# The Gravitational Waveform

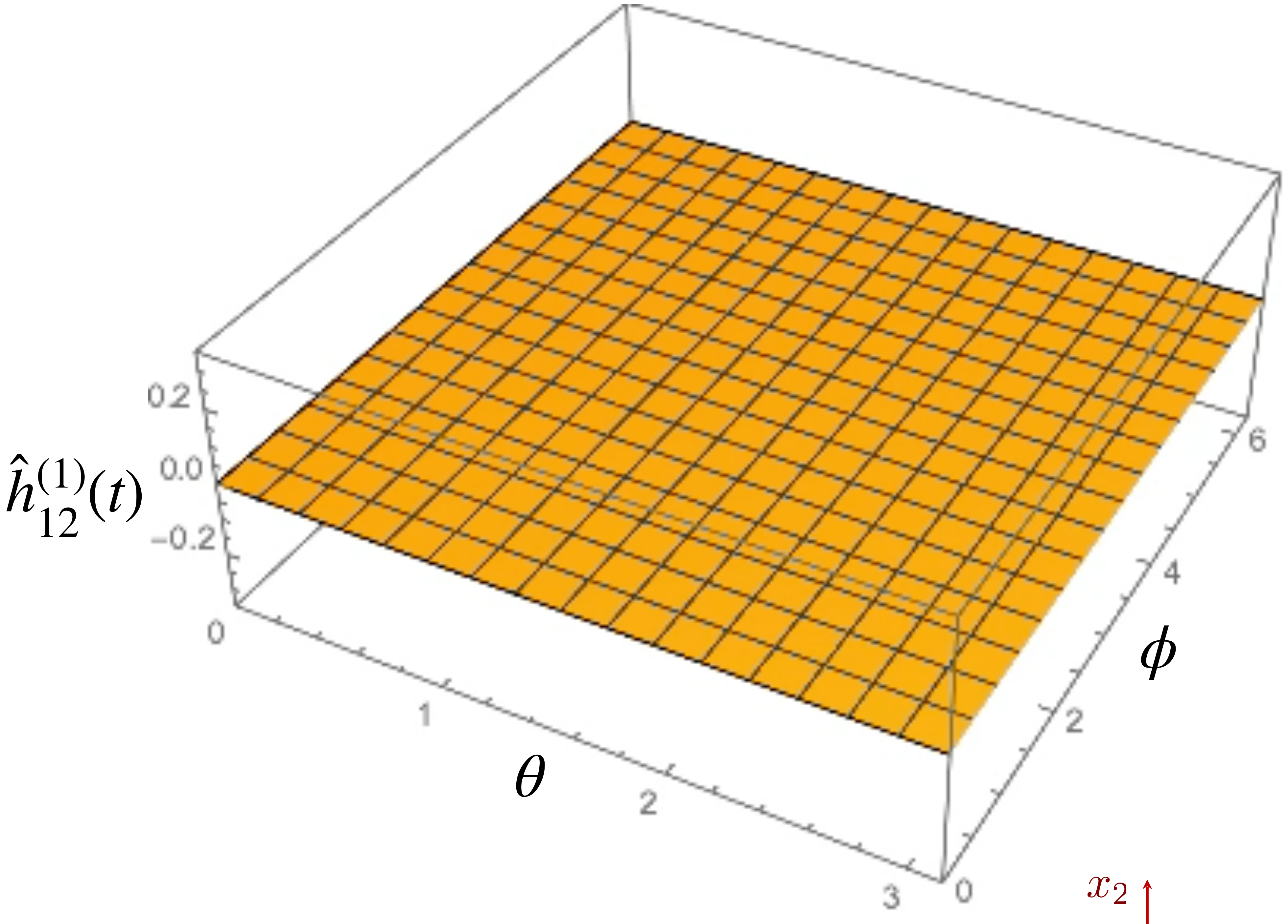
## From Scattering Amplitudes to General Relativity

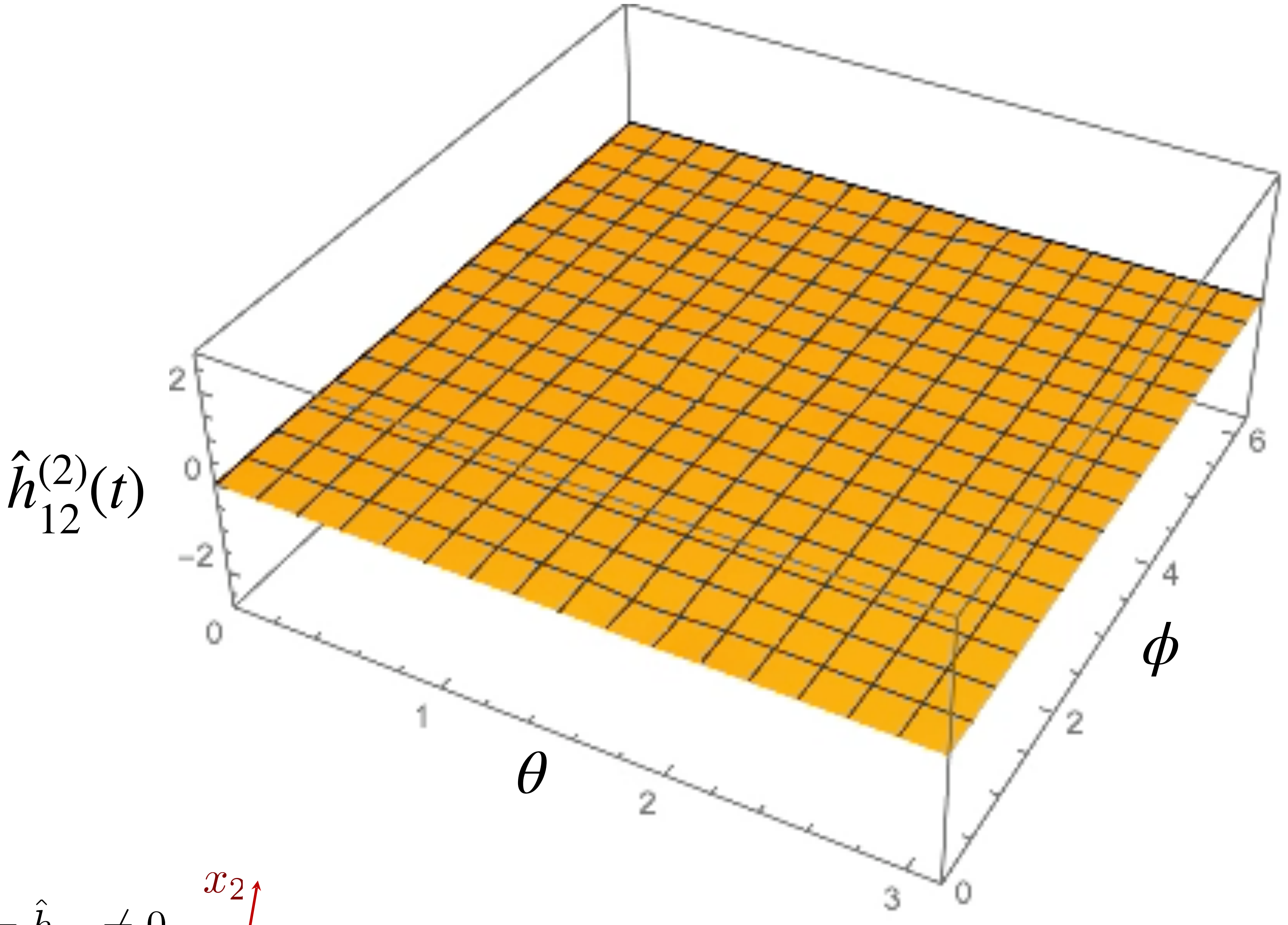
- Brandhuber, Brown, Chen, SDA, Gowdy, Travaglini
- SDA, Gonzo, Novichkov
- Bini, Damour, SDA, Geralico, Herdershee, Roiban, Teng
- Brunello, SDA

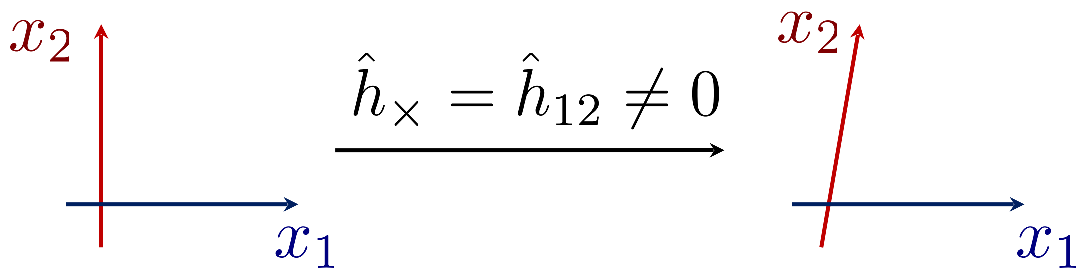
**Stefano De Angelis** - Amplitudes 2024, Institute for Advanced Study - June 13th, 2024

- [Kovacs, Thorne] ’78
- [Jakobsen, Mogull, Plefka, Steinhoff]
- [Mougiakakos, Riva, Vernizzi]
- [Jakobsen, Mogull, Plefka, Steinhoff]
- [SDA, Gonzo, Novichkov]
- [Brandhuber, Brown, Chen, Gowdy, Travaglini]
- [Aoude, Haddad, Heissenberg, Helset]

$$h_{\mu\nu}\Big|_{|\boldsymbol{x}|\to\infty} = \eta_{\mu\nu} + \frac{\kappa^2 M}{8\pi|\boldsymbol{x}|}\left[\frac{\kappa^2 M}{\sqrt{-b^2}}\hat{h}^{(1)}_{\mu\nu} + \left(\frac{\kappa^2 M}{\sqrt{-b^2}}\right)^2 \hat{h}^{(2)}_{\mu\nu} + \ldots\right]$$




$$u_{1,2} = \frac{1}{\sqrt{24}}(5,0,0,\pm 1)$$

$$\hat{h}_+ = \frac{1}{2}\left(\hat{h}_{11} - \hat{h}_{22}\right)$$

$$\hat{h}_\times = \hat{h}_{12}$$

Plot authors: Aidan Herdershee, Radu Roiban, Fei Teng

# Why Gravitational Waveforms?

## Why Scattering?

- GW templates for matched-filtering analyses
- Analytic continuation from unbound to bound   Talk by Zvi Bern
- Scattering setups are interesting on their own
- Analytic properties of scattering amplitudes

# How do we compute waveform?

## KMOC as on-shell in-in formalism

- Compute expectation values from scattering amplitudes

$$\Delta\langle\mathcal{O}\rangle = \langle\mathcal{O}\rangle_{\text{out}} - \langle\mathcal{O}\rangle_{\text{in}} = {}_{\text{out}}\langle\psi|\,\mathcal{O}\,|\psi\rangle_{\text{out}} - {}_{\text{in}}\langle\psi|\,\mathcal{O}\,|\psi\rangle_{\text{in}} = {}_{\text{in}}\langle\psi|\,S^\dagger[\mathcal{O}, S]\,|\psi\rangle_{\text{in}}$$

- The initial state of the two-body problem

$$|\psi\rangle_{\text{in}} = \int \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_2)\,\phi_1(p_1)\phi_2(p_2)\,e^{i(b_1\cdot p_1 + b_2\cdot p_2)}\,|p_1, p_2\rangle_{\text{in}}$$

[Kosower, Maybee, O'Connell]

- Pick the wavefunctions

$$1/m_i \ll G\,m_i \ll b$$

  - $1/m_i$ Compton wavelengths
  - $G\,m_i$ Schwarzshild radii
  - $b$ impact parameter

- Pick the operator

$$\mathcal{O} = \mathcal{W}_{GR} = \varepsilon_h^{\mu\nu} h_{\mu\nu} \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

[Cristofoli, Gonzo, Kosower, O'Connell]
[Bini, Damour, SDA, Geralico, Herdershee, Roiban, Teng]

# The Waveform from Scattering Amplitudes

[Caron-Huot, Giroux, Hannesdottir, Mizera]

The waveform is an in-in observable, while the amplitude is an in-out observable

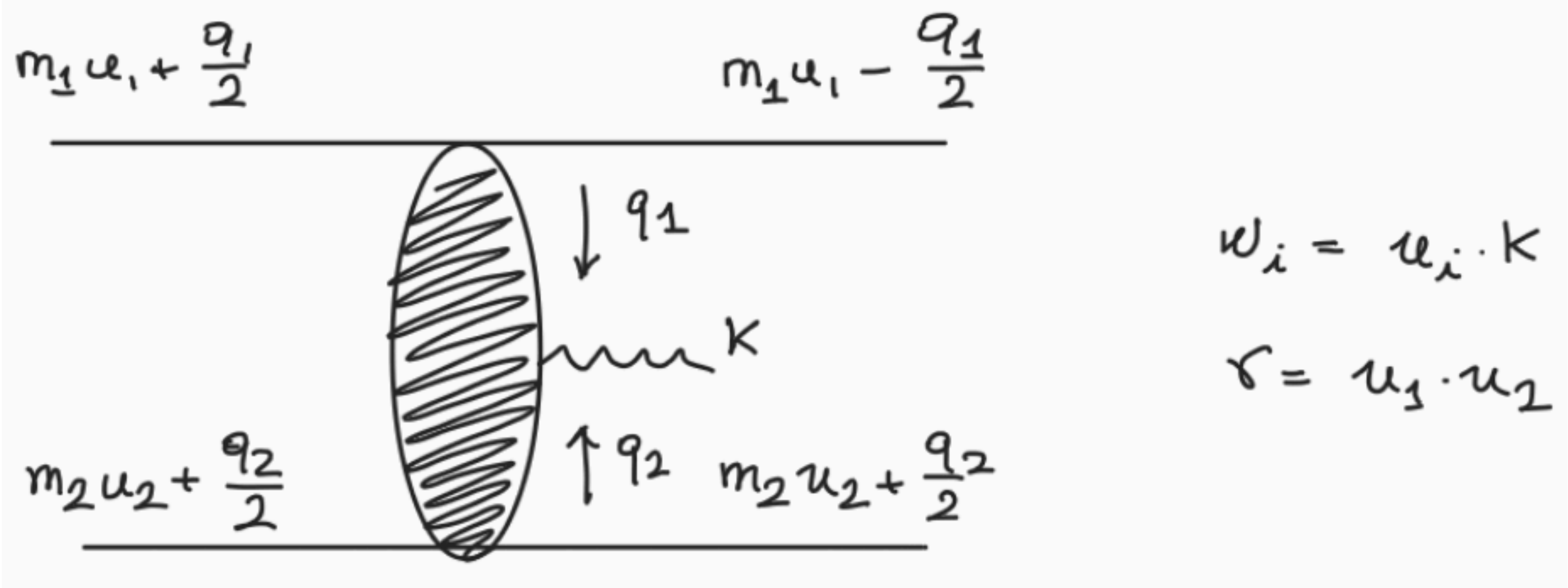
Fourier transform to *impact parameter* space

$$d\mu = \prod_{i=1}^{2} \hat{d}^D q_i \, \delta(2p_i \cdot q_i + q_i^2)\, e^{ib_i \cdot q_i}$$

$$\Delta\langle \mathscr{W}_h \rangle(u, \vec{n}) = \frac{1}{4\pi r}\int_0^\infty \hat{d}\omega \int d\mu \left\{ \hat{\delta}^D(q_1 + q_2 - k) e^{-i\omega u} \left[ \mathscr{A}(p_1 p_2 \to p_1' p_2' k^{-h}) - i\mathscr{A}^*(\tilde{p}_1' \tilde{p}_2' \to \tilde{X}) \otimes \mathscr{A}(p_1 p_2 \to X k^{-h}) \right] + \text{c.c.} \right\}$$

The five-point scattering amplitude

[Cristofoli, Gonzo, Kosower, O'Connell]



# The five-point result @ NLO

[Brandhuber, Brown, Chen, SDA, Gowdy, Travaglini]
[Herdershee, Roiban, Teng]
[Georgoudis, Heissenberg, Vazquez-Holm]
[Bohnenblust, Ita, Kraus, Schlenk]

Amplitudecraft:

- Heavy-mass EFT [Damgaard, Haddad, Helset], [Brandhuber, Chen, Travaglini, Wen]
- Generalised Unitarity
- Integrations of loops

$$\mathcal{M}^{(1)}_{5,\bar{m}_1^3\bar{m}_2^2} = \frac{\mathfrak{d}_{\mathrm{IR}}}{\tilde{\epsilon}} + \mathfrak{R} + i\pi\,\mathfrak{i}_1 + \frac{i\pi}{\sqrt{\gamma^2-1}}\mathfrak{i}_2 - c_{1,0}\frac{i\pi - 2\log\frac{\sqrt{-q_2^2+w_1^2}+w_1}{\sqrt{-q_2^2}}}{16\pi\sqrt{-q_2^2+w_1^2}} - c_{2,0}\frac{i}{16\sqrt{-q_1^2}} + \mathfrak{l}_{w_1}\log\frac{w_1^2}{\mu^2} + \mathfrak{l}_{w_2}\log\frac{w_2^2}{\mu^2} + \mathfrak{l}_q\log\frac{q_1^2}{q_2^2} + \mathfrak{l}_\gamma\frac{\log\left(\sqrt{\gamma^2-1}+\gamma\right)}{\sqrt{\gamma^2-1}} + \mathcal{O}(\epsilon)$$

$$\mathcal{M}^{(1)}_{5,\bar{m}_1^3\bar{m}_2^2,\mathrm{cut}} = \frac{d_{\mathrm{IR}}}{\sqrt{\gamma^2-1}}\times\frac{1}{\tilde{\epsilon}} + c_{q_1}\frac{\log\frac{-q_1^2}{\mu^2}}{\sqrt{\gamma^2-1}} + c_{q_2}\frac{\log\frac{-q_2^2}{\mu^2}}{\sqrt{\gamma^2-1}} + c_{w_1}\frac{\log\frac{w_1^2}{\mu^2}}{\sqrt{\gamma^2-1}} + c_{w_2}\frac{\log\frac{w_2^2}{\mu^2}}{\sqrt{\gamma^2-1}} + c_y^1\frac{\log\left(\gamma^2-1\right)}{\sqrt{\gamma^2-1}} + c_y^2\log\left(\sqrt{\gamma^2-1}+\gamma\right) + \frac{R}{\sqrt{\gamma^2-1}} + \mathcal{O}(\epsilon)$$

- $\hbar$ power-couting parameter [Damour]
- Singular terms in $1/\hbar$ cancel between the amplitude and the cut
- The result must be symmetrised $m_1 \leftrightarrow m_2$

IR divergences exponentiate!

- Shapiro time-delay of the observed radiation [Weinberg]
- The logarithmic drift of the worldlines [Caron-Huot, Giroux, Hannesdottir, Mizera]

- The result is infrared divergent!
- Each rational coefficient has $\sim 10^2 - 10^3$ terms, with spurious poles [Talk by Abreu and Tancredi]
- Many terms are polynomials in $q_1^2$ and $q_2^2$, the Fourier transform will *kill* them

# The problem of spurious poles

## Impact parameter space and the small-velocity limit

$$I_q = \frac{\text{3049 terms}}{q_1^2 q_2^2 \left[(q_1^2)^2 - 2q_1^2 q_2^2 + (q_2^2)^2 + 4q_1^2 w_1^2\right]^4 \left[(q_1^2)^2 w_1^2 + (q_2^2)^2 w_2^2 - 2q_1^2 q_2^2 w_1 w_2 \gamma\right]^2} \sim \frac{1}{p_\infty^{12}}$$

Fake divergences! We need to expand all the variables at $\mathcal{O}(p_\infty^{13})$ to get the waveform at the first relevant order

Partial fractioning and/or finite-field to disentangle divergences.

- [Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov]
- MultivariateApart.wl [Heller, von Manteuffel]
- [Brandhuber, Brown, Chen, SDA, Gowdy, Travaglini]

The problem is not just analytical: the spurious poles sit in the physical region.

$$\hat{\gamma}_1 = \frac{w_1^2 + w_2^2}{2w_1 w_2} > 1 \qquad \hat{\gamma}_2 = \frac{(q_1^2 w_1)^2 + (q_2^2 w_2)^2}{2q_1^2 q_2^2 w_1 w_2} > 1 \qquad \hat{w}_1 = \frac{\left|q_1^2 - q_2^2\right|}{2\sqrt{-q_1^2}} > 0$$

Integration-contour deformation to improve numerical convergence:
- [Herdershee, Roiban, Teng]
- [Bohnenblust, Ita, Kraus, Schlenk]

Introducing a new set of functions which are smooth on the physical sheet.

$$L_q = \frac{1}{\Delta_2^2}\left\{2\log\frac{q_2^2}{q_1^2} + 2\log\frac{w_2}{w_1} - \frac{\left(q_2^2 w_2\right)^2 - \left(q_1^2 w_1\right)^2}{q_1^2 q_2^2 w_1 w_2}\left[\left(1 + \frac{\gamma \Delta_2}{\gamma^2 - 1}\right)\frac{\operatorname{arccosh}\gamma}{\sqrt{\gamma^2 - 1}} - \frac{\Delta_2}{\gamma^2 - 1}\right]\right\}$$

- [Bern, Dixon, Kosower]
- [Bini, Damour, SDA, Geralico, Herdershee, Roiban, Teng]

# The waveform in General Relativity

## The Multipolar-Post-Minkowskian formalism

- [Bini, Damour, Geralico]
- Review: [Blanchet]

- The MPM formalism computes the time-domain waveform as a sum over irreducible multipolar contributions, keyed by their multipole order and their spatial parity:

$$W^{\mathrm{MPM}}(T_r, \theta, \phi) = U_2 + \frac{1}{c}(V_2 + U_3) + \frac{1}{c^2}(V_3 + U_4) + \frac{1}{c^3}(V_4 + U_5) + \frac{1}{c^4}(V_5 + U_6) + \frac{1}{c^5}(V_6 + U_7) + \cdots$$

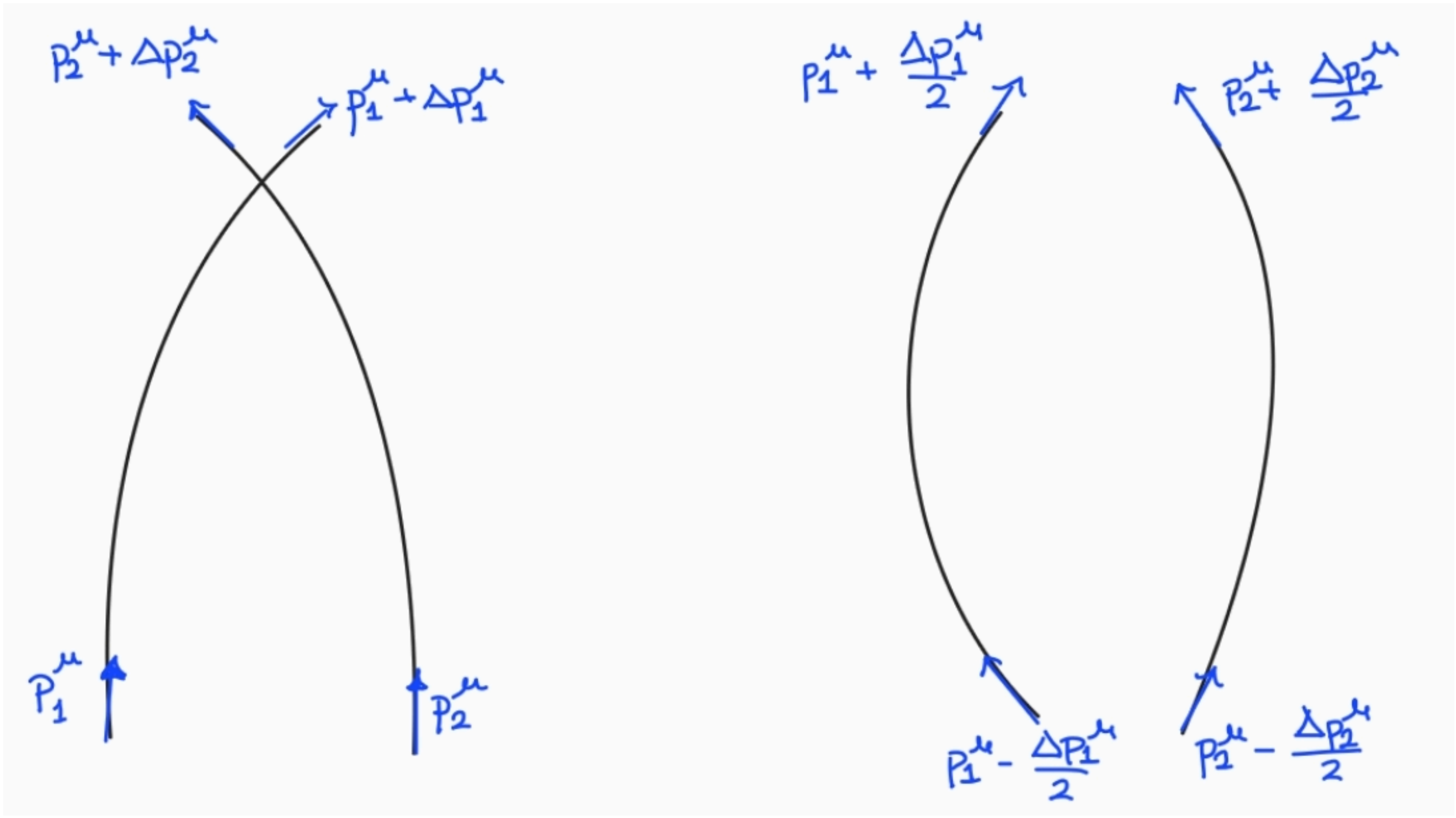
- It computes each radiative multipole moment in terms of the stress-energy tensor of the material source. Each radiative multipole is given by a sum of contributions involving both source variables at a time and hereditary integrals over the past behaviour of the source:

$$U_{ij}(t) = \frac{d^2 I_{ij}(t)}{dt^2} + \frac{2GM}{c^3}\int_0^\infty d\tau I^{(4)}_{ij}(t-\tau)\left(\log\left(\frac{\tau}{2b_0}\right) + \frac{11}{12}\right) + \frac{G}{c^5}\left(\frac{1}{7}I^{(5)}_{a\langle i}I_{j\rangle a} - \frac{5}{7}I^{(4)}_{a\langle i}I^{(1)}_{j\rangle a} - \frac{2}{7}I^{(3)}_{a\langle i}I^{(2)}_{j\rangle a}\right) + \cdots$$

$$I^{\leq 2.5PN}_{ij}(t) = \nu M\left(1 + \frac{a_2}{c^2} + \frac{a_4}{c^4} - \frac{24}{7}\frac{\nu}{c^5}\frac{G^2M^2}{r^2}\dot{r}\right)x^{\langle i}x^{j\rangle} + \ldots$$

# A Tale of Two Formalisms

## Matching KMOC and MPM




- The two computations are set up in “different frames”
  [Bini, Damour, Geralico]

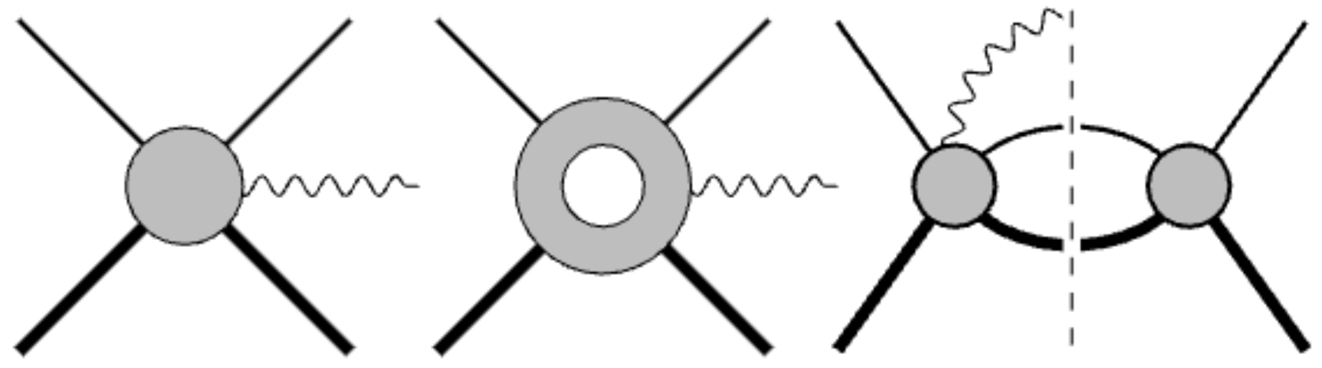
- The rotation $\phi \to \phi + \Delta\chi^{1\text{PM}}/2$ is equivalent to the KMOC cut terms [Georgoudis, Heissenberg, Russo]

- In conventional (or ’t Hooft) dimreg scheme, we need to take into account $\epsilon/\epsilon$ contributions (not in FDH) [Bini, Damour, SDA, Geralico, Herdershee, Roiban, Teng]

# A Tale of Two Formalisms
## Backreaction on the Static Background

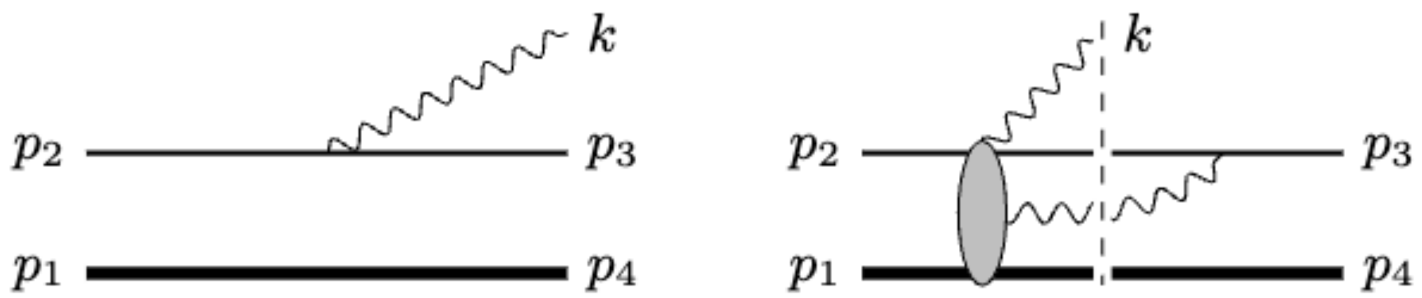
- In MPM, there is a constant-in-time leading contribution. We match it by introducing one-particle graviton emission at zero energy:

$$\mathscr{W}_{\text{const}}(k) \propto \frac{\hat{\delta}(\omega)}{r}\left[m_1\frac{(\varepsilon\cdot u_1)^2}{u_1\cdot n}+m_2\frac{(\varepsilon\cdot u_2)^2}{u_2\cdot n}\right]$$



- We can divide the momentum-space matrix element into three categories:

$$\mathscr{W}(k,q_1,q_2)=\mathscr{W}_{\text{const}}(k,q_1,q_2)+\mathscr{W}_{\text{conn}}(k,q_1,q_2)+\mathscr{M}_{\text{disc}}(k,q_1,q_2)$$

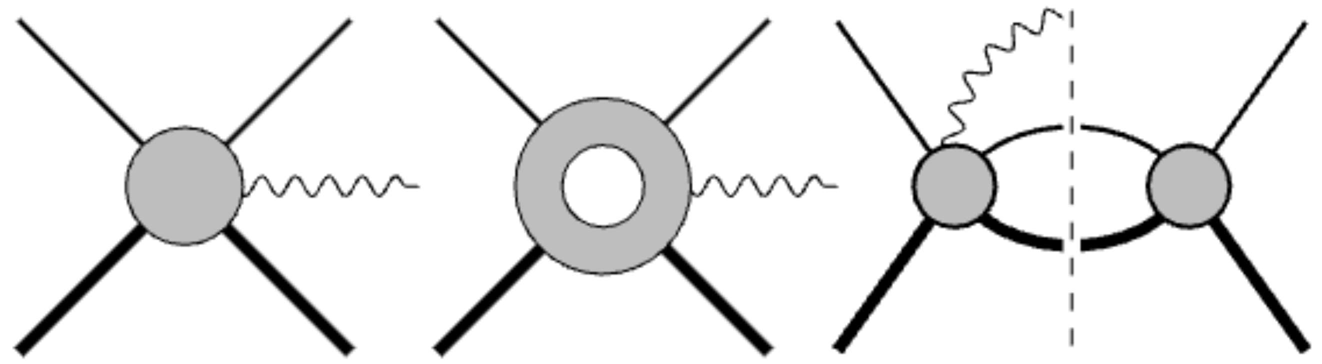


(a) (b)

- The waveform's constant part depends on the choice of the BMS frame.

# A Tale of Two Formalisms

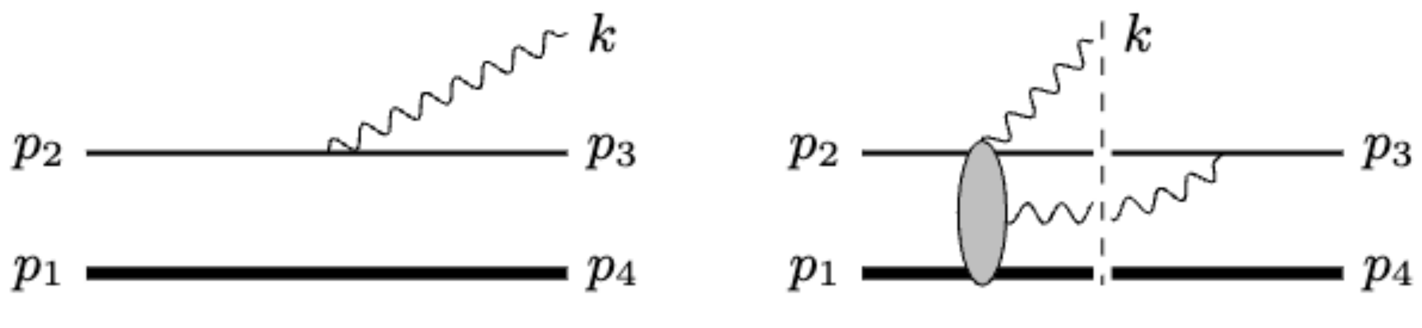
## Backreaction on the Static Background



- If zero-energy gravitons are kept in the spectrum, the three-point amplitude generates time-dependent terms through the cut term.
- Since the zero energy graviton is supported only at the origin of phase space, we must regularize it so that this point, $|\ell| = 0$, remains in the integration domain.
- We find that the cut gives a finite contribution:

$$\mathscr{M}_{\text{disc}} = \mathscr{S}^{1\text{ loop disc}} - i\omega GE\left[\frac{1}{\epsilon} - \log\frac{\beta^2}{\pi}\right]\mathscr{M}^{\text{tree}}$$

$$\mathscr{S}^{1\text{ loop disc}} = iG\left[m_1 w_1 \log\frac{w_1^2}{\omega^2} + m_2 w_2 \log\frac{w_2^2}{\omega^2}\right]\mathscr{M}^{\text{tree}}$$



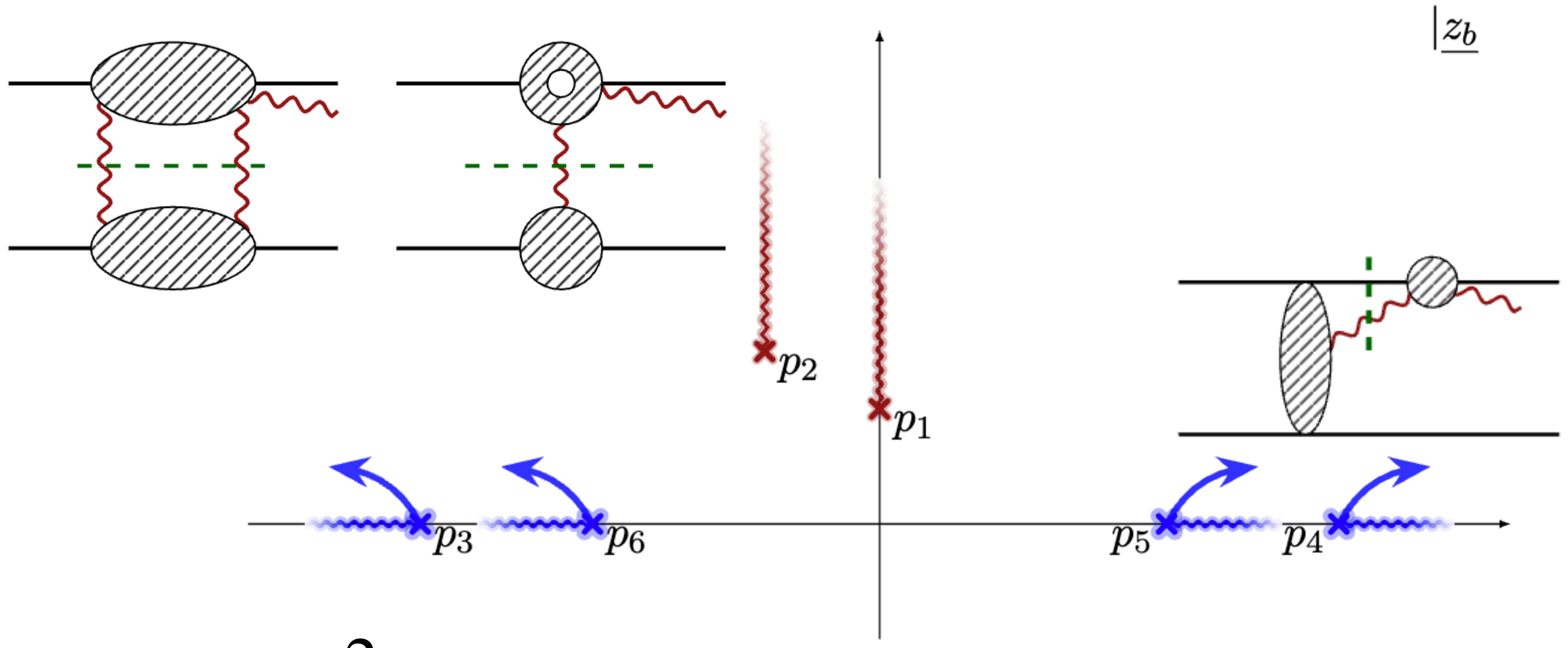
(a) (b)

- The second term can be removed by a finite time shift. The first term $\mathscr{S}^{1\text{ loop disc}}$ coincides with a BMS supertranslation. [Veneziano, Vilkovisky], [Georgoudis, Heissenberg, Russo]

# An Improved Framework for Waveforms [Brunello, SDA]

- From the analytic properties of scattering amplitudes, we isolate long-range interactions.

Generalised unitarity for Fourier + loop integrand, followed by a suitable change of $i\varepsilon$ prescription




At tree level, the $q_i^2$-integration is trivialised (factorisation in the complex $q_i^2$ channels):
- [SDA, Gonzo, Novichkov]

Talk by Andrew McLeod

- We avoid spurious poles in the $q_i^2$ integrals by performing tensor-to-scalar decomposition at the level of Fourier + loop integrals. [Anastasiou, Karler, Vicini]
- We introduce IBP relations for Fourier + loop integrals. The final waveform is a sum of Master Integrals — the Fourier transforms of loop MIs and their derivatives w.r.t. the impact parameter.

$$\mathcal{I}_{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11}} = \int_{\hat{q},\hat{\ell}} e^{D_1} \prod_{i=1}^{11} \left(\frac{1}{D_i^{a_i}}\right)$$

$$\int_{\hat{q},\hat{\ell}} \frac{\partial}{\partial\{\ell^\mu, q^\mu\}} \left( e^{D_1} \frac{v^\mu}{\prod_{i=1}^{11} D_i^{a_i}} \right) = 0$$

$$\mathrm{IBP}[a_1,\dots,a_{11}] + \mathrm{IBP}[a_1-1,\dots,a_{11}] - D_1\,\mathrm{IBP}[a_1,\dots,a_{11}] = 0$$

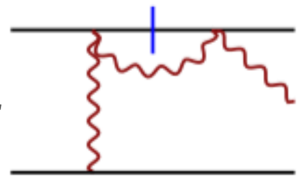
We can use packages developed for loops:
- LiteRed [Lee]
- LiteIBP [Peraro]

# The Fourier integrals [Brunello, SDA]

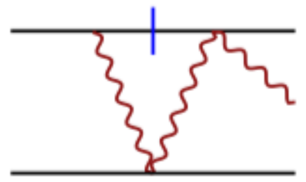
$$\mathscr{F}\left[(-q^2)^\alpha\right] \propto K_{\alpha+\frac{D}{2}-1}\left(\sqrt{-b^2}\,\hat{w}_2\right)$$

$$\frac{i}{16\pi\sqrt{-b^2 p_\infty}}\left\{z\int_0^\infty \mathrm{d}x\left[e^{-z\cosh x}\,\mathbf{H}_{-1}\left(z\sqrt{p_\infty}\sinh x\right)\right] - i\,\frac{e^{-z\sqrt{1+p_\infty}}}{\sqrt{p_\infty}}\right\}$$

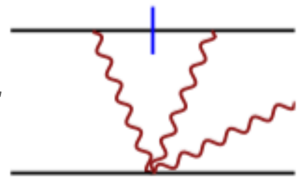
Struve-H function



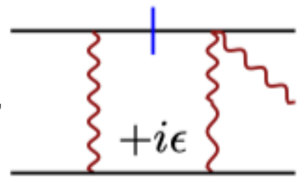
$$= -\frac{i\,w_1}{4\pi(-q_2^2)} + \mathcal{O}\left(\epsilon^1\right),$$



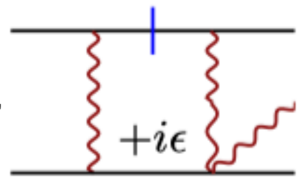
$$= \frac{\pi + 2i\log\frac{w_1+\sqrt{w_1^2-q_2^2}}{\sqrt{-q_2^2}}}{8\pi\sqrt{w_1^2-q_2^2}} + \mathcal{O}\left(\epsilon^1\right),$$



$$= \frac{1}{8\sqrt{(-q_1^2)}} + \mathcal{O}\left(\epsilon^1\right),$$




$$= \frac{i}{8\pi(-q_2^2)\sqrt{\gamma^2-1}}\left(\frac{-q_2^2}{w_1\mu_{\mathrm{IR}}}\right)^{-2\epsilon}\left[\frac{1}{\tilde{\epsilon}} - \log\left(\gamma^2-1\right) - 2\log\left(\gamma+\sqrt{\gamma^2-1}\right) + i\pi\right] + \mathcal{O}\left(\epsilon^1\right)$$




$$= \frac{i}{8\pi(-q_1^2)\sqrt{\gamma^2-1}}\left(\frac{-q_1^2}{w_2\mu_{\mathrm{IR}}}\right)^{-2\epsilon}