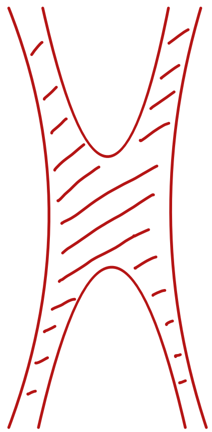
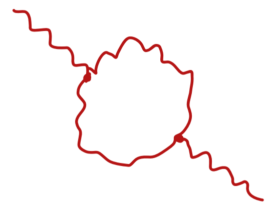
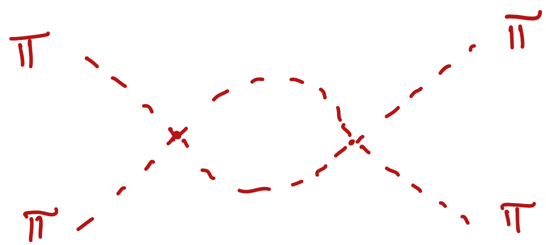
Real World Amplitudes from Curves on Surfaces





Carolina Figueiredo



Amplitudes 2024
IAS
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w/ N. Arkani-Hamed
Q. Cao, J. Dong
S. He

Outline...

I. Curve Integral Formalism for "Stringy" $\mathrm{Tr}\,\phi^3$.

[Kinematics as curves on surfaces]

II. Revealing Qualitatively New Features

* Kinematic connection between different ths.: $\mathrm{Tr}\,\phi^3 \leftrightarrow \mathrm{NLSM} \leftrightarrow \mathrm{YM}$

* Hidden Factorizations away from Poles + Zeros.

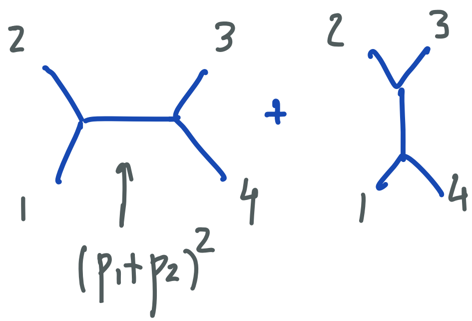
* "Perfect" Integrands for real world amplitudes.

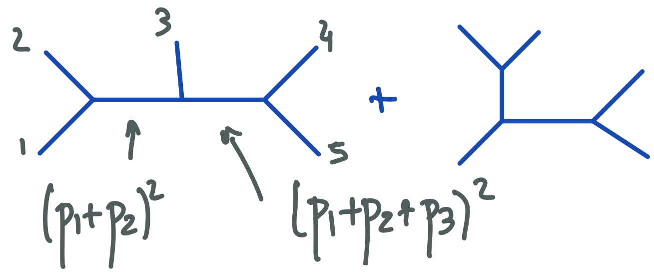
$\mathrm{Tr}\,\varphi^3$ Theory and the Curve Integral Formalism

$$\mathcal{L}_{\mathrm{Tr}\varphi^3} = \frac{1}{2}\mathrm{Tr}(\partial\varphi)^2 + \frac{g}{3}\mathrm{Tr}(\varphi^3)$$

$\varphi_I{}^J$ $N \times N$ matrix.

$$A^{\mathrm{Tr}\varphi^3} = \sum_{\mathcal{D},\ \text{diagrams}} \left(\prod_{P \in \mathcal{D}} \frac{1}{P^2} \right)$$

$A_4 =$  $+ \dots$

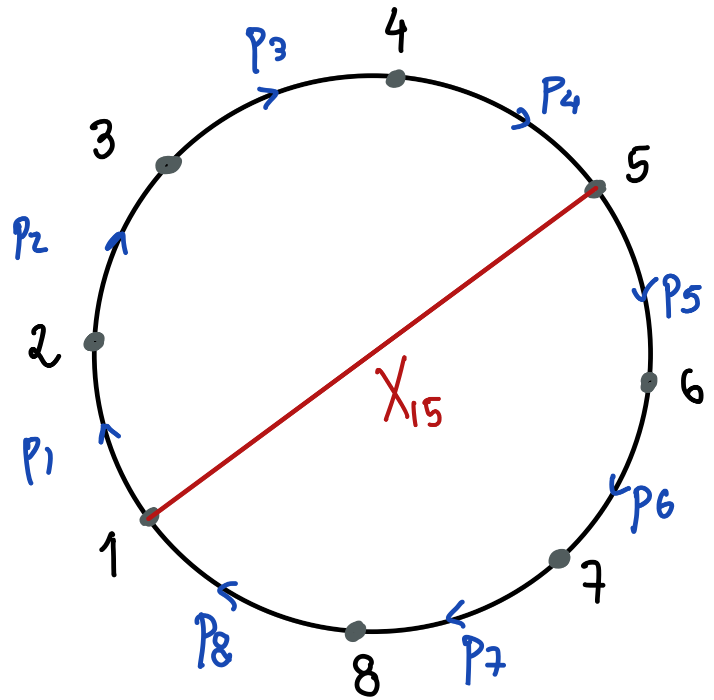
$A_5 =$  $+ \dots$

$$\text{Singularities} \equiv (p_i + p_{i+1} + \dots + p_{j-1})^2 = 0$$

[Kinematics and curves on Surfaces]

Color-ordered amplitude:

Singularities $\equiv$ $(p_i + p_{i+1} + \cdots + p_{j-1})^2 = 0$



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Curves, $\mathcal{C}$, on the Surface. $\mathcal{S}$

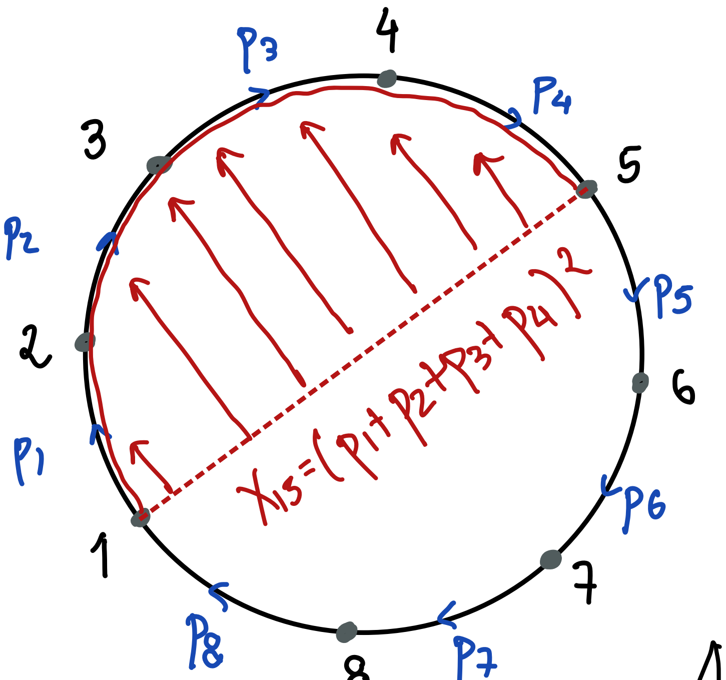
$X_{\mathcal{C}}$ = Kinematics associated to the curve read off by Homology!

|||

$$X_{i,j} = (p_i + p_{i+1} + \cdots + p_{j-1})^2$$

[Momentum $\leftrightarrow$ Homology]

Color-ordered amplitude: Singularities $\equiv$ $(p_i + p_{i+1} + \cdots + p_{j-1})^2 = 0$



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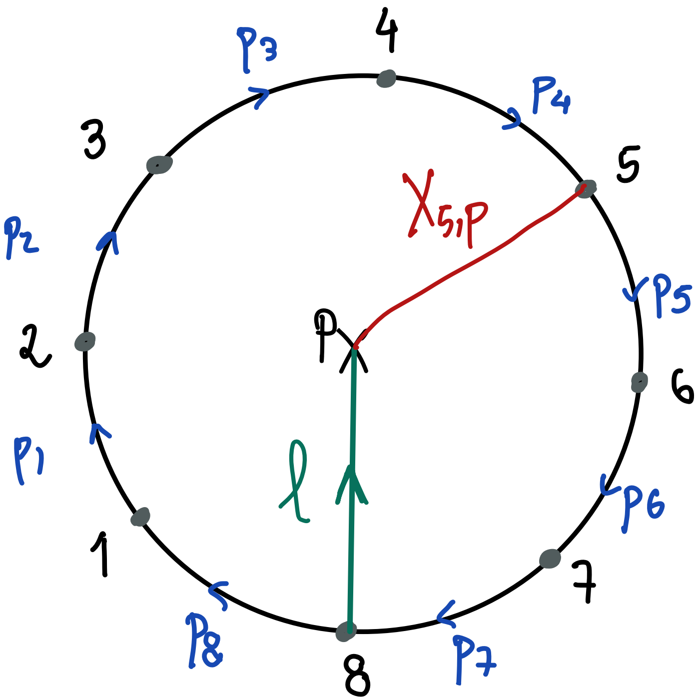
Curves, $\mathcal{C}$, on the Surface. $\mathcal{S}$

$X_{\mathcal{C}}$ = Kinematics associated to the curve read off by Homology!

$\mathcal{A}^{\mathrm{Tr}\,\phi^3}\left[X_{\mathcal{C}} \equiv X_{i,j}\right]$ manifest singularities. ✓

[Momentum $\leftrightarrow$ Homology]

Loop-level : $\mathcal{S}$ punctures + more interesting topology.

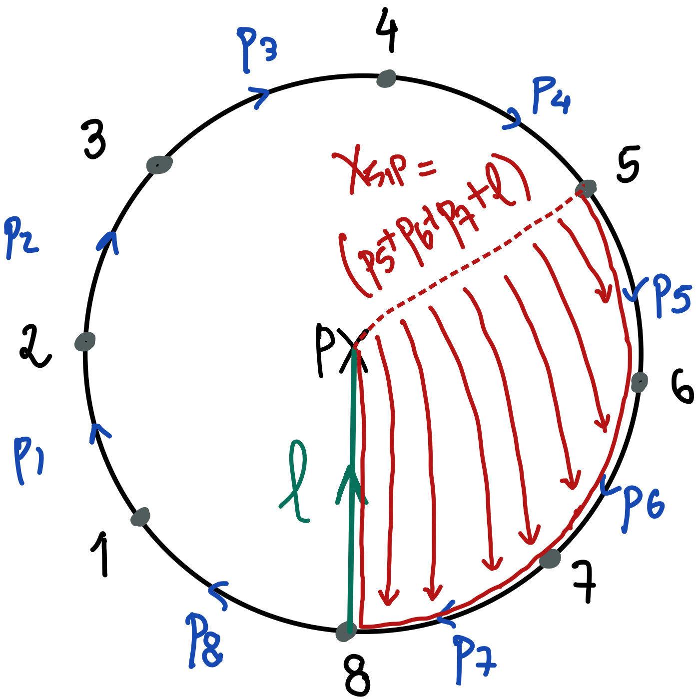


To each curve $\mathcal{C}$ on $\mathcal{S}$ associate:

$X_{\mathcal{C}}$ = Homology!

[Momentum $\leftrightarrow$ Homology]

Loop-level : $\mathcal{S}$ punctures + more interesting topology.



