Amplitudes and Cross Sections in Four Dimensions Through Local Factorizations

Amplitudes, IAS June 12, 2024
C. Anastasiou, R. Haindl, GS, Z. Yang and M. Zeng
JHEP 04, 222 (2021) [arXiv:2008.12293]
C. Anastasiou & GS, JHEP (05) 2023 242, arXiv 2212.12162
GS & Aniruddha Venkata, JHEP (02) 2024 101, arXiv 2309.13023
C. Anastasiou, Julia Karlen, GS, A. Venkata arXiv 2403.13712
and to appear someday.

Sidestepping IR regularization through local cancellations in momentum space

1. Variant Factorization for Hadronic Cross Sections

2. NNLO process-dependence in complex EW annihilation amplitudes

3. Cancellation of Final-State Collinear and Soft for EW Inclusive

1. Variant Factorization for Hadronic Cross Sections

- **Familiar factorized cross section:**

$$\sigma_{AB\to Q} \;=\; \int dx_1 dx_2\, \phi_{q/A}(x_1)\phi_{\bar{q}/B}(x_2)\, \hat{\sigma}_{q\bar{q}\to Q}(x_1p_1, x_2p_2, Q)$$

- Q: for us, some collection of momenta of observed EW bosons, $\{q_i\}$, which we assume to be massive:

$$\int \prod_{i=1}^{n} \frac{d^3\vec{q}_i}{2\omega_i}\, 2\pi\delta\left(\left(\sum_{i=1}^{n} q_i\right)^2 - Q^2\right) \;=\; \int \frac{d^3\vec{Q}}{(2\pi)^3 2\omega_Q} \int \prod_{i=1}^{n} \frac{d^3\vec{q}_i}{2\omega_i}(2\pi)^4\delta^4\left(\vec{Q} - \sum_{i=1}^{n} \vec{q}_i\right)$$

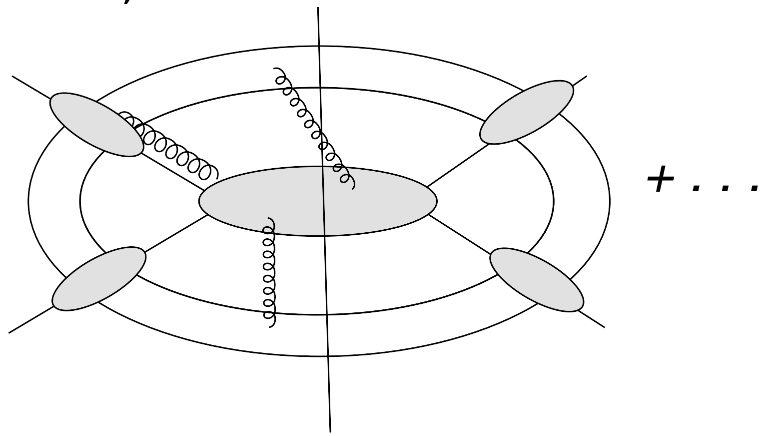
- Can be $V = W, Z$ (Drell-Yan). VV, VH, HH, VVV . . . .
- "wishlist" sorts of final states . . . recoiling jets are also possible (Huss, Huston, Jones, Pellen, J. Phys. G (2023) 043001)

- The individual q_i may be fixed, or partly integrated over. We can also fix the rapidity, $y_Q = (1/2)\ln Q^+/Q^-$ on both sides of this relation.

- **Here's our variant factorization ("Higgs scheme"):**

$$\sigma_{AB\to Q} \; = \; \int dx_1 dx_2 \, \mathbb{F}_{q\bar{q};C}(x_1,x_2)\, \hat{\tau}_{q\bar{q}\to Q;C}(x_1 p_1, x_2 p_2, Q)$$

- **After final-state cancellations, the cross section look like**



- $\mathbb{F}_{q\bar{q};C}(x_1,x_2)$ **matches singularities to everything outside the short-distance (middle) part.**

- $\mathbb{F}_{q\bar{q};C}(x_1,x_2)$ **can be defined by the inclusive cross section for** $q\bar{q} \to$ **scalar, either a singlet (**$C=1$**) or adjoint (**$C=8$**), of invariant mass squared** Q^2**, and at fixed** Q **and rapidity. (Analog to the old "DIS scheme" for parton distributions.)**

- **We normalize the** $\mathbb{F}$ **at lowest order to the singlet case,**

$$\mathbb{F}^{(0)}_{q\bar{q};C}(x_1,x_2) \; = \; \delta(1-x_1)\,\delta(1-x_2)\delta_{C1}\,.$$

