.6, 12.3	17.5 ± 0.3	$18.0 \pm 0.2\ (\times 10^{-4})$
$a_2^{(2)}$	0	2.87, 2.81	1.70 ± 0.13	$2.2 \pm 0.2\ (\times 10^{-4})$

Low energy parameters: pion charge radii

threshold expansion of the form factors:

scalar form factor: $$F_0^0(s) = F_0^0(0)\left[1 + \frac{1}{6}s\langle r^2\rangle_S^\pi + \ldots\right]$$

vector form factor: $$F_1^1(s) = 1 + \frac{1}{6}s\langle r^2\rangle_V^\pi + \ldots$$

	GTB	Exp. fits
$\langle r^2\rangle_S^\pi$	0.64, 0.61	$0.61 \pm 0.04\,\mathrm{fm}^2$
$\langle r^2\rangle_V^\pi$	0.388, 0.381	$0.439 \pm 0.008\,\mathrm{fm}^2$

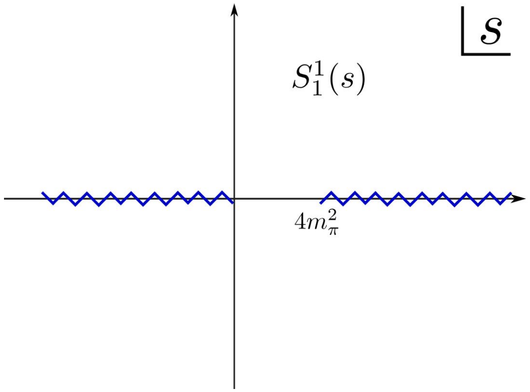
Low energy parameters: chiral Lagrangian coefficients

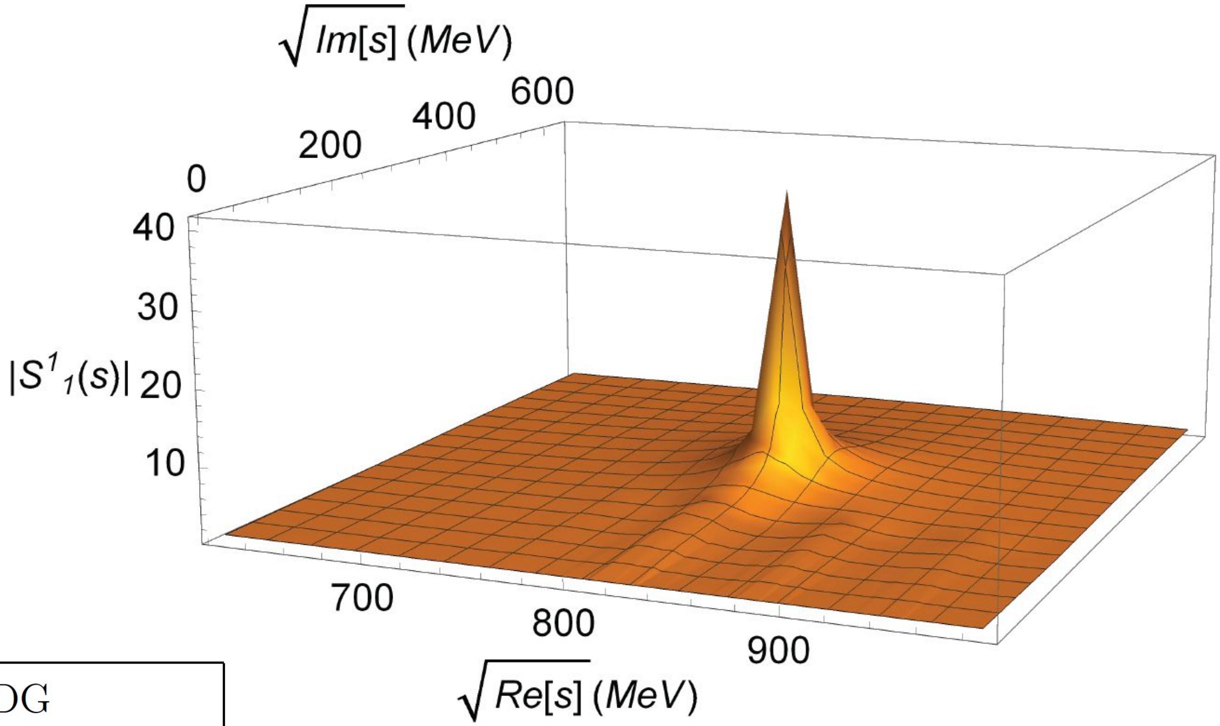
calculate the chiral Lagrangian coefficients