To each curve $\mathcal{C}$ on $\mathcal{S}$ associate:

$$X_{\mathcal{C}} = \text{Homology!}$$

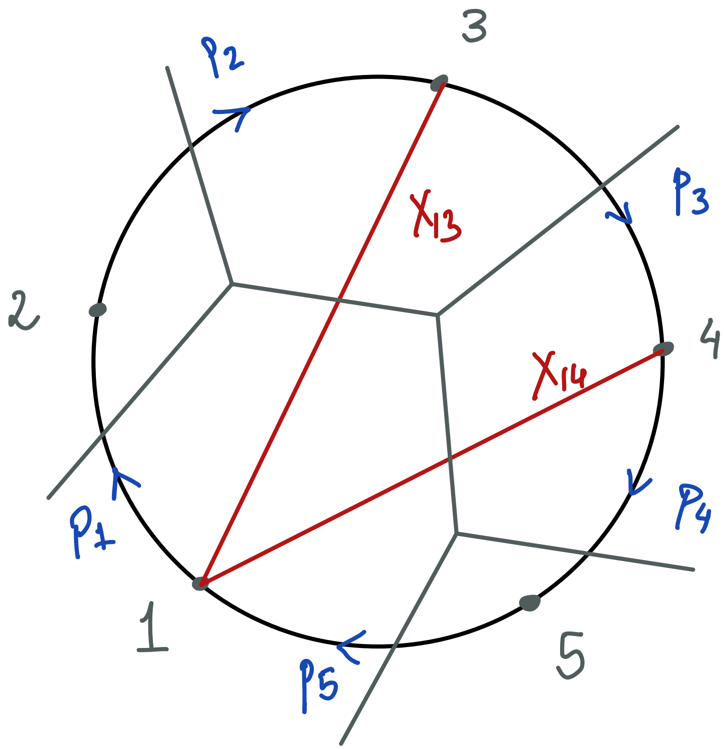
$$|||$$

Propagators in Feynman Diagrams.

$$A^{\mathrm{Tr}\varphi^3}\left[X_{\mathcal{C}} \equiv X_{i,j}\right]$$ manifest singularities. ✓

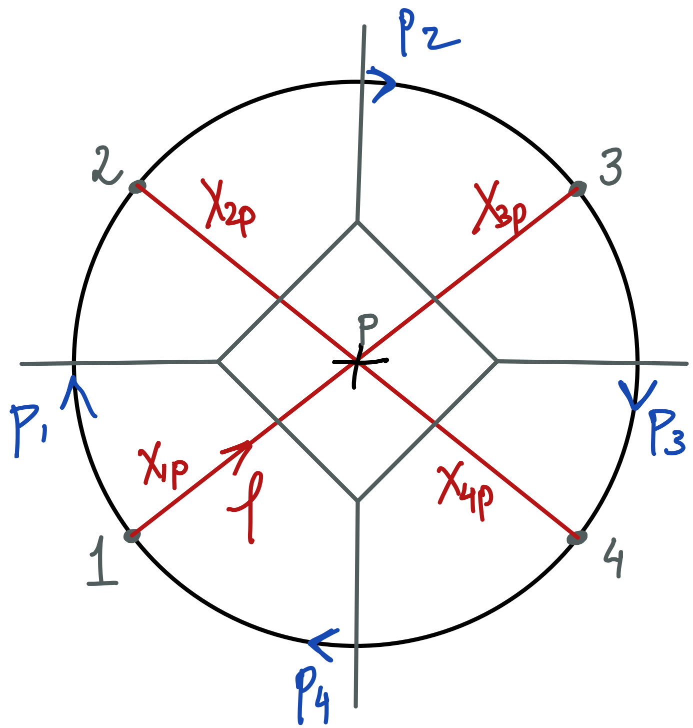
[Feynman Diagrams as Triangulations of S]

5-point Tree



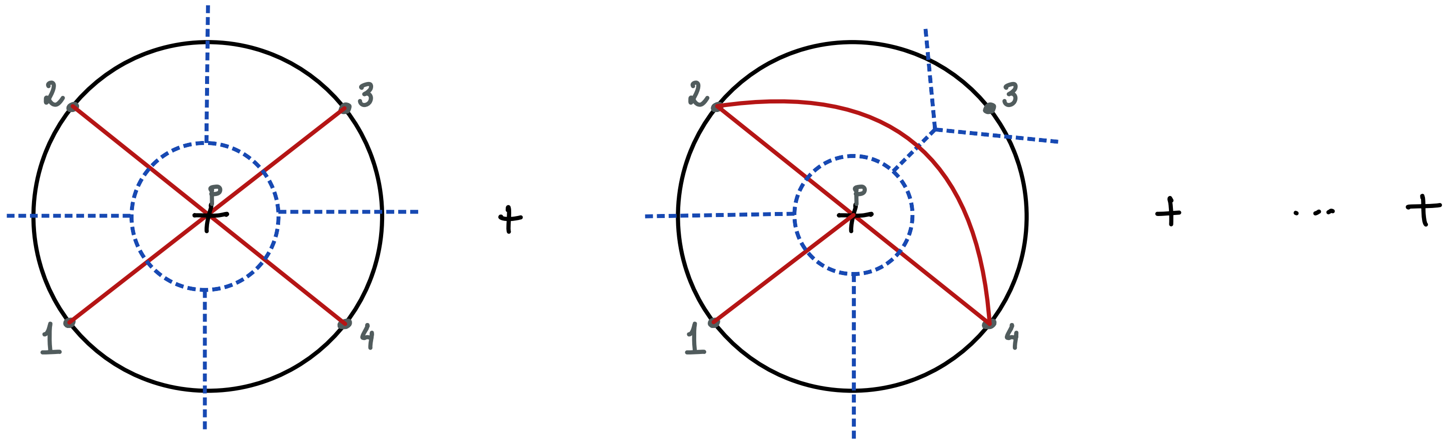
$$\frac{1}{X_{1,3}\,X_{1,4}}$$

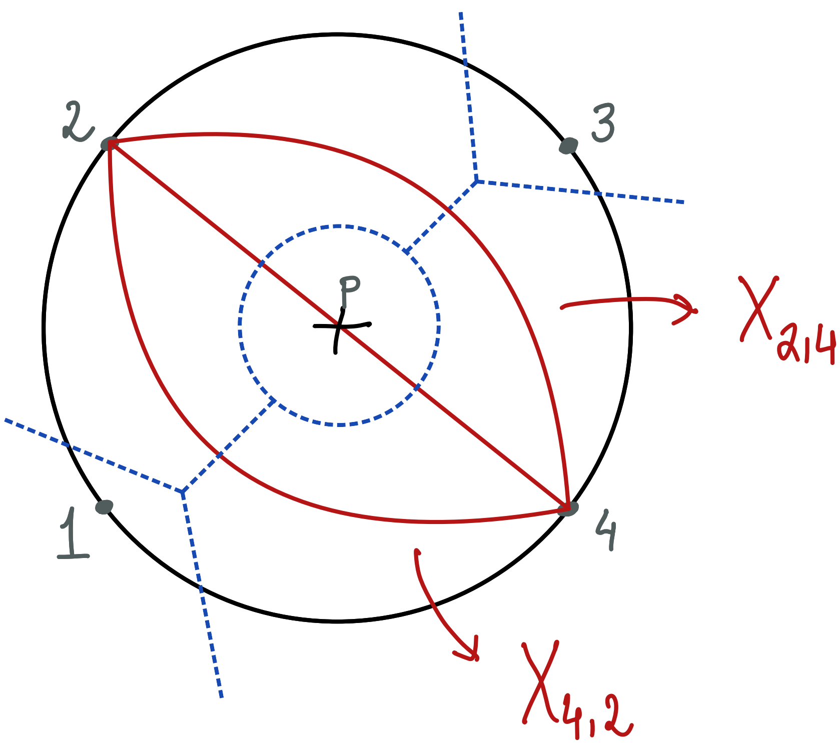
4-point 1-loop



$$\frac{1}{X_{1,p}\,X_{2,p}\,X_{3,p}\,X_{4,p}}$$

[Amplitude $\leftrightarrow$ Σ Triangulations of S]





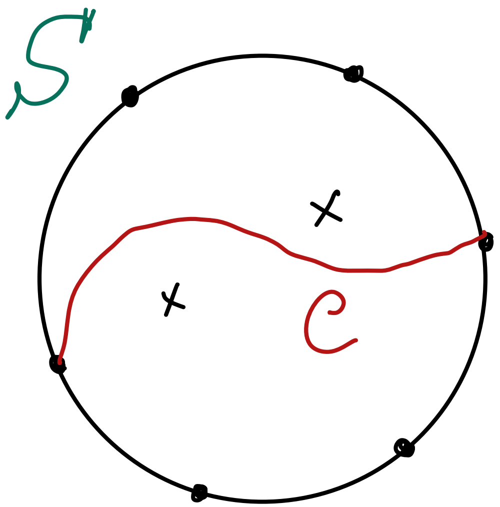
Keep Curves up to **Homotopy**!

$X_{2,4} \neq X_{4,2}$ as CURVES

Generalizing away from Momenta $\leftrightarrow$ Homology.

$(X_{24} = X_{42})$.

Curve Integral Formalism



u-variables:

$$u_C[y] = \frac{F^{1,2}(y)\, F^{2,1}(y)}{F^{1,1}(y)\, F^{2,2}(y)}$$

Positive coordinates y

Counting Problem associated to C on S.

$$A = \int_0^{+\infty} \underbrace{\left(\prod \frac{dy}{y}\right)}_{\text{dlog form}} \times \prod_{C\in S} u_C[y]^{\alpha' X_C}$$

Product over all curves on S

[N. Arkani-Hamed, H. Frost, G. Salvatori, P-G. Plamondon, H. Thomas 23']

Curve Integral Formalism: Tree-level.

$$A_n^{\text{tree}} = \underbrace{\int \frac{d^n z}{SL(2,\mathbb{R})} \times \frac{1}{z_{12}\, z_{23} \cdots z_{n1}}}_{\prod_{i=1}^{n-3} \frac{dy_i}{y_i}} \times \underbrace{\prod_{i<j} z_{ij}^{-2\alpha' p_i \cdot p_j}}_{\prod_{i<j} u_{ij}^{\alpha' X_{ij}}} \qquad [\text{Worldsheet}]$$

$$= \prod_{i=1}^{n-3} \frac{dy_i}{y_i} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}$$

Curve Integral Formalism: Tree-level.

$$A_n^{\text{tree}} = \underbrace{\int \frac{d^n z}{SL(2,\mathbb{R})} \times \frac{1}{z_{12}\, z_{23} \cdots z_{n1}}}_{= \prod_{i=1}^{n-3} \frac{dy_i}{y_i}} \times \underbrace{\prod_{i<j} z_{ij}^{-2\alpha' p_i \cdot p_j}}_{\prod_{i<j} u_{ij}^{\alpha' X_{ij}}} \qquad \text{[Worldsheet]}$$

$$u_{ij} = \frac{z_{i-1,j}\, z_{i,j-1}}{z_{i,j}\, z_{i-1,j-1}}$$

Manifest all poles

$[u_{ij} \to 0]$

Curve Integral Formalism: Tree-level.

$$A_n^{\text{tree}} = \int \frac{d^n z}{SL(2,\mathbb{R})} \times \frac{1}{z_{12}\, z_{23} \cdots z_{n1}} \times \prod_{i<j} z_{ij}^{-2\alpha' p_i \cdot p_j} \quad [\text{Worldsheet}]$$

$$\underbrace{\frac{d^n z}{SL(2,\mathbb{R})} \times \frac{1}{z_{12}\, z_{23} \cdots z_{n1}}}_{=\ \prod_{i=1}^{n-3} \frac{dy_i}{y_i}} \times \underbrace{\prod_{i<j} z_{ij}^{-2\alpha' p_i \cdot p_j}}_{=\ \prod_{i<j} u_{ij}^{\alpha' X_{ij}}}$$

$$u_{ij} = \frac{z_{i-1,j}\, z_{i,j-1}}{z_{i,j}\, z_{i-1,j-1}}$$

$$\underbrace{\prod_{i=1}^{n-3} \frac{dy_i}{y_i}}$$

Blows up ALL Singularities

Manifest all poles $[u_{ij} \to 0]$

Curve Integral Formalism: Tree-level.

$$A_n^{\text{tree}} = \underbrace{\int \frac{d^n z}{SL(2,\mathbb{R})} \times \frac{1}{z_{12}\, z_{23} \cdots z_{n1}}}_{\displaystyle \overset{\shortparallel}{\underbrace{\prod_{i=1}^{n-3} \frac{dy_i}{y_i}}_{\text{Blows up ALL Singularities}}}} \times \underbrace{\prod_{i<j} z_{ij}^{-2\alpha' p_i \cdot p_j}}_{\displaystyle \prod_{i<j} u_{ij}^{\alpha' X_{ij}}} \qquad [\text{Worldsheet}]$$

$$u_{ij} = \frac{z_{i-1,j}\, z_{i,j-1}}{z_{i,j}\, z_{i-1,j-1}}$$

$\alpha' X_{ij}$ $\downarrow$ Manifest all poles $[u_{ij} \to 0]$

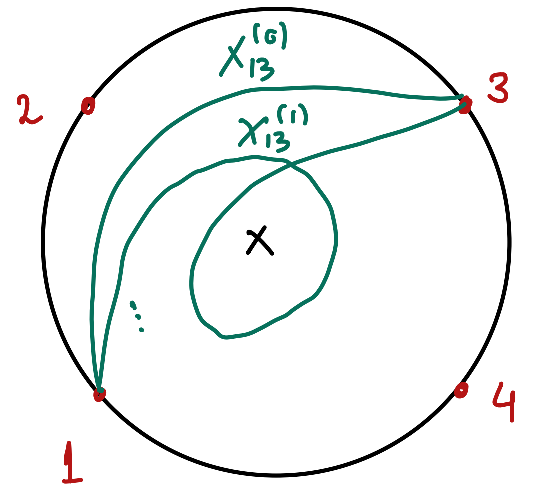
* No Gauge Redundancy / Fixing

* Trivial to extract Field Theory Limit.

Curve Integral Formalism: Loop-level.

$$A_n = \int_0^{+\infty} \underbrace{\prod \frac{dy_p}{y_p}}_{\text{dlog form}} \times \prod_{C\in S} u_C[y_p]^{\alpha' X_C} \longrightarrow$$

Now ∞ many Curves!



∞ self intersection

$$\prod_{C^{(0)}\in S} u_C^{\alpha' X_{C^{(0)}}} \times \underbrace{\prod_{C^{(q)}\in S} u_C^{\alpha' X_{C^{(q)}}}}_{\infty \text{ product}}$$

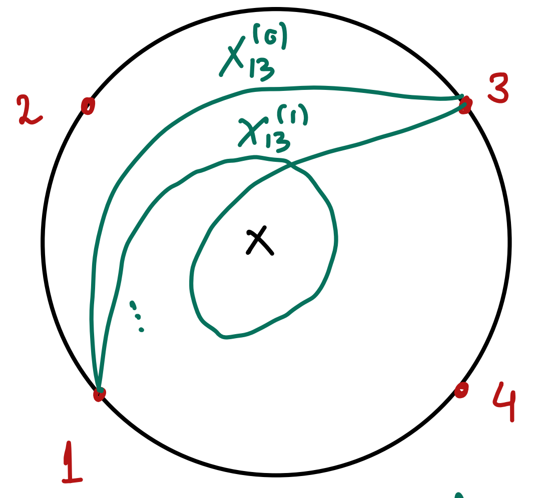
$$\equiv$$

1-loop Bosonic String.

Curve Integral Formalism: Loop-level.

$$A_n = \int_0^{+\infty} \underbrace{\prod \frac{dy_p}{y_p}}_{\text{dlog form}} \times \underbrace{\prod_{C\in S} u_C[y_p]^{\alpha' X_C}}$$

Now ∞ many Curves!



∞ self intersection

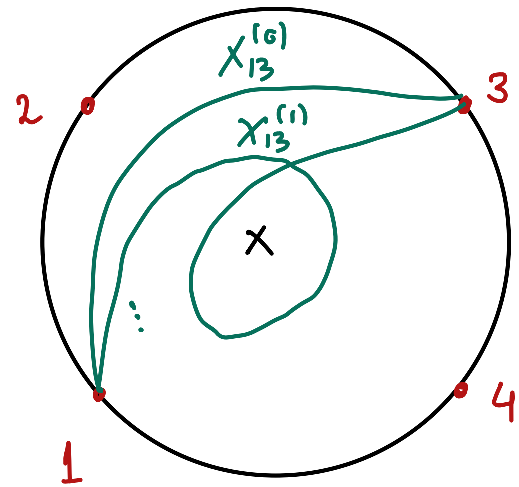
$$\prod_{C^{(0)}\in S} u_C^{\alpha' X_{C^{(0)}}} \times \underbrace{\cancel{\prod_{C^{(q)}\in S} u_C^{\alpha' X_{C^{(q)}}}}}_{\infty \text{ product}}$$

a) Low-energies:

$(\mathrm{Tr}\,\varphi^3)$

Curve Integral Formalism: Loop-level.

$$A_n = \int_0^{+\infty} \underbrace{\prod \frac{dy_p}{y_p}}_{\text{dlog form}} \times \underbrace{\prod_{C \in S'} u_C[y_p]^{\alpha' X_C}}$$ → Now ∞ many Curves!



∞ self intersection

$$\prod_{C^{(0)} \in S} u_C^{\alpha' X_{C^{(0)}}} \times \cancel{\underbrace{\prod_{C^{(q)} \in S'} u_C^{\alpha' X_{C^{(q)}}}}_{\infty \text{ product}}}$$

@ low-energies: $(\mathrm{Tr}\,\varphi^3)$

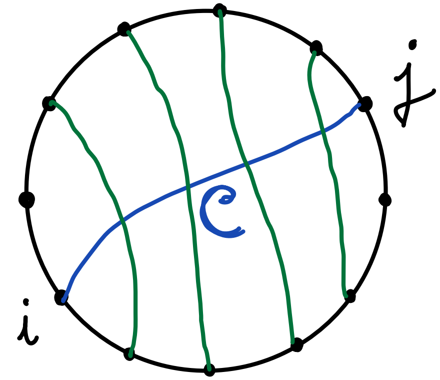
$$\int_0^{+\infty} \prod \frac{dy_p}{y_p} \prod_{C^{(0)} \in S} u_C^{\alpha' X_{C^{(0)}}}$$

* Much Simpler!

* "Stringy" UV regularization

Factorization $\leftrightarrow$ U-variables

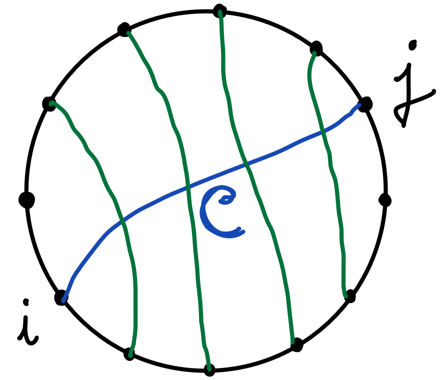
$$\boxed{u_e + \prod_{e' \in S} u_{e'}^{\mathrm{Int}(e';e)} = 1}$$



U- equations $\qquad u_e \geqslant 0 \Rightarrow u_e \in [0,1]$

Factorization ↔ U-variables

$$\boxed{u_e + \prod_{e' \in S} u_{e'}^{\mathrm{Int}(e';e)} = 1}$$

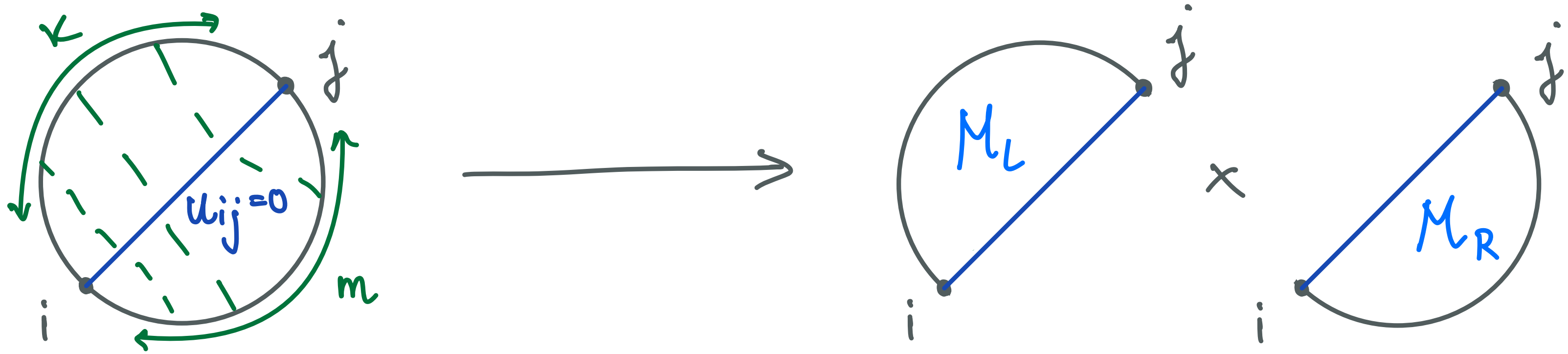


U-equations $\quad u_e \geq 0 \Rightarrow u_e \in [0,1]$

* Factorization: $[X_{ij} = -n \Rightarrow$ singularity $u_{ij} = 0]$

$$u_{ij} \to 0 \Rightarrow \forall_{(km)\ \text{incomp.}\ (ij)} \ u_{km} \to 1$$ [Binary]

$$\prod_c u_c^{\alpha' X_c}$$



Revealing New Features.

* Kinematic Connection between different theories: $\mathrm{Tr}\,\varphi^3 \leftrightarrow \mathrm{NLSM} \leftrightarrow \mathrm{YM}$

[2401.00041, 2401.05483, 2403.048261

w/ N. Arkani-Hamed, Q. Cao, J. Dong, S. He]

δ-Shifted Try^3

$$A^{\alpha'}_{\mathrm{Try}^3}\left[X_{i,j}+\delta_{i,j}\right]=\int_0^{+\infty}\prod\frac{dy}{y}\prod u_{i,j}^{\alpha' X_{i,j}}\times\underbrace{\left(\frac{\prod u_{e,e}}{\prod u_{o,o}}\right)^{\alpha'\delta}}_{\delta\text{-shift}}$$

$$\delta_{ij}=\begin{cases}+\delta & (i,j)\text{ even}\\ -\delta & (i,j)\text{ odd}\\ 0 & \text{otherwise}\end{cases}$$

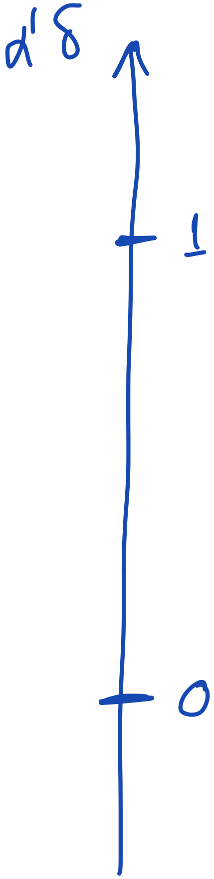
$u_{i,j}[y]$

$$=\int_0^{+\infty}\prod\frac{dy}{y}\prod u_{i,j}^{\alpha' X_{i,j}}\times\left(\frac{1}{\prod y}\right)^{\alpha'\delta}$$

Only Changing Measure !

δ-Shifted $\mathrm{Tr}\,\varphi^3$

$$\mathcal{A}_{2n}[\alpha' X_{ij}] \propto \underbrace{\int \prod \frac{dy}{y} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}}_{\mathrm{Tr}\,\varphi^3} \times \underbrace{\left(\frac{\prod u_{\text{even,even}}}{\prod u_{\text{odd,odd}}} \right)^{\alpha'\delta}}_{\delta\text{-shift}}$$



$\rightarrow \alpha'\delta = 0$ $\quad$ $\mathrm{Tr}\,\varphi^3$ theory

① low energies

δ-Shifted $\mathrm{Tr}\,\varphi^3$

$$\mathcal{A}_{2n}[\alpha' X_{ij}] \propto \underbrace{\int \prod \frac{dy}{y} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}}_{\mathrm{Tr}\,\varphi^3} \times \underbrace{\left(\frac{\prod u_{\text{even},\text{even}}}{\prod u_{\text{odd},\text{odd}}} \right)^{\alpha'\delta}}_{\delta\text{-shift}}$$

$\alpha'\delta$

1

$\alpha'\delta \in \mathbb{R}^+ \setminus \mathbb{Z}$ NLSM @ low energies

$0 \rightarrow \alpha'\delta = 0$ $\mathrm{Tr}\,\varphi^3$ theory

δ-Shifted $\mathrm{Tr}\,\varphi^3$

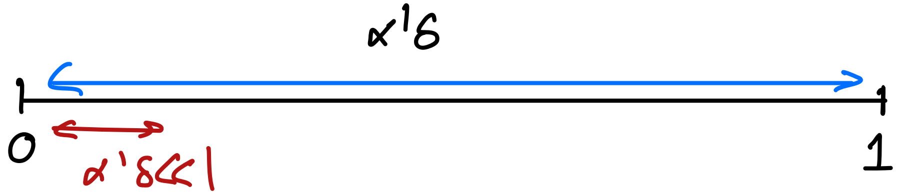
$$\mathcal{A}_{2n}[\alpha' X_{ij}] \propto \underbrace{\int \prod \frac{dy}{y} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}}_{\mathrm{Tr}\,\varphi^3} \times \underbrace{\left(\frac{\prod u_{\text{even,even}}}{\prod u_{\text{odd,odd}}} \right)^{\alpha'\delta}}_{\delta\text{-shift}}$$

$\alpha'\delta$

$1 \rightarrow \alpha'\delta = 1$ YM theory

$\alpha'\delta \in \mathbb{R}^+ \setminus \mathbb{Z}$ NLSM @ low energies

$0 \rightarrow \alpha'\delta = 0$ $\mathrm{Tr}\,\varphi^3$ theory

Claim: 

$\Longrightarrow$ NLSM

$\alpha'\delta \ll 1$, @ low energies:

$$A_\delta^{\mathrm{tr}\varphi^3}[X_{ij} \to X_{ij} + \delta_{ij}]$$

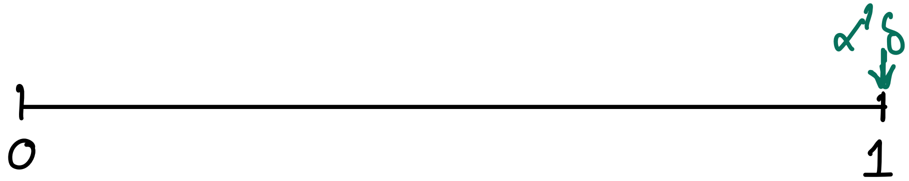
Ex. 4 pts

$$A_4^\delta = \frac{1}{X_{13} - \delta} + \frac{1}{X_{24} + \delta} \xrightarrow[X \ll \delta]{} -1/\delta^2 \underbrace{(X_{13} + X_{24})}_{A_4^{NLSM}} + \mathcal{O}(\delta^{-3})$$

* Lagrangian derivation of $\mathrm{Tr}\varphi^3 \leftrightarrow$ NLSM.

* Useful (Tree + Loop-level)!