- **Collinear factorization implies that our "long-distance" function** $\mathbb{F}$ **is related by convolution to the normal pdfs:**

$$\mathbb{F}_{q\bar{q};C}(x_1,x_2) \; = \; \int dy_1 dy_2\, \phi_q(y_1)\phi_{\bar{q}}(y_2)\, \hat{\mathbb{F}}_{q\bar{q},C}\left(\frac{x_1}{y_1},\frac{x_2}{y_2}\right)$$

- **Why are we doing all this?**
- **It helps us organize the computation of $\hat{\tau}$, and hence $\hat{\sigma}$, up to two loops, directly in four dimensions and at fixed values of the full set of q_i.**
- **Perturbatively, we use $\mathbb{F}_{q\bar{q};C}(x_1, x_2)$ to determine $\hat{\tau}_{q\bar{q},C}$. Assuming Q represents a color-singlet final state, the relevant terms are**

$$
\begin{aligned}
\hat{\tau}^{(0)}_{q\bar{q}\to Q,C} &= \sigma^{(0)}_{q\bar{q}\to Q,1}\delta_{C1} \\
\hat{\tau}^{(1)}_{q\bar{q}\to Q,C} &= \sigma^{(1)}_{q\bar{q}\to Q,C} - \mathbb{F}^{(1)}_{q\bar{q};1} \otimes \hat{\tau}^{(0)}_{q\bar{q}\to Q,1}\delta_{C1} \\
\hat{\tau}^{(2)}_{q\bar{q}\to Q,C} &= \sigma^{(2)}_{q\bar{q}\to Q,C} - \mathbb{F}^{(2)}_{q\bar{q};1} \otimes \hat{\tau}^{(0)}_{q\bar{q}\to Q,1}\delta_{C1} - \mathbb{F}^{(1)}_{q\bar{q};C} \otimes \hat{\tau}^{(1)}_{q\bar{q}\to Q,C}
\end{aligned}
$$

- **All of these relations are in four dimensions, and both soft and collinear infrared singular terms cancel algebraically, LOCALLY IN MOMENTUM SPACE, between the terms shown. We'll have to add some terms to make it work, but these all integrate to zero.**
- **All mass-dependence and dependence on $\{q_i\}$ can be computed numerically, at least in principle.**
- **We can illustrate the method with NNLO corrections to EW annihilation for quarks and gluons.**

2. NNLO process-dependence in complex EW annihilation amplitudes

- We've looked in detail at two loops for processes like:

$$q(p_1)\bar{q}(p_2) \to W^+(q_1)\, W^-(q_2)\ \ldots, [2212.12162]$$
$$gg \to H(q_1)\, H(q_2)\, H(q_3) \ldots [2403.13712]$$

- These are important, but complicated analytically,
- and it might be nice to be able to compute them numerically efficiently,
- which would require momentum space integrals that are infrared finite locally. (And UV convergent.)
- We work in Feynman gauge to avoid non-causal singularities.

- These are treated as above, relying on the IR factorization of these amplitudes:

$$M_{a\bar{a}\to Q}(p_1+p_2\to q_1+q_2+\dots,\epsilon) \;=F_{a\bar{a}}(p_1,p_2,\epsilon)\;H_{a\bar{a}\to Q}(p_1,p_2;q_1,q_2\dots)$$

Here, $M_{a\bar{a}\to Q}$ is a virtual contribution to $\sigma_{a\bar{a}\to Q}$, and $F_{a\bar{a}}$ is a virtual amplitude in $\mathbb{F}_{a\bar{a}}$.

- All dependence of the final state is in

$$H_{a\bar{a}\to Q}(p_1,p_2;q_1,q_2\dots) \;=\; \frac{M_{a\bar{a}\to n}(p_1+p_2\to q_1+q_2+\dots,\epsilon)}{F_{a\bar{a}}(p_1,p_2,\epsilon)}$$

- All true infrared singularities are absorbed into $F_{a\bar{a}}$.

- The "hard" function H is complex and complicated, and includes dynamics of intermediate states at momentum configurations that are not "soft" or "collinear". These "threshold" momentum configurations are amenable to numerical analysis on deformed momentum contours or by other means

- See talk by Matilde Vicini at 2024 Loops and Legs for implementation at NNLO n_F, in work with Dario Kemanschah.