$$
\begin{aligned}
a_{D0} &= \frac{1}{1440\pi^3 f_\pi^4}\left\{\bar{l}_1 + 4\bar{l}_2 - \frac{53}{8}\right\} + \ldots \\
a_{D2} &= \frac{1}{1440\pi^3 f_\pi^4}\left\{\bar{l}_1 + \bar{l}_2 - \frac{103}{40}\right\} + \ldots \\
F_0(s) &= 1 + \frac{s}{16\pi^2 f_\pi^2}\left(\bar{l}_4 - \frac{13}{12}\right) + \ldots \\
F_1(s) &= 1 + \frac{s}{96\pi^2 f_\pi^2}(\bar{l}_6 - 1) + \ldots
\end{aligned}
$$

	GTB	GL	Bij	CGL
$\bar{l}_1$	0.92, 0.93	-2.3 ± 3.7	-1.7 ± 1.0	-0.4 ± 0.6
$\bar{l}_2$	4.1, 4.0	6.0 ± 1.3	6.1 ± 0.5	4.3 ± 0.1
$\bar{l}_4$	4.7, 4.6	4.3 ± 0.9	4.4 ± 0.3	4.4 ± 0.2
$\bar{l}_6$	14.3, 14.1	16.5 ± 1.1	$16.0 \pm 0.5 \pm 0.7$	

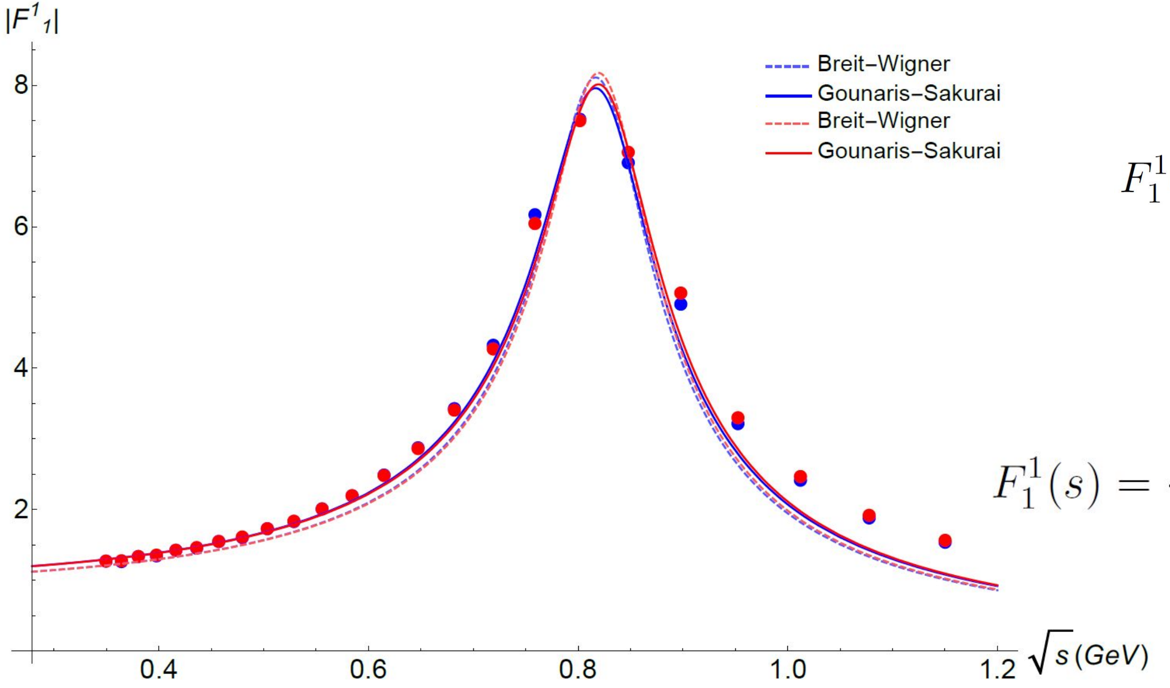
ρ(770) meson as pole on the second sheet of $S_1^1(s)$





	GTB	PDG
$\mathrm{Re}(\sqrt{s_\rho})$	829, 832	$761 - 765 \pm 0.23$ MeV
$\mathrm{Im}(\sqrt{s_\rho})$	63, 64	$71 - 74 \pm 0.8$ MeV