Claim:



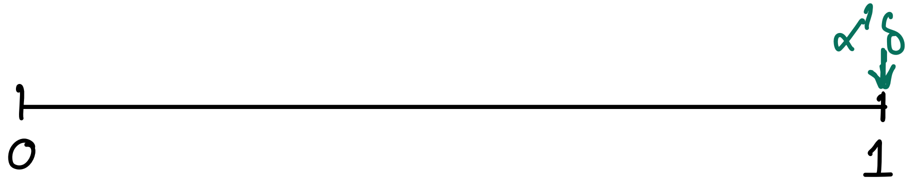
$\Longrightarrow$ YM

$$\alpha'\delta = 1$$

$$A_{2n}[\alpha' X_{ij}] \propto \int \prod \frac{dy}{y^2} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}$$

different measure

Claim:



$\Rightarrow$ YM

$$\alpha'\delta = 1$$

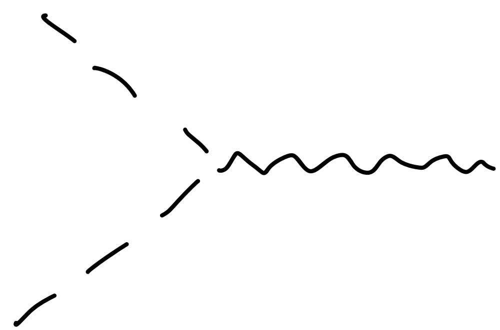
$$A_{2n}[\alpha' X_{ij}] \propto \int \prod \frac{dy}{y^2} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}$$

different measure

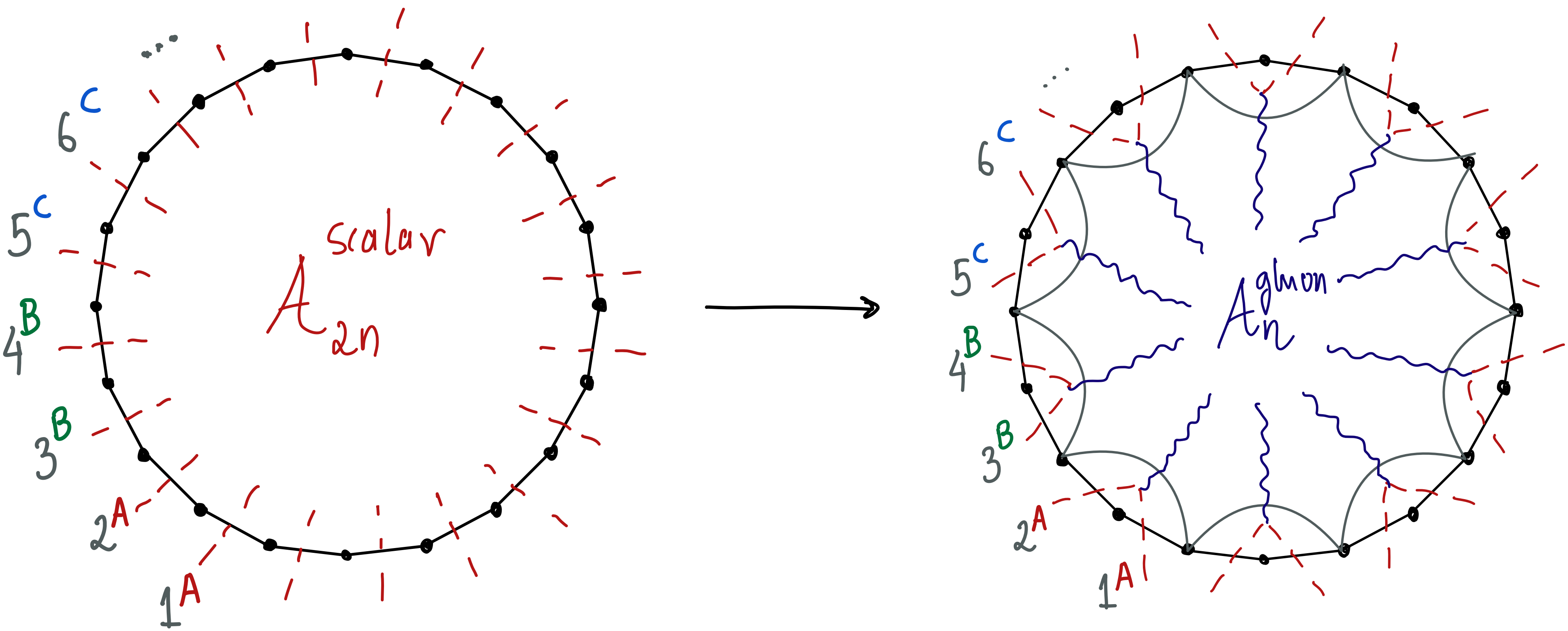
Q: Where are the polarizations?

$A_{2n}[X] \equiv$ scattering of scalars

that couple to gluons



General 2n-scattering scalars ⟶ n-gluons



$$A_{2n} = \int \prod \frac{dy}{y_2} \prod u_{ij}^{\alpha' X_{ij}} \longrightarrow A_n^{gluon} = \mathrm{Res}\left(A_{2n} \right)[X_{ij}].$$

Scaffolding X=0

Scaffolded Gluons * Tree-level : match to Bosonic String ✓

Scaffolded Gluons

* Tree-level: match to Bosonic String ✓
* Loop-level

$$A^{\mathrm{Tr}\,\phi^3}_{\mathrm{Loop}} = \int \prod \frac{dy}{y} \prod_{C \in S} u_C^{\alpha' X_C} \longrightarrow \underbrace{\int \prod \frac{dy}{y^2} \prod_{C \in S} u_C^{\alpha' X_C}}_{\text{Surface Gluon Integrands.}} \equiv A^{\mathrm{Gluons}}_{\mathrm{Loop}}.$$

* General Dimensions
* Leading Singularities ✓ [Sérgio's GongShow].
  (2 loops)

– X14 X15 X27 X36 + X15^2 X27 X36 + X14 X15 X28 X36 – X15^2 X28 X36 + X14 X15 X27 X37 – X15^2 X27 X37 – X14 X16 X27 X37 + X15 X16 X27 X37 – X14 X15 X28 X37 + X15^2 X28 X37 +
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X15 X26 X37 Y4 + X15 X28 X37 Y4 – X16 X28 X37 Y4 – X16 X25 X38 Y4 + X15 X26 X38 Y4 – X15 X27 X38 Y4 + X16 X27 X38 Y4 + X16 X27 X58 Y4 – X27 X36 X58 Y4 – X16 X37 X58 Y4 +
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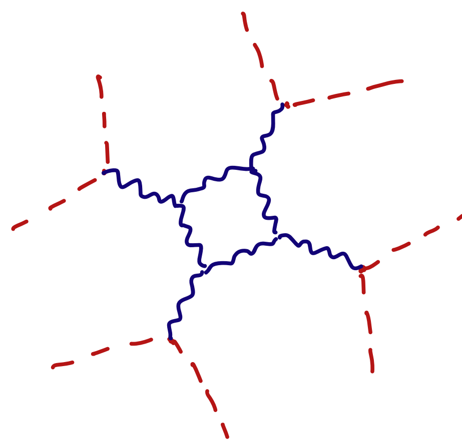
X15 X24 X38 Y6 + X14 X25 X38 Y6 + X14 X27 X38 Y6 – X15 X27 X38 Y6 – X15 X28 X47 Y6 + X15 X38 X47 Y6 – X25 X38 X47 Y6 + X15 X27 X48 Y6 – X15 X37 X48 Y6 + X25 X37 X48 Y6 –
X14 X27 X58 Y6 + X14 X37 X58 Y6 – X24 X37 X58 Y6 + 2 X14 X37 Y2 Y6 – 2 X15 X37 Y2 Y6 – 2 X14 X38 Y2 Y6 + 2 X15 X38 Y2 Y6 + 2 X15 X47 Y2 Y6 + 2 X38 X47 Y2 Y6 – 2 X15 X48 Y2 Y6 –
2 X37 X48 Y2 Y6 + 2 X14 X58 Y2 Y6 + 2 X37 X58 Y2 Y6 – 2 X47 X58 Y2 Y6 + 2 X28 X37 Y4 Y6 – 2 X27 X38 Y4 Y6 – X15 X24 X36 Y8 + X14 X25 X36 Y8 – X14 X27 X36 Y8 + X15 X27 X36 Y8 +
X15 X24 X37 Y8 – X16 X24 X37 Y8 – X14 X25 X37 Y8 + X16 X25 X37 Y8 + X14 X26 X37 Y8 – X15 X26 X37 Y8 – X15 X27 X46 Y8 + X15 X37 X46 Y8 – X25 X37 X46 Y8 – X16 X25 X47 Y8 +
X15 X26 X47 Y8 – X15 X36 X47 Y8 + X25 X36 X47 Y8 + 2 X37 X46 Y2 Y8 – 2 X36 X47 Y2 Y8 + 2 X16 X25 Y4 Y8 – 2 X15 X26 Y4 Y8 + 2 X15 X27 Y4 Y8 – 2 X16 X27 Y4 Y8 + 2 X15 X36 Y4 Y8 –
2 X25 X36 Y4 Y8 + 2 X27 X36 Y4 Y8 – 2 X15 X37 Y4 Y8 + 2 X16 X37 Y4 Y8 + 2 X25 X37 Y4 Y8 – 2 X26 X37 Y4 Y8 + 2 X15 X24 Y6 Y8 – 2 X14 X25 Y6 Y8 – Y2 Y4 Y6 Y8 – Y2 Y4 Y6 Y8 △

≡ Box Leading Sing.



A_4^{Gluon}

Revealing New Features.

* Hidden Factorizations away from poles

  + Zeros of string/particle amplitudes

[2312.16282, 2405.09608 w/ N. Arkani-Hamed, Q. Cao, J. Dong, S. He]

New Kind of Factorizations ($\Rightarrow$ Zeros)

$$A_S\left[\{p_i \cdot p_j \to 0\}\right] \longrightarrow A_{S_1} \times A_{S_2}$$

w/ i, j not-adjacent

(no poles)

New Kind of Factorizations ($\Rightarrow$ Zeros)

$$A_S\left[\{p_i \cdot p_j \to 0\}\right] \longrightarrow A_{S_1} \times \boxed{A_{S_2}}$$

w/ i,j not-adjacent

(no poles)

$\longrightarrow 0$

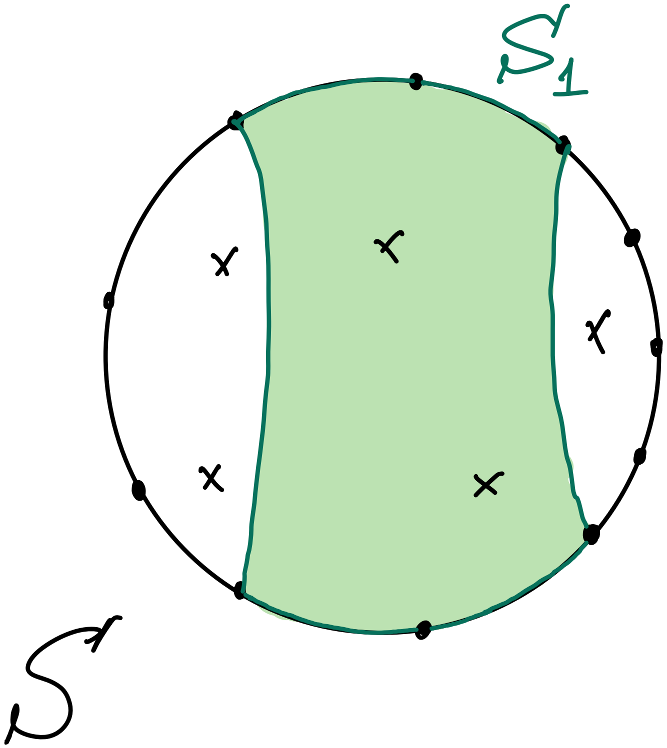
$p'_i \cdot p'_j \to 0$

SPLITs: AWAY FROM Poles (Near Zeros)

u-variables for Subsurfaces

Simple fact about u-variables!

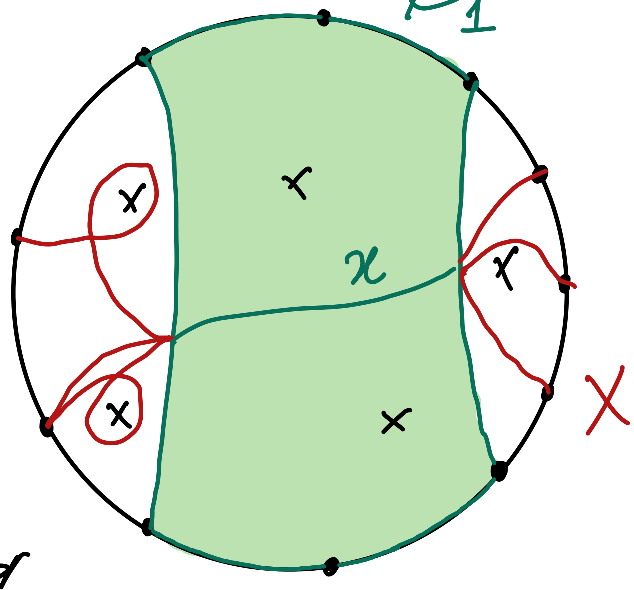
u-variables for subsurface $S_1' \subset S'$ can be written in terms of u-variables of S'



u-variables for Subsurfaces

Simple fact about u-variables!

u-variables for subsurface $S_1 \subset S$ can be written in terms of u-variables of S

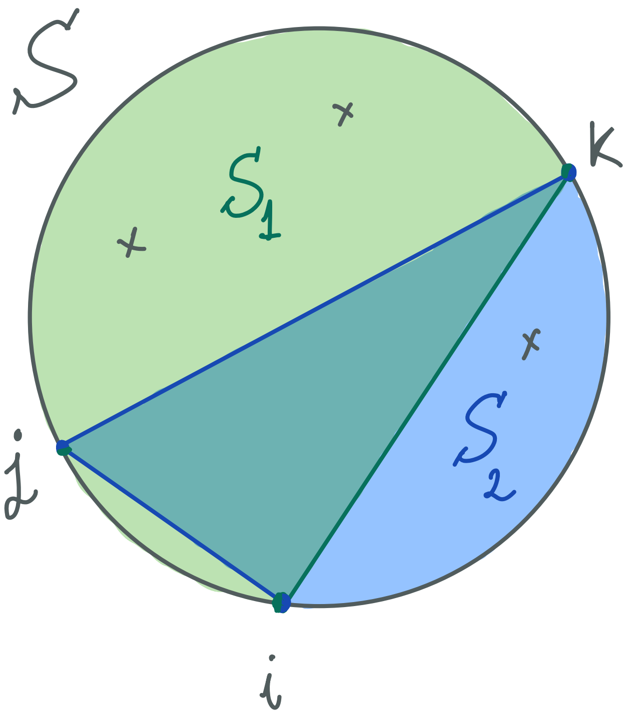


$$u_x = \prod U_X^{\#(x \subset X)}$$

Extension Formula

(all ways of extending x into a curve in S)

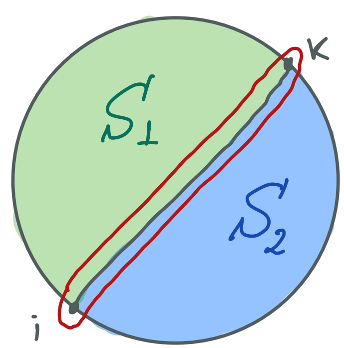
Split Factorization: Choose $\mathcal{S}_1$ and $\mathcal{S}_2$ such that all curves X in $\mathcal{S}$ belong to at least one of subsurfaces.



OVERLAP ON TRIANGLE!

$$A_{\mathcal{S}_1} = \int \prod \frac{dy}{y} \times \prod_{c_1 \in \mathcal{S}_1} u_{c_1}^{\alpha' x_{c_1}} \to \text{Kinematics of } \mathcal{S}_1$$

$$A_{\mathcal{S}_2} = \int \prod \frac{dy}{y} \times \prod_{c_2 \in \mathcal{S}_2} u_{c_2}^{\alpha' y_{c_2}} \to \text{Kinematics of } \mathcal{S}_2$$

[


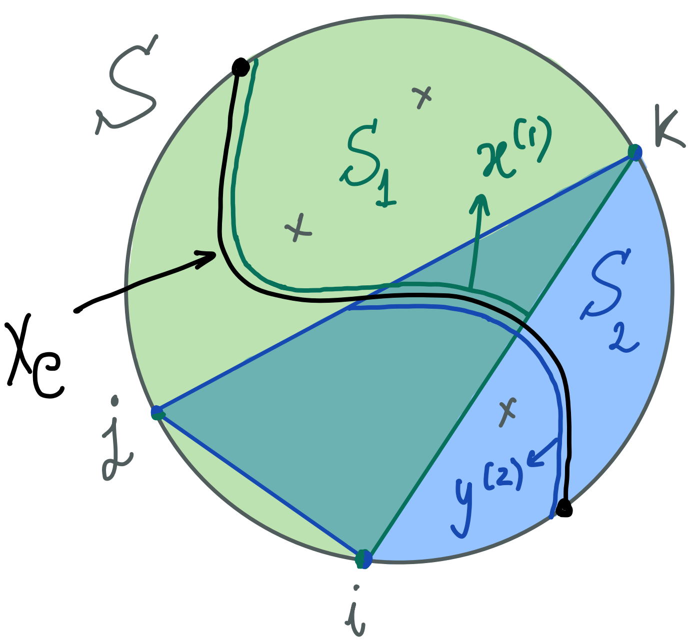
Factorization.

$X_{i,k}$ NOT in $\mathcal{S}_1 / \mathcal{S}_2$.
]

Extension Formula $\Rightarrow$ Split Kinematics

$$A_{S_1} \times A_{S_2} = \left(\int \prod \frac{dy}{y} \prod_{c_1 \in S_1} u_{c_1}^{\alpha' x_{c_1}} \right) \times \left(\int \prod \frac{dy}{y} \prod_{c_2 \in S_2} u_{c_2}^{\alpha' y_{c_2}} \right) = \int \prod \frac{dy}{y} \prod_{c \in S} u_c^{\alpha' \tilde{X}_c}$$

$$\text{w/} \quad \tilde{X}_c = \sum_{c_1 \, c_2} [\# c_1 \subset c]\, x_{c_1} + [\# c_2 \subset c]\, y_{c_2}$$



$$\tilde{X}_c \to x^{(1)} + y^{(2)}$$

$(\Rightarrow \tilde{X}_c \neq 0$ NO POLES$)$

$$A_S(\tilde{X}_c) \to A_{S_1}(x_{c_1}) \times A_{S_2}(y_{c_2})$$

From Splits to Zeros:

Consider a Split in which one subsurface is a 4 point:

$$A_4(s,t) \times A_{S_2} \quad \text{or} \quad A_4(s,t) \times A_{S_1} \times A_{S_2} \times \cdots$$

$$\longrightarrow \left(\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \right) \longleftarrow$$

which vanishes for $s+t=0$ (or $s+t=-n$)

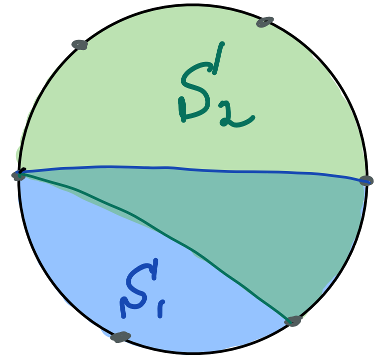
$\Rightarrow$ Broad class of Zeros for particle / string amplitudes

* Standard Physical Interpretation

Zeros?

Splits?

Splits @ Tree level



Split Kinematics:

$$X_e = \sum_{c_1\, c_2} [\#c_1 \subset e]\, x_{c_1} + [\#c_2 \subset e]\, y_{c_2}$$

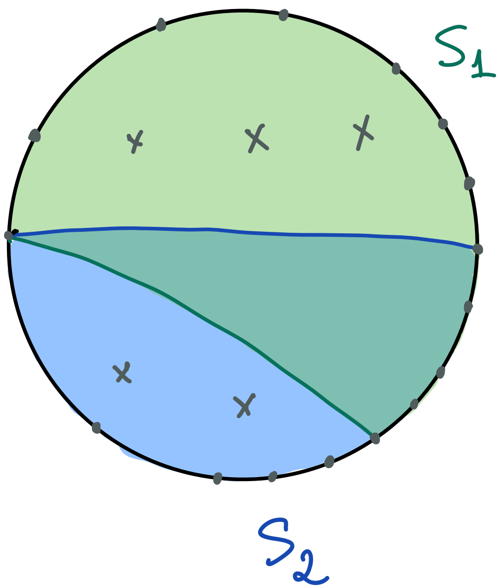
$$\Updownarrow$$

particular patterns of $\{ p_i \cdot p_j = 0 \}$ (i,j) not adjacent

F. Cachazo, N. Early, B. Umbert 22' [Splitting Amplitudes]

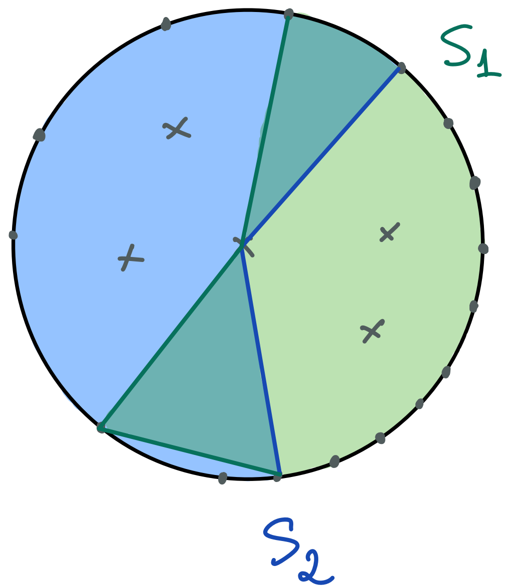
* Consecutive Splits ⇒ Product of 4-points ⇒ Minimal Kinematics [Bernd's Talk]

Splits for the Loop Integrand $\mathcal{I}$



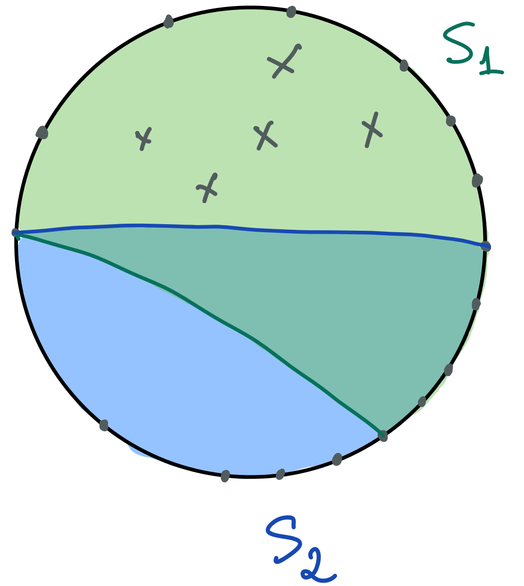
$$\mathcal{I}_n^{\ell} \to \mathcal{I}_{n_1}^{\ell_1} \times \mathcal{I}_{n_2}^{\ell_2}$$

$$\ell = \ell_1 + \ell_2$$



$$\mathcal{I}_n^{\ell} \to \mathcal{I}_{n_1}^{\ell_1} \times \mathcal{I}_{n_2}^{\ell_2}$$

$$\ell = \ell_1 + \ell_2 + 1$$



$$\mathcal{I}_n^{\ell} \to \mathcal{I}_{n_1}^{\ell} \times A^{\text{tree}}_{S_2}$$

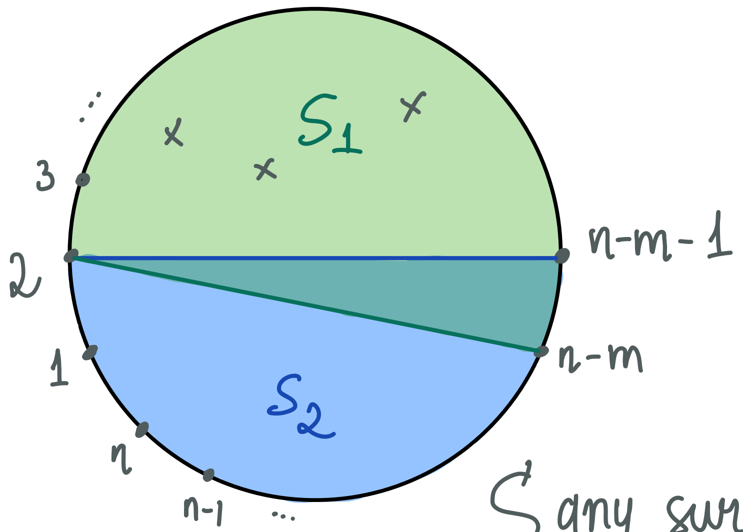
Soft limits

$\updownarrow$

Post-loop integration

× Do Splits fix the Integrand ?

From Splits to Soft Limits.



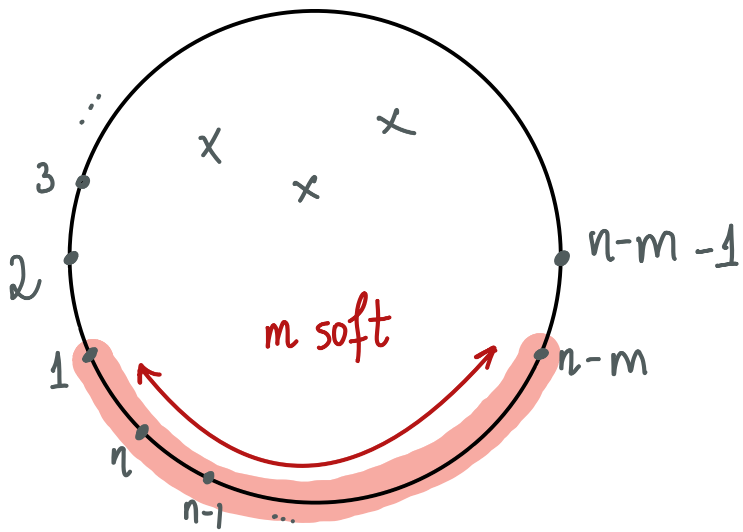
S any surface.

Split Kinematics $X_S[x_{ij}, y_{ij}]$

$$\mathcal{I}^S \longrightarrow A^{S_2}[y_{ij}] \times \mathcal{I}^{S_1}[x_{ij}]$$

$$\Big\downarrow \; y_{ij} \to 0$$

m-soft limit, $p_n, p_{n-1}, \dots, p_{n-m} \to 0$.



$$\mathcal{I}^S \to \text{Soft}^m[y_{ij}] \times \mathcal{I}^{\text{lower}}$$

$$A^S \to \text{Soft}^m[y_{ij}] \times A^{\text{lower}}$$

From $\mathrm{Tr}\,\phi^3$ to Pions and Gluons.

(the δ-shift)

$$A^{\delta}_{\mathrm{Tr}\phi^3}\left[X_{ij} \to X_{ij} + \delta_{ij}\right] \quad \text{w/} \quad \delta_{ij} = \begin{cases} +\delta & i,j \text{ even} \\ -\delta & i,j \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

! δ-shift commutes w/ Split kinematics !

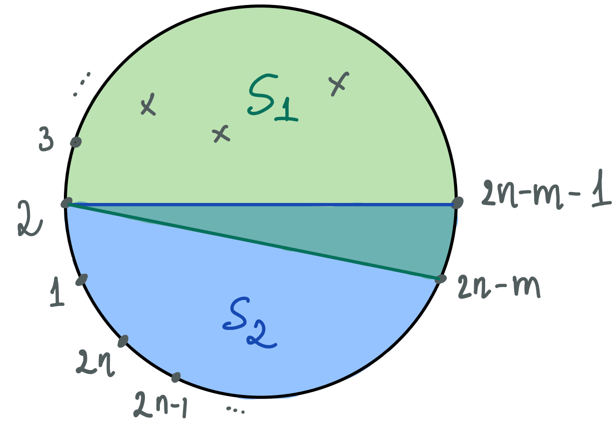
$\Downarrow$

Same splits are there for pions + gluons

[All orders]

↳ access to multisoft limits ($y_{ij} \to 0$)

Pions (NLSM): [All-Orders]



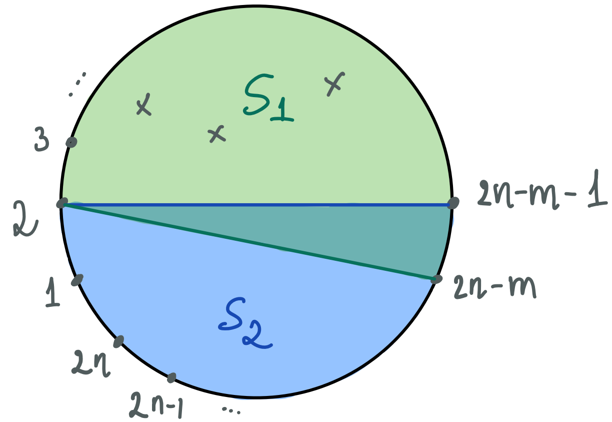
* $\forall$ (odd)-soft limit

$$A_{2n}^{\text{NLSM}} \rightarrow \cancel{A^{\text{NLSM}}_{S_2}[y_{ij}]}^{\,0} \times A_{S_1} \longrightarrow 0$$

integrand vanishes !

(generalizes Adler Zero)

Pions (NLSM): [All-Orders]



* $\forall$ (odd)-soft limit

$$A_{2n}^{NLSM} \longrightarrow \cancel{A_{S_2}^{NLSM}[y_{ij}]}^{\,0} \times A_{S_1} \longrightarrow 0$$

integrand vanishes!

(generalizes Adler Zero)

* (even)-soft limits:

loop momentum

$$A_{2n}^{NLSM} \longrightarrow A_{S_2}^{tree}[y_{ij}] \times A_{S_1}^{NLSM} \longrightarrow \text{Soft}^{(2m)} \times \boxed{A_{2n-2m}^{NLSM}}$$

w/

$$\text{Soft}^{(2m)} = A_{S_2}^{\text{Tr}\phi^3}\Big|_{\delta\text{-shift}} \overset{y_{ij}\to 0}{=} A_{S_2}^{tree}\left(1^{\phi}, 2^{\phi}, (2n-2m-1)^{\phi}, (2n-2m)^{\pi}, \ldots, (2n)^{\pi}\right)$$

Revealing New Features.

* Surface Kinematics → Determining "Perfect" Integrands for real world amplitudes.

[2403.04826, 2401.00041, 2401.05483, to appear

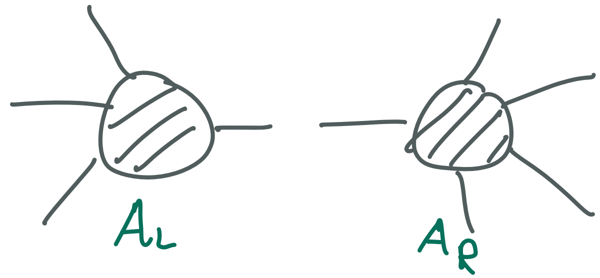
w/ N. Arkani-Hamed, Q. Cao, J. Dong, S. He]

CHALLENGE: Determine real world amplitudes from

SINGLE CUTS $\equiv$ single poles ?

Tree-level ✓

Rational Function + On poles:



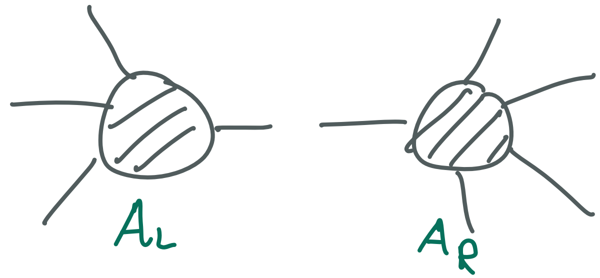
$\Rightarrow$ Can be used to recursevely construct A. [BCFW].

CHALLENGE: Determine real world amplitudes from

SINGLE CUTS $\equiv$ single poles ?

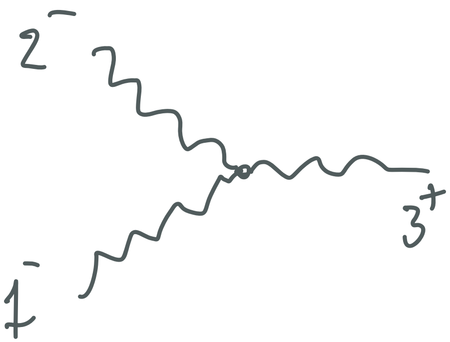
<u>Tree-level</u> ✓

Rational Function + On poles:



$\Rightarrow$ Can be used to recursevely construct A. [BCFW].

BUT, Needs extension of momenta $\mathbb{R} \to \mathbb{C}$.



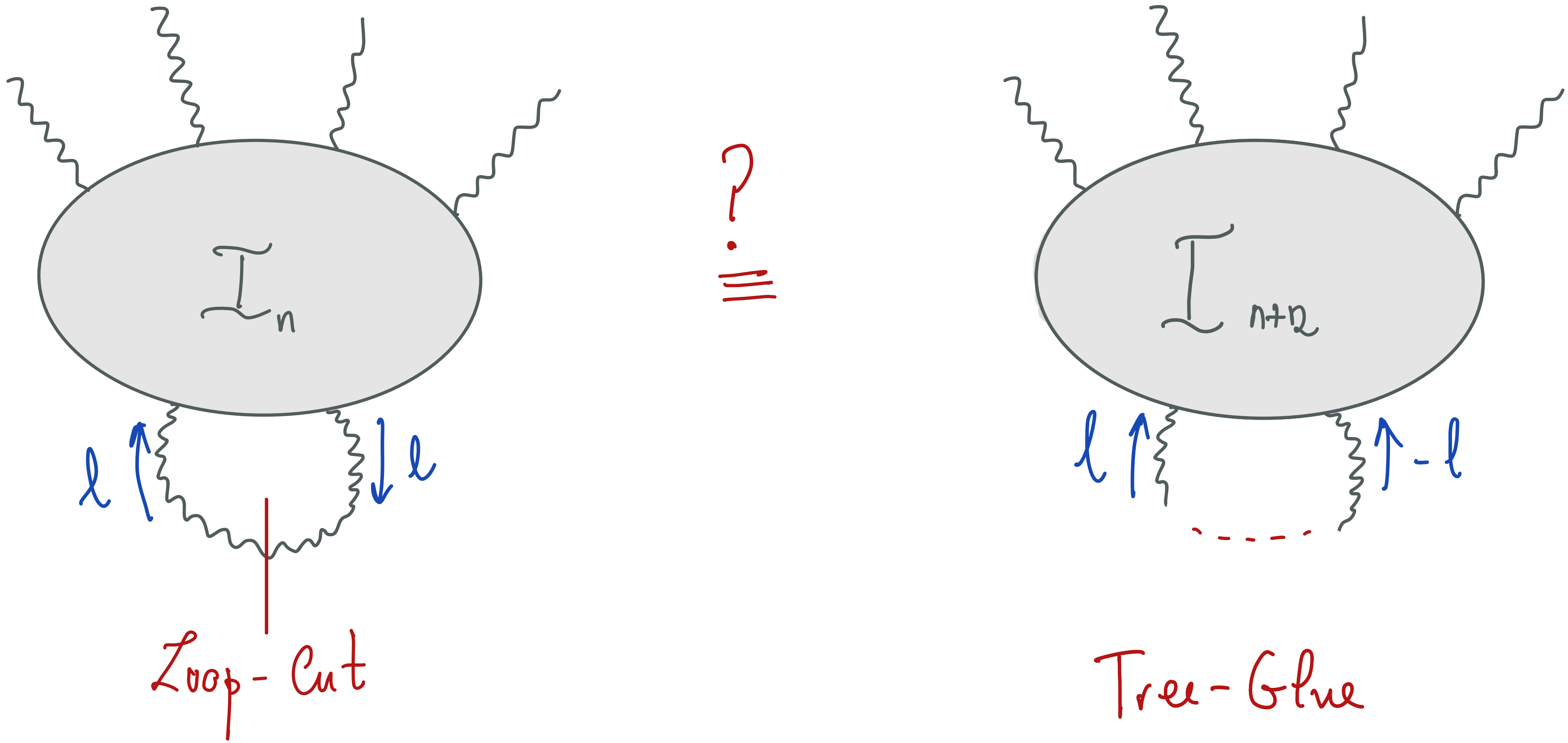
3 point amplitude $= 0$ Lorentzian

$\neq 0$ Complex Kinematics!

CHALLENGE: Determine real world amplitudes from

SINGLE CUTS ≡ single poles ?

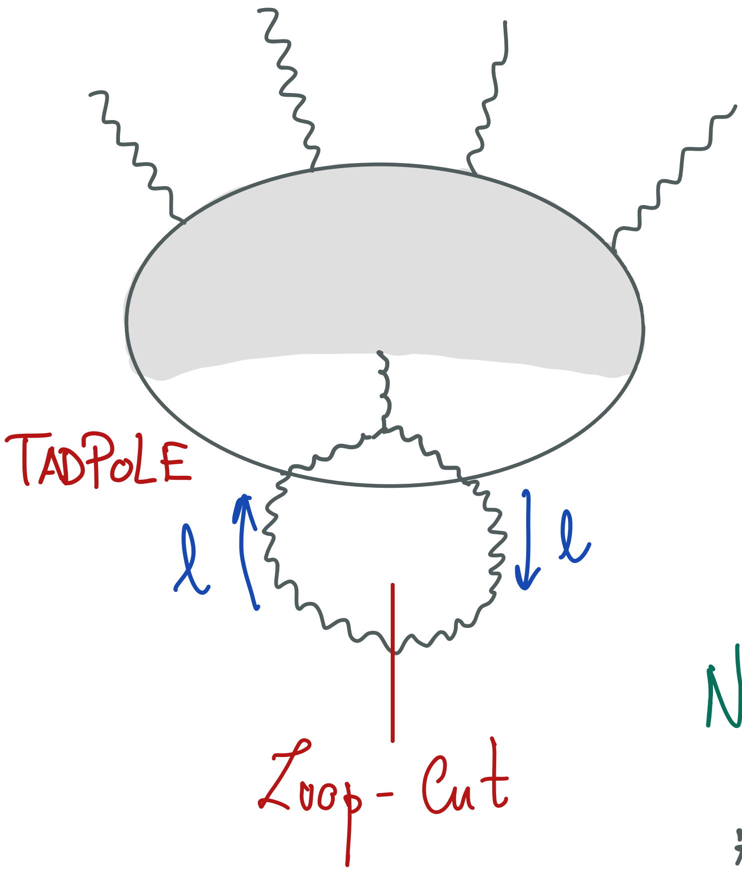
Loop-level? No!



CHALLENGE: Determine real world amplitudes from

SINGLE CUTS ≡ single poles ?

Loop-level? No!



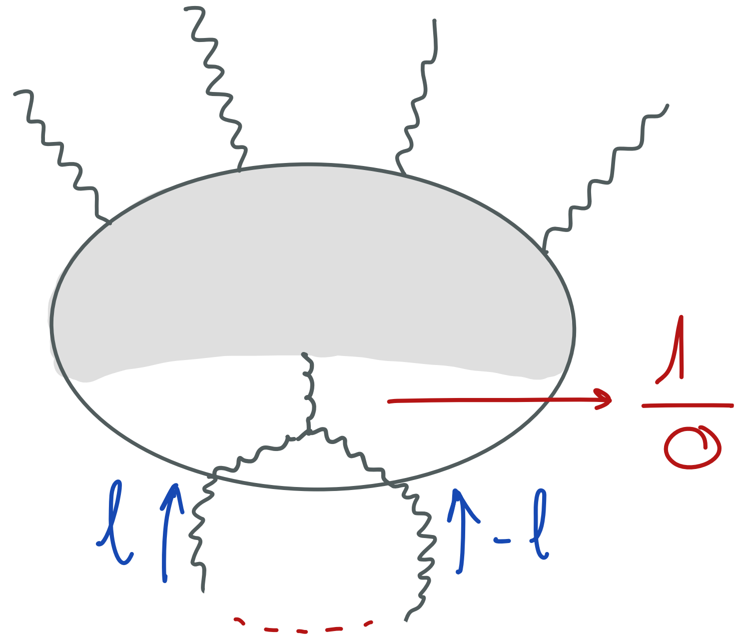
? ≡

DIVERGENT!

⇓

No "The" Integrand.

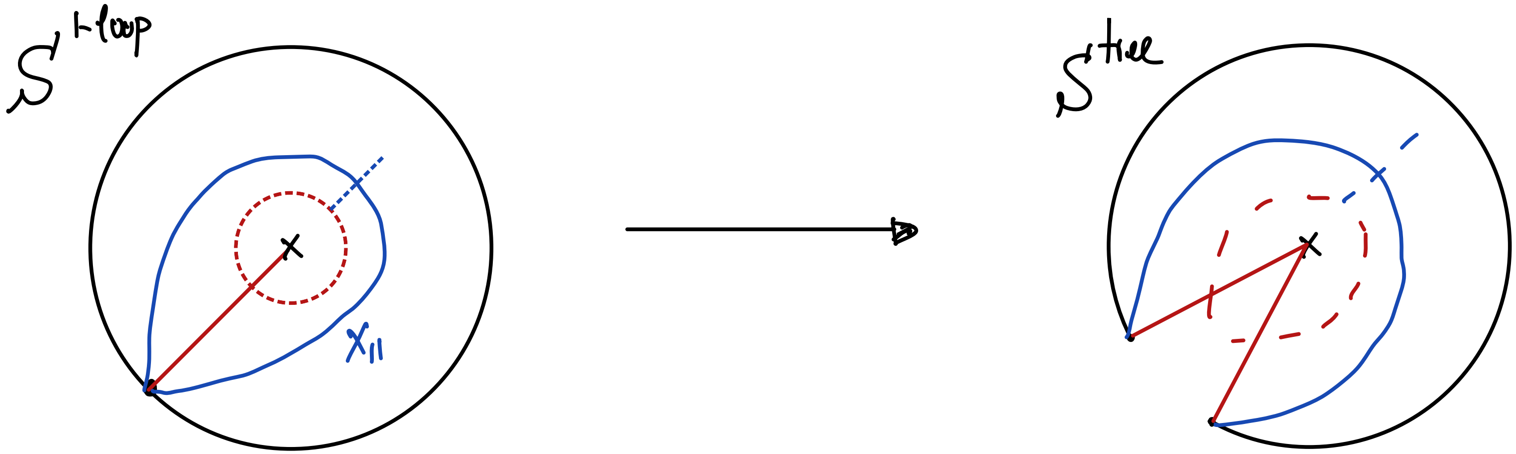
* Adler Zero ✗
* Gauge Invariant ✗



Tree-Glue

[cancels in maximal SUSY]

In Surface language: Loop-cut

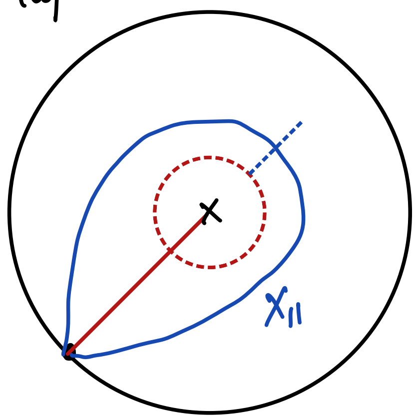


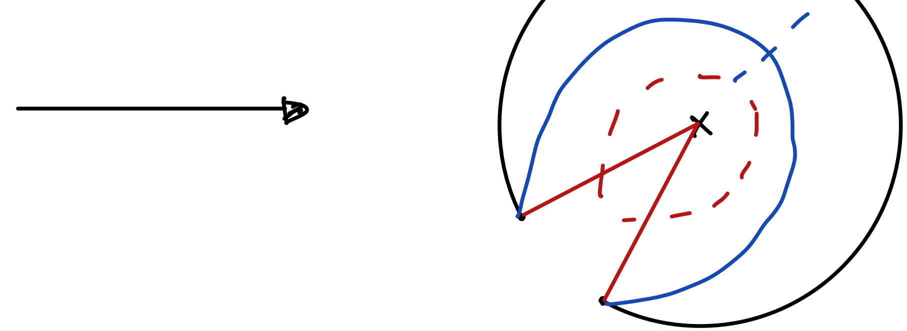
$X_{11} \equiv 0$ in momentum space

$A^{\text{tree}}\big|_{X_{11}=0}$ blows up!

In Surface language: Loop-cut

$S^{1\text{-loop}}$ $\qquad\qquad\qquad\qquad$ S^{tree}





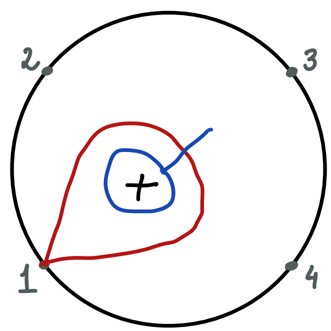
$X_{11} \equiv 0$ in momentum space $\qquad\qquad$ $A^{\text{tree}}\big|_{X_{11}=0}$ blows up!

BUT,

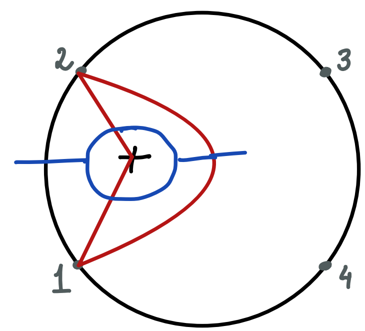
in Curve integral CAN keep curves w/o standard momentum

$$A^{\delta}_{2n} = \int \prod \frac{dy}{y^{1+\delta}} \prod_{C \in S} u_{ij}^{X_C}$$

HOMOLOGY → HOMOTOPY!



X_{11} ~ tadpoles



$X_{2,1}$ ~ Bubbles

Lorentzian $\xrightarrow{\text{Tree}}$ Complex $\xrightarrow{\text{Loop}}$ "Surface" Kinematics
(Curves on surfaces)

"The" (Surface) Integrand for Pions & Gluons $\mathcal{I}^{\delta}_{2n} = \int \prod \frac{dy}{y^{1+\delta}} \prod_{C \in S} u_{ij}^{X_C}$

Pions: * All-loop order [Recursively]

e.g. Shifting $\mathrm{Tr}\,\phi^3$.

* Adler Zero + Splits (Multi-soft limits)

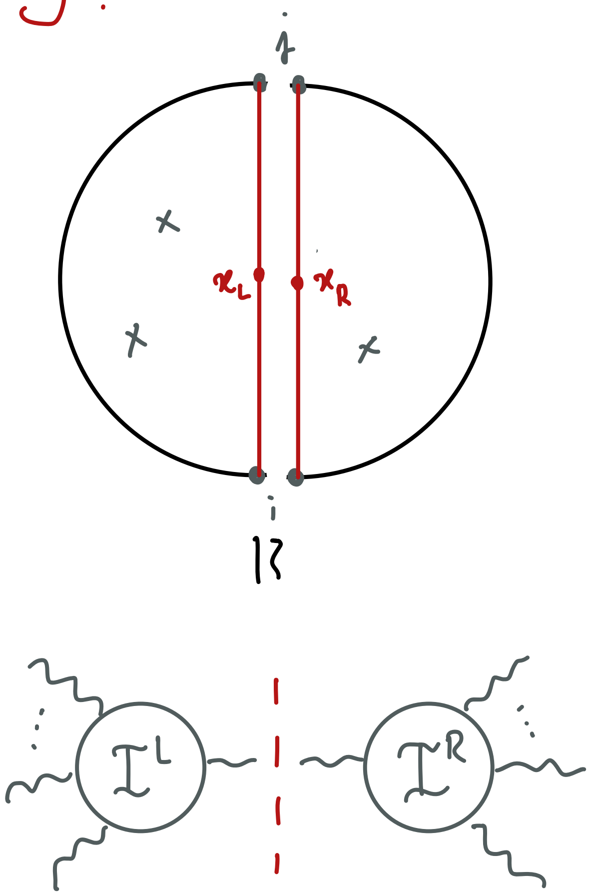
Gluons: All SINGLE CUTS match

+ Gauge-Invariant !

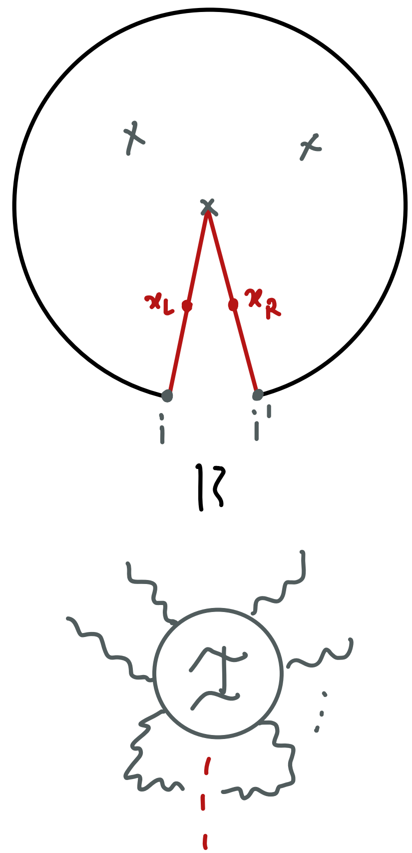
"The" (Surface) Integrand for Gluons

$$\mathcal{I}^{\delta=1}_{2n} = \int \prod \frac{dy}{y^2} \prod_{C \in S} u_{ij}^{X_C}$$

CUTS:



(Tree Cut)

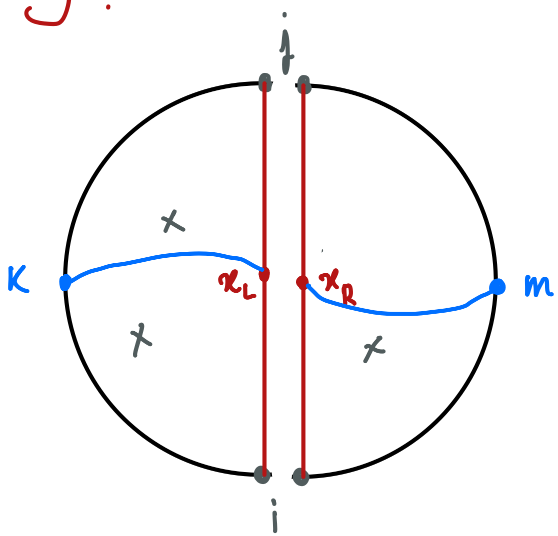


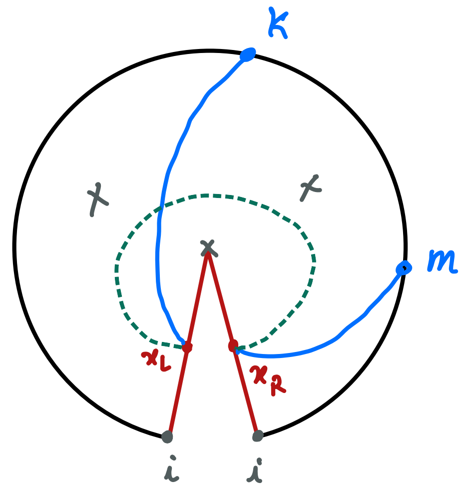
(Loop Cut)

"The" (Surface) Integrand for Gluons

$$\mathcal{I}^{\delta=1}_{2n} = \int \prod \frac{dy}{y^2} \prod_{C \in S} u_{ij}^{X_C}$$

CUTS:





$$\sum_{k,m} \left(X_{k,m} - X_{k,j} - X_{m,i} \right) \frac{\partial \mathcal{I}^L}{\partial X_{k,x_L}} \times \frac{\partial \mathcal{I}^R}{\partial X_{x_R,m}}$$

$$\sum_{k,m} \left(X_{k,m} - X_{i,k} - X_{m,i} \right) \frac{\partial^2 \mathcal{I}}{\partial X_{k,x_L}\, \partial X_{m,x_R}} - D \frac{\partial^2 \mathcal{I}}{\partial X^2_{x_L,x_R}}$$

¡Crucially defined for full surface kinematics!

"The" (Surface) Integrand for Gluons

$$\mathcal{I}^{\delta=1}_{2n} = \int \prod \frac{dy}{y^2} \prod_{C \in S} u_{ij}^{X_C}$$

CUTS $\Rightarrow$ Determined Recursively!

* Tree-level: vanishes under analogue of δ-shift @ $X \to +\infty$.
* One-loop: vanishes under $X_{i,p} \to X_{i,p} + t$, $t \to +\infty$.

$\Rightarrow$ Recursion as sum over SINGLE CUTS! (NOT BCFW).

(correct integrated results)

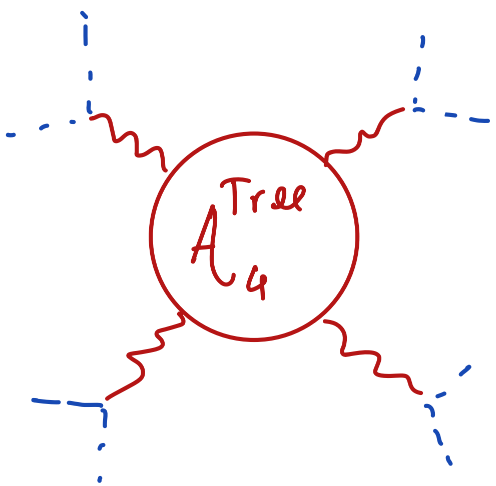
* higher loops: in progress

* Gauge-Invariants ✓ * Splits/Multi-soft ✓.

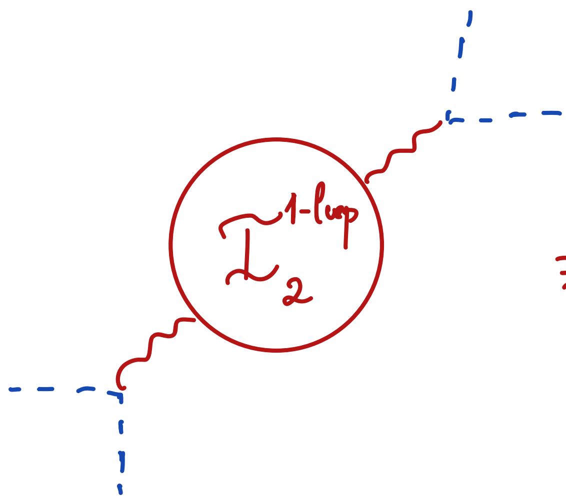
"The" (Surface) Integrand for Gluons

$$\mathcal{I}_{2n}^{\delta=1} = \int \prod \frac{dy}{y^2} \prod_{C \in S} u_{ij}^{X_C}$$

Examples:



$$= \frac{1}{X_{1,5}\,X_{3,7}} \Big(X_{1,5}^2\,(X_{2,7} - X_{2,8} - X_{3,7} + X_{3,8})\,(X_{3,6} - X_{3,7} - X_{4,6} + X_{4,7}) +$$
$$X_{1,5}\,\big(X_{2,5}\,X_{3,6}\,X_{3,7} - X_{2,5}\,X_{3,7}^2 - X_{2,5}\,X_{3,6}\,X_{3,8} + X_{2,5}\,X_{3,7}\,X_{3,8} - X_{2,5}\,X_{3,7}\,X_{4,6} + X_{2,8}\,X_{3,7}\,X_{4,6} +$$
$$X_{2,5}\,X_{3,8}\,X_{4,6} - X_{2,7}\,X_{3,8}\,X_{4,6} + X_{1,6}\,(X_{2,7} - X_{2,8} - X_{3,7} + X_{3,8})\,(X_{3,7} - X_{4,7}) -$$
$$X_{2,8}\,X_{3,6}\,X_{4,7} + X_{2,5}\,X_{3,7}\,X_{4,7} - X_{2,5}\,X_{3,8}\,X_{4,7} + X_{2,6}\,X_{3,8}\,X_{4,7} +$$
$$X_{2,7}\,X_{3,6}\,X_{4,8} - X_{2,6}\,X_{3,7}\,X_{4,8} - X_{2,7}\,X_{3,6}\,X_{5,8} + X_{2,7}\,X_{3,7}\,X_{5,8} + X_{3,6}\,X_{3,7}\,X_{5,8} -$$
$$X_{3,7}^2\,X_{5,8} + X_{2,7}\,X_{4,6}\,X_{5,8} - X_{3,7}\,X_{4,6}\,X_{5,8} - X_{2,7}\,X_{4,7}\,X_{5,8} + X_{3,7}\,X_{4,7}\,X_{5,8} +$$
$$(-X_{2,7} + X_{3,7})\,(X_{3,7} - X_{4,7})\,X_{6,8} + X_{2,4}\,(-((X_{3,6} - X_{3,7})\,(X_{3,7} - X_{3,8})) + X_{3,7}\,X_{6,8})\big) +$$
$$X_{1,4}\,(X_{1,5}\,(X_{3,6} - X_{3,7})\,(-X_{2,7} + X_{2,8} + X_{3,7} - X_{3,8}) + X_{3,7}\,(X_{1,6}\,(-X_{2,7} + X_{2,8} + X_{3,7} - X_{3,8}) +$$
$$(X_{2,6} - X_{2,7} - X_{3,6} + X_{3,7})\,X_{5,8} - (X_{2,5} - X_{2,7} + X_{3,7})\,X_{6,8})) +$$
$$X_{3,7}\,(-X_{1,6}\,(X_{2,5}\,(-X_{3,7} + X_{3,8} + X_{4,7} - X_{4,8}) + X_{2,4}\,(X_{3,7} - X_{3,8} + X_{5,8})) +$$
$$X_{2,4}\,((X_{3,6} - X_{3,7})\,X_{5,8} + X_{3,7}\,X_{6,8}) +$$
$$X_{2,5}\,((-X_{3,6} + X_{3,7} + X_{4,6} - X_{4,7})\,X_{5,8} + (-X_{3,7} + X_{4,7})\,X_{6,8}))\Big)$$



$$= -1 - \Delta + \frac{Y_2}{Y_1} + \frac{\Delta Y_2}{Y_1} - \frac{Y_1}{Y_3} - \frac{\Delta Y_1}{Y_3} + \frac{Y_2}{Y_3} + \frac{\Delta Y_2}{Y_3} - \frac{Y_3}{Y_1} - \frac{\Delta Y_3}{Y_1} + \frac{Y_4}{Y_1} + \frac{\Delta Y_4}{Y_1} + \frac{Y_4}{Y_3} + \frac{\Delta Y_4}{Y_3} - \frac{Y_2 Y_4}{Y_1 Y_3} - \frac{\Delta Y_2 Y_4}{Y_1 Y_3} - \frac{X_{1,2}}{Y_1} + \frac{Y_4 X_{1,2}}{Y_1 Y_3} + \frac{X_{1,4}}{Y_1} -$$
$$\frac{Y_2 X_{1,4}}{Y_1 Y_3} - \frac{Y_2 X_{1,4}}{Y_1 X_{1,1}} - \frac{\Delta Y_2 X_{1,4}}{Y_1 X_{1,1}} + \frac{Y_3 X_{1,4}}{Y_1 X_{1,1}} + \frac{\Delta Y_3 X_{1,4}}{Y_1 X_{1,1}} + \frac{X_{1,2} X_{1,4}}{Y_1 X_{1,1}} + \frac{X_{2,1}}{Y_1} - \frac{Y_4 X_{2,1}}{Y_1 Y_3} + \frac{Y_3 X_{2,1}}{Y_1 X_{1,1}} + \frac{\Delta Y_3 X_{2,1}}{Y_1 X_{1,1}} - \frac{Y_4 X_{2,1}}{Y_1 X_{1,1}} - \frac{\Delta Y_4 X_{2,1}}{Y_1 X_{1,1}} +$$
$$\frac{X_{1,4} X_{2,1}}{Y_1 X_{1,1}} + \frac{\Delta X_{1,4} X_{2,1}}{Y_1 X_{1,1}} - \frac{X_{2,3}}{Y_3} + \frac{Y_4 X_{2,3}}{Y_1 Y_3} - \frac{3 X_{2,4}}{Y_1} - \frac{\Delta X_{2,4}}{Y_1} + \frac{X_{2,4}}{Y_3} + \frac{X_{2,4}}{X_{1,1}} + \frac{\Delta X_{2,4}}{X_{1,1}} - \frac{Y_3 X_{2,4}}{Y_1 X_{1,1}} - \frac{\Delta Y_3 X_{2,4}}{Y_1 X_{1,1}} + \frac{X_{3,2}}{Y_3} - \frac{Y_4 X_{3,2}}{Y_1 Y_3} -$$
$$\frac{X_{1,1} X_{3,2}}{Y_1 Y_3} + \frac{X_{1,4} X_{3,2}}{Y_1 Y_3} + \frac{Y_1 X_{3,2}}{Y_3 X_{3,3}} + \frac{\Delta Y_1 X_{3,2}}{Y_3 X_{3,3}} - \frac{Y_4 X_{3,2}}{Y_3 X_{3,3}} - \frac{\Delta Y_4 X_{3,2}}{Y_3 X_{3,3}} + \frac{X_{1,1} X_{3,3}}{Y_1 Y_3} - \frac{X_{1,4} X_{3,3}}{Y_1 Y_3} - \frac{X_{2,1} X_{3,3}}{Y_1 Y_3} + \frac{X_{2,4} X_{3,3}}{Y_1 Y_3} - \frac{X_{3,4}}{Y_3} +$$
$$\frac{Y_2 X_{3,4}}{Y_1 Y_3} + \frac{X_{3,2} X_{3,4}}{Y_3 X_{3,3}} - \frac{X_{4,1}}{Y_1} + \frac{Y_2 X_{4,1}}{Y_1 Y_3} + \frac{X_{2,1} X_{4,1}}{Y_1 X_{1,1}} + \frac{X_{4,2}}{Y_1} - \frac{3 X_{4,2}}{Y_3} - \frac{\Delta X_{4,2}}{Y_3} + \frac{X_{1,1} X_{4,2}}{Y_1 Y_3} + \frac{X_{4,2}}{X_{3,3}} + \frac{\Delta X_{4,2}}{X_{3,3}} - \frac{Y_1 X_{4,2}}{Y_3 X_{3,3}} - \frac{\Delta Y_1 X_{4,2}}{Y_3 X_{3,3}} +$$
$$\frac{X_{4,3}}{Y_3} - \frac{Y_2 X_{4,3}}{Y_1 Y_3} - \frac{X_{1,1} X_{4,3}}{Y_1 Y_3} + \frac{X_{2,1} X_{4,3}}{Y_1 Y_3} + \frac{Y_1 X_{4,3}}{Y_3 X_{3,3}} + \frac{\Delta Y_1 X_{4,3}}{Y_3 X_{3,3}} - \frac{Y_2 X_{4,3}}{Y_3 X_{3,3}} - \frac{\Delta Y_2 X_{4,3}}{Y_3 X_{3,3}} + \frac{X_{2,3} X_{4,3}}{Y_3 X_{3,3}} + \frac{X_{3,2} X_{4,3}}{Y_3 X_{3,3}} + \frac{\Delta X_{3,2} X_{4,3}}{Y_3 X_{3,3}}$$

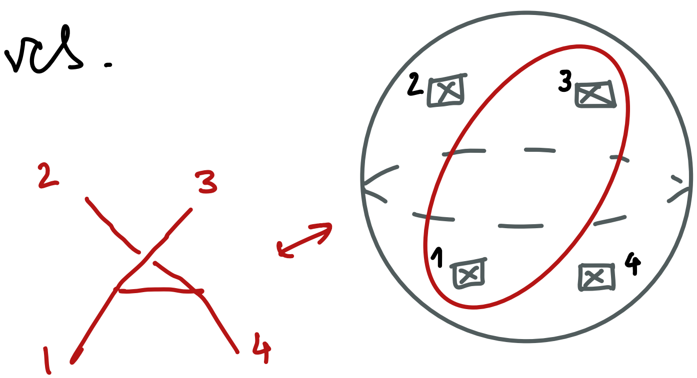
Outlook...

* Fermions.

* Realistic particle content. (Beyond) SM

$$\phi^i_j \underset{\leftarrow}{\overset{\rightarrow}{=}}^{u(N)} \longrightarrow \begin{matrix} SU(3)\times SU(2)\times U(1) \\ Q\,(3,2,+1/6) \\ H\,(1,2,-1/2) \end{matrix} \longrightarrow \text{Extension of SM?}$$

* Gravity from Closed Curves.



Thank You !