- The essential point is that the singlet QCD form factor enjoys the same factorization with the same jet subdiagrams:

$$F_{a\bar{a}\to 1}(p_1 + p_2 \to 1) \;=\; F_{a\bar{a}}(p_1, p_2, \epsilon)\; H_{a\bar{a}\to 1}(p_1, p_2, Q)$$

- As above, we can use this knowledge to simplify a procedure for IR subtraction

$$H_{a\bar{a}\to Q}(p_1, p_2; q_1, q_2 \ldots) \;\equiv\; \frac{M_{a\bar{a}\to Q}(p_1 + p_2 \to q_1 + q_2 + \ldots, \epsilon)}{F_{a\bar{a}\to 1}(p_1, p_2, Q)}$$

- Just expand, each $L = (\alpha_s/\pi)^n L^{(n)}$, and then solve for $H^{(n)}$

$$F_{a\bar{a}\to 1}(p_1 + p_2 \to 1)\, H_{a\bar{a}\to Q}(p_1, p_2; q_1, q_2 \ldots) \;=\; M_{a\bar{a}\to 1}(p_1 + p_2 \to q_1 + q_2 + \ldots, \epsilon)$$

- or

$$\begin{aligned} H^{(1)} &= M^{(1)} - F^{(1)}\, H^{(0)} \\ H^{(2)} &= M^{(2)} - F^{(1)}\, H^{(1)} - F^{(2)}\, H^{(0)} \end{aligned}$$

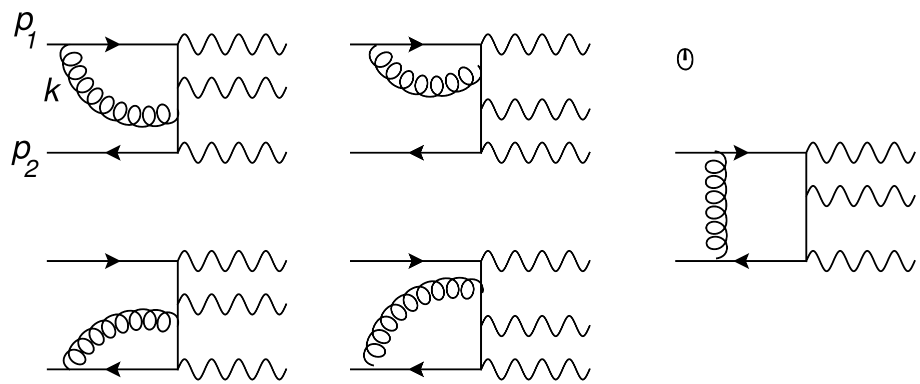
- **This construction for the hard-scattering is surely true for the full functions, but we want a result for the** *integrands*, $\mathcal{L} = \mathcal{M}$, $\mathcal{F}$, $\mathcal{H}$:

$$L_{a\bar{a}\to Q}(p_1, p_2; q_1, q_2 \dots) \;=\; \mathcal{L}^{(0)} + \int \frac{d^D k}{(2\pi)^D} \mathcal{L}_1(k) \;+ \int \frac{d^D k}{(2\pi)^D} \frac{d^D l}{(2\pi)^D} \mathcal{L}^{(2)}(k, l) \;+ \dots$$

- **To be able to "give $\mathcal{H}$ to a computer", what we want is to show is:**

$$\begin{aligned} \mathcal{H}^{(1)} &= \mathcal{M}^{(1)} - \mathcal{F}^{(1)} \mathcal{H}^{(0)} \\ \mathcal{H}^{(2)} &= \mathcal{M}^{(2)} - \mathcal{F}^{(1)} \mathcal{H}^{(1)} - \mathcal{F}^{(2)} \mathcal{H}^{(0)} \end{aligned}$$

- **Let's first loop look at what happens at one loop:** $\mathcal{H}^{(1)} = \mathcal{M}^{(1)} - \mathcal{F}^{(1)}\mathcal{H}^{(0)}$.
- **IR singularities arise when $k \to 0$ and in the $k \propto p_{1,2}$ *collinear* limits:**

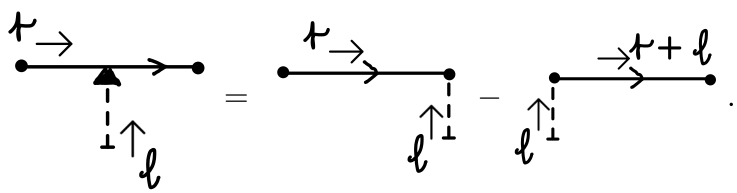


- **When k gets collinear to p_1, singular behavior comes from**

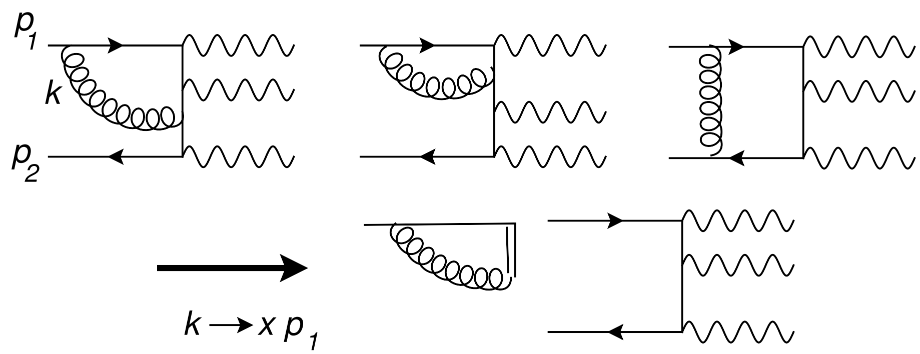
$$u(p_1)\gamma_\nu\,(\not p - \not k)\;\frac{-\eta^{\mu\nu}}{k^2+i\epsilon}\;\Rightarrow_{k\to xp_1}\; -u(p_1)\frac{(p_1-k)\cdot p_2}{p_2\cdot k}\,\frac{1}{k^2+i\epsilon}\,k^\nu$$

- **Then in the collinear limit the gluon k is scalar-polarized and the "Feynman identity" applies, resulting in lots of pairwise cancellations,**

$$\frac{i}{\not r+\not \ell}\,[-ig\not \ell]\,\frac{i}{\not r} \;=\; \frac{ig}{\not r} \;-\; \frac{ig}{\not r+\not \ell}$$



- And all k-dependence separates from the EW bosons . . . and in the sum,



- This is an algebraic relation, which is automatic when we add the integrands of the original amplitude. (The double line is $\sim p_{2\nu}/p_2 \cdot k$)

- The only k-dependent factor on the right equals the one-loop form factor in the k collinear to p_1 region, and $\mathcal{H}^{(1)} = \mathcal{M}^{(1)} - \mathcal{F}^{(1)}\,\mathcal{H}^{(0)}$ is confirmed locally. The same is true for the "soft" $k \rightarrow 0$ and collinear-p_2 limits.

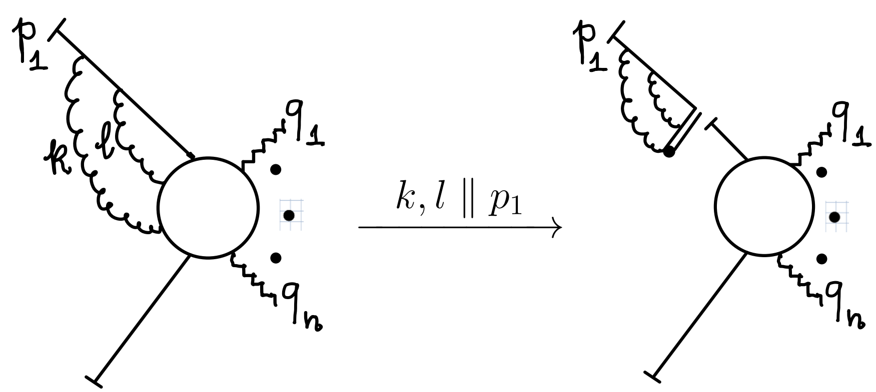
- The single term $\mathcal{F}^{(1)}\,\mathcal{H}^{(0)}$ serves as a local IR subtraction for the full set of (5 for a VVV final state) diagrams of the original amplitude.

- The same holds for any EW final state of heavy bosons with this initial state, like $q\bar{q} \rightarrow W^+W^-W^+W^-$, or more.

- Could something like this work for two loops? Yes.

- To get these local relations at two loops, however, it is sometimes necessary to modify the integrand by adding some IR "counterterms".

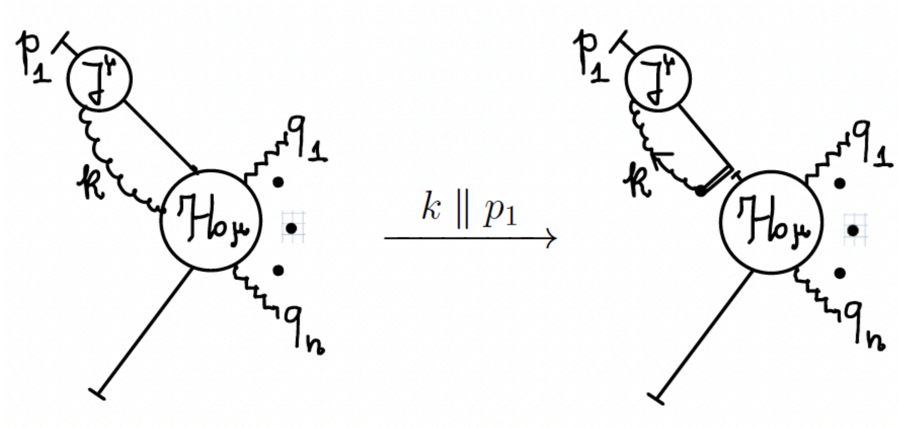
- Actually, when both gluons are collinear to either of the incoming quarks, or when one or both are soft, everything works just as at one loop. For example, in the double-collinear limit:



  is algebraic and hence completely local.

- Through the Ward, Taylor-Slavnov (BRST) identities, the hard scattering "expels" unphysical polarizations without subtractions.

- **Things get a little complicated when we try to see how a "single-collinear" gluon separates from the hard subdiagram at the** *integrand* **level,**



- **Compared to one loop, we encounter two qualitative complications, associated with an extra loop, either in the jet or hard part:**

  1. **"loop polarizations" when $\mathcal{J}^\mu$ is a one-loop vertex or self energy.**

  2. **"shift mismatches", when $\mathcal{H}_\mu$ has the extra loop, and the Ward identity requires a shift in loop momentum.**

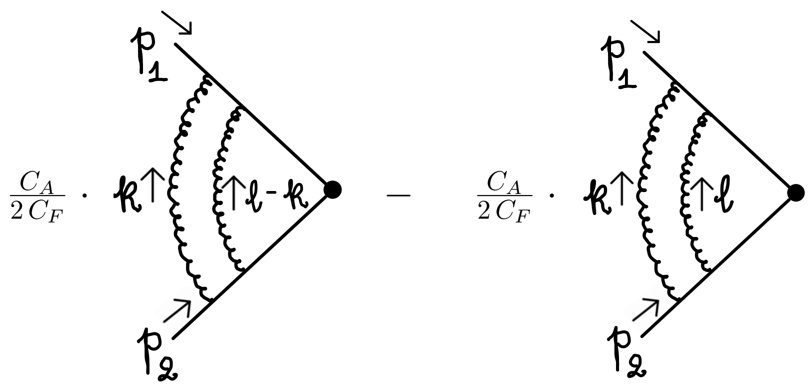
- **These complications are addressed by counterterms that integrate to zero, but reorganize the integrand. I'll give basic examples that illustrate the detailed approach for our treatment of this region.**

1. Collinear-singular loop polarizations. Add counterterm that integrates to zero. l is the loop momentum:

$$\delta\mathcal{J}^{\mu}(k,l) = \frac{2(1-\epsilon)}{(p_1+k+l)^2}\left[\frac{2l^{\mu}+p_1^{\mu}+k^{\mu}}{l^2} - \frac{2(l+p_1)^{\mu}+k^{\mu}}{(l+p_1)^2}\right]\frac{\not{\eta}_1}{2p_1\cdot\eta_1}$$

where we can take $\eta_1 = p_2$. We just add this to the integrand before integrating.

2. Shift mismatches require another counterterm that integrates to zero, but cancels the singularities of the unwanted "shift" terms locally – in both the k collinear to p_1 and p_2 regions:



- **The same "exotic-color planar" counterterms apply to arbitrary EW state.**

Summary for "initial states in EW production":

- There are more counterterms (including UV), but the ones we've seen illustrate the method.

- With a limited number of counterterms (roughly one per diagram) we can derive a hard function $\mathcal{H}^{(2)}$ that is free of IR divergences at NNLO, and can be computed numerically (the latter in progress).

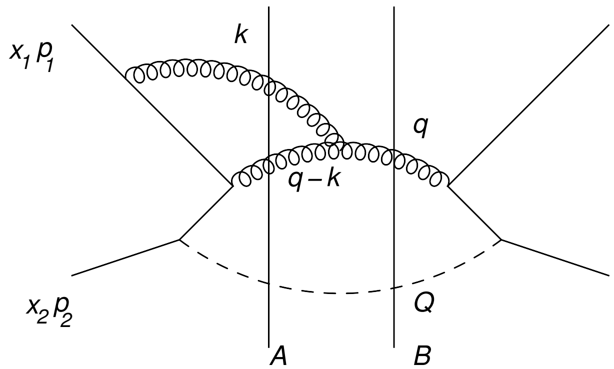
- Applications with color in the final state remain to be investigated in detail, but follow the extension to final states sketched above and are made possible by the cancellation of final state interactions.

- as well as the possibilities of $\mathrm{N^3LO}$ extensions. These may require further insight.

- For "practical" implementation, it is natural to do one integral per loop (Loop-tree duality, Cross-Free Family & Time-ordered perturbation theory, for example.)
(Capatti, Hirschi, Kermanshah, Pelloni, Ruijl [1912.0929],) Kermanshah [2110.06869])

3. Cancellation of Final-State Collinear and Soft for EW Inclusive

A simple but quite general example from TOPT



- **Let's look at one gluon, q, recoiling at high-p_T from our EW boson(s) Q – IR singularities at NNLO inclusive from $k \to 0$ and $k \propto q$.**

- **After sum over the two time orders,**

$$\int dx_1 dx_2 \int \frac{d^3\vec{k}}{(2\pi)^3}\, I(x_1,x_2,k,q,Q)\ F(x_1,x_2,k,q,Q)$$

- **"initial-state factors" for k CO to p_1 or soft:**

$$\begin{aligned} I &= \mathbb{F}(x_1,x_2)\, \frac{1}{2\omega_{p_1-k}\,(2\omega_{p_2-Q})^2}\, \frac{1}{2\omega_k\, 2\omega_{q-k}\, 2\omega_q\, 2\omega_Q} \\ &\times \frac{1}{x_1\sqrt{s}/2 - \omega_{p_1-k} - \omega_k} \left(\frac{1}{x_2\sqrt{s}/2 - \omega_{p_2-Q} - \omega_Q} \right)^2 \end{aligned}$$

- **Final-state factors for k CO to p_1 or soft, when 3-particle E_A and 2-particle E_B are degenerate with initial state:**

$$F = \left[\delta\left((x_1 + x_2)\sqrt{s}/2 - \omega_Q - \omega_{q-k} - \omega_k\right) \frac{1}{(x_1 + x_2)\sqrt{s}/2 - \omega_Q - \omega_q - i\epsilon} \right.$$
$$\left. \frac{1}{(x_1 + x_2)\sqrt{s}/2 - \omega_Q - \omega_{q-k} - \omega_k + i\epsilon}\, \delta\left((x_1 + x_2)\sqrt{s}/2 - \omega_Q - \omega_q\right) \right]$$

- **Singularities happen when**

$$\omega_{q-k} + \omega_k \rightarrow \omega_q$$

where $E_A = E_B$. i.e. at $k = 0$ (soft) or $k \propto q$.

- After change of variables: $E_0 = (x_1 + x_2)\sqrt{s}/2, \Delta E = (x_1 - x_2)\sqrt{s}$, the integral takes the form

$$\int dE_0 \; F(E_0, \Delta E) \left\{ \delta(E_0 - E_B) \, \frac{1}{E_0 - E_A + i\epsilon} \; + \; \frac{1}{E_0 - E_B - i\epsilon} \, \delta(E_0 - E_A) \right\}$$

$$= \; \frac{F(E_B, \Delta E)}{E_B - E_A + i\epsilon} \; + \; \frac{F(E_A, \Delta E)}{E_A - E_B - i\epsilon}$$

$$= \; \frac{F(E_B, \Delta E) \; - \; F(E_A, \Delta E)}{E_B - E_A + i\epsilon}$$

- And the integral is finite. All we have to do is add the terms. This is all we need for NNLO EW inclusive. It can also serve for EW plus jet at NLO.

- So the cancellation of final states is also local, as has been exploited recently in LTD formats in

Capetti, Hirschi, Pelloni, Ruijl, [2010.01068]
Uribe, Dhani, Sborini, Rodrigo [2404.05491]

Summary and for the Future

- General "Higgs scheme" factorization for multi-EW boson cross sections

- Locality at NLO and NNLO for virtual corrections

- Cancellation of induced final-state IR divergences in TOPT

- Pending and for the future:
  - Full expressions for cross sections
  - Complete NNLO numerical integrations
  - Extensions beyond NNLO