Vector (electromagnetic) form factor and *ρ(770)* meson



Breit-Wigner form

$$F_1^1(s) = -\frac{m_\rho^2}{s - m_\rho^2 + i m_\rho \Gamma_\rho \theta(s - 4m_\pi^2)}$$

Gounaris-Sakurai form

$$F_1^1(s) = \frac{m_\rho^2[1 + d\Gamma_\rho/m_\rho]}{(m_\rho^2 - s) - i m_\rho \Gamma_\rho (q/q_\rho)^3 (m_\rho/\sqrt{s})}$$

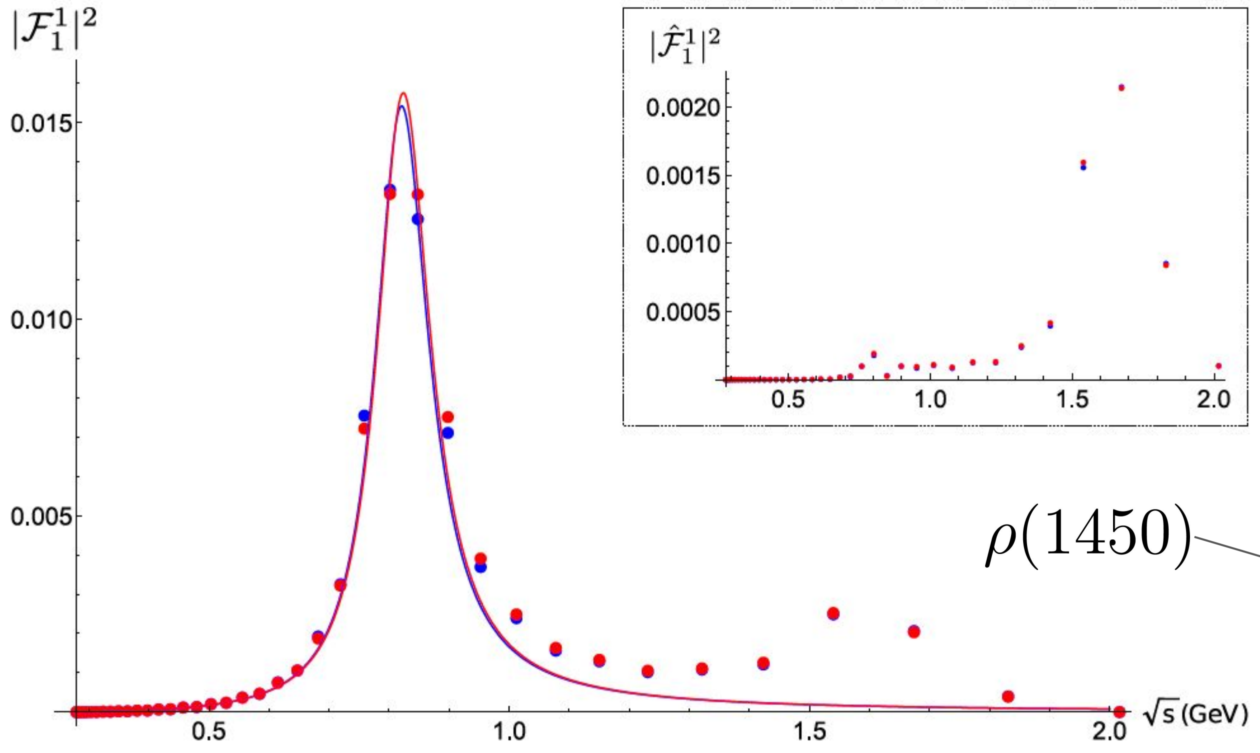
couplings $\quad \Gamma_\rho = g_{\rho\pi\pi}^2 \frac{m_\rho}{48\pi}\left[1 - \frac{4m_\pi^2}{m_\rho^2}\right]^{\frac{3}{2}}$

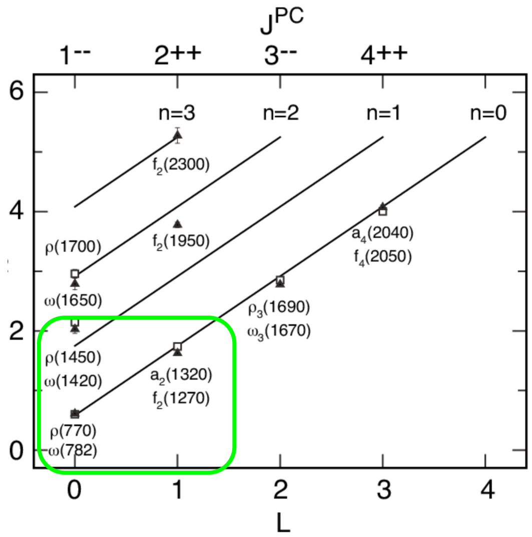
$g_{\rho\pi\pi} = 4.9, \quad 4.9$

$g_{\rho\pi\pi} = 6$

	GTB	PDG
m_ρ	836, 839	775 ± 0.23 MeV
Γ_ρ	111, 111	149.1 ± 0.8 MeV

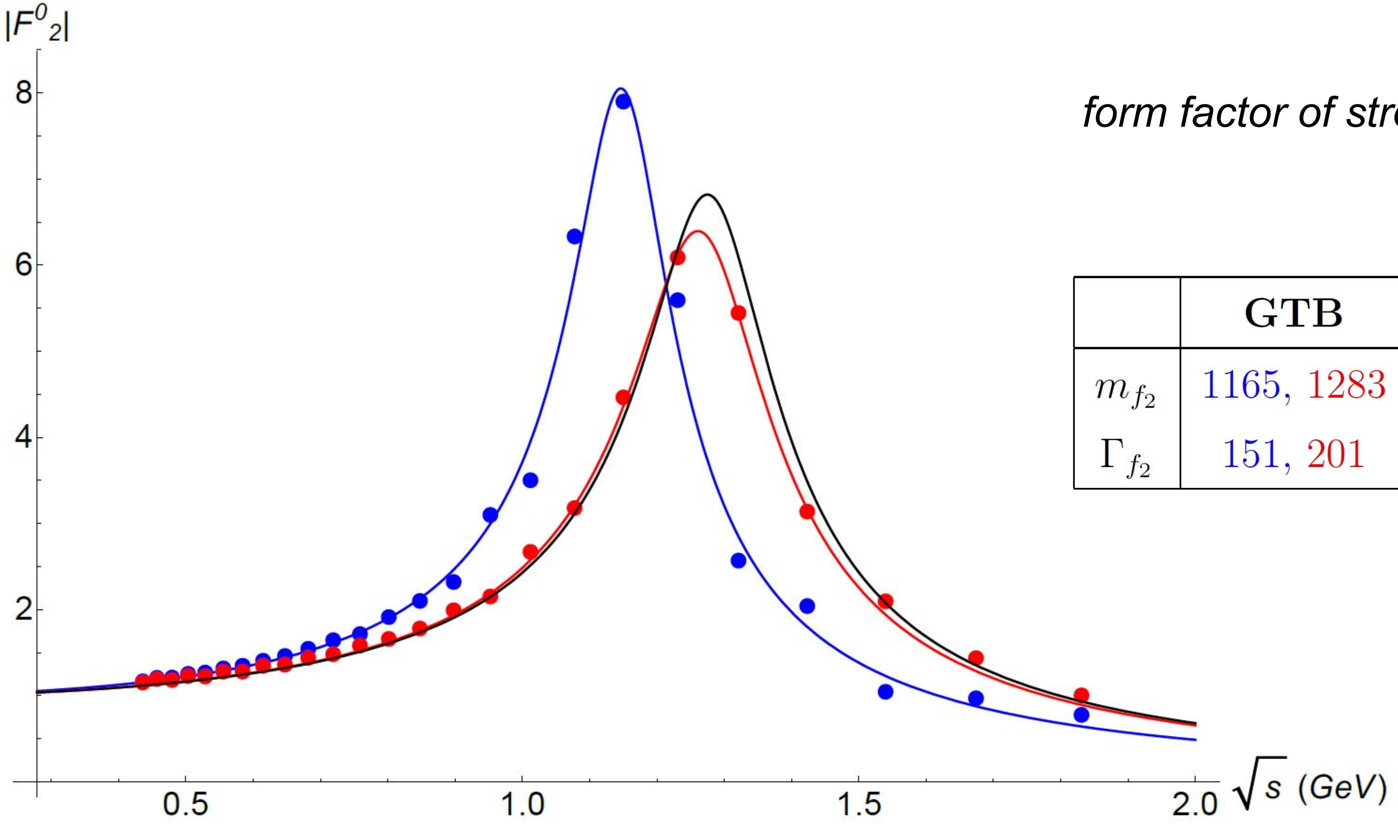
Vector (electromagnetic) form factor and *ρ(770)* meson





$\rho(1450)$ — leading meson of the second Regge trajectory

Gravitational form factor and f_2 meson



form factor of stress energy tensor

	GTB	PDG
m_{f_2}	1165, 1283	1275.4 ± 0.6 MeV
Γ_{f_2}	151, 201	186.6 ± 2.3 MeV

Conclusions

- Gauge Theory Bootstrap: theoretical/numerical computation

  only input: $N_c \; N_f \; m_q \quad \alpha_s \qquad\qquad f_\pi \quad m_\pi$

  strongly coupled low energy physics of asymptotically free gauge theories

Conclusions

- Gauge Theory Bootstrap: theoretical/numerical computation

  only input: N_c N_f m_q α_s f_π m_π

  strongly coupled low energy physics of asymptotically free gauge theories

- Numerical test with $N_f = 2$ $N_c = 3$ find good agreement with experiments

  Results suggest: we are on the right track for solving QCD (gauge theories)

Conclusions

- Gauge Theory Bootstrap: theoretical/numerical computation

  only input: $N_c \; N_f \; m_q \quad \alpha_s \qquad\qquad f_\pi \quad m_\pi$

  strongly coupled low energy physics of asymptotically free gauge theories

- Numerical test with $N_f = 2 \;\; N_c = 3$ find good agreement with experiments

  Results suggest: we are on the right track for solving QCD (gauge theories)

- Fast machine precision numerics (~20min on average laptop)

  need refinement/improvement to be more robust

Ancillary files (details)**:**
- GTB_numerics.m
- GTB_numerics.nb

Prospects

- many future explorations in this framework:

change gauge theory parameters → strongly coupled low energy dynamics

gauge theory vs. chiral dynamics (e.g. S0 vs. P1)

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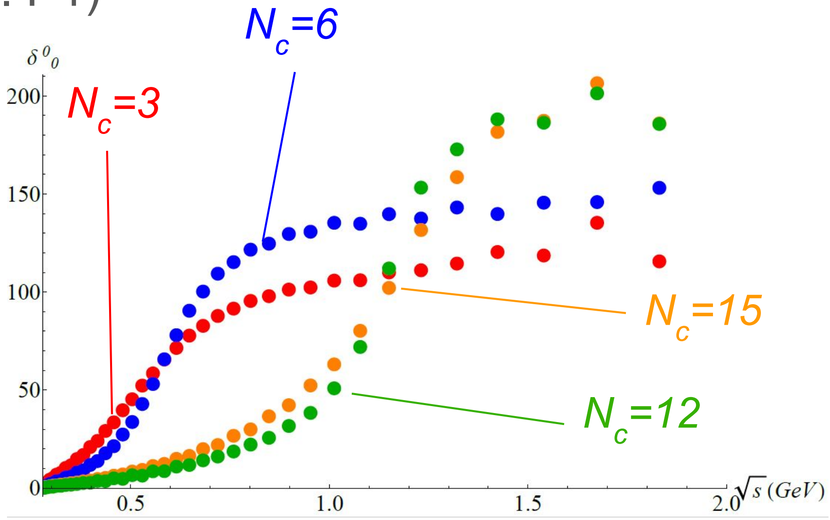
[WIP with Kruczenski]

very preliminary results:



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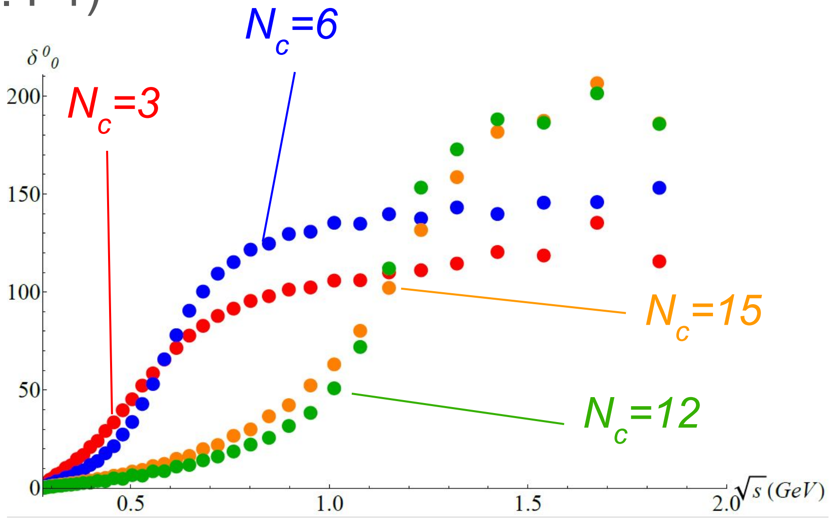

- analytic understanding?

convex geometry of ACU+pQCD → strongly coupled amplitudes of physical theory

Thank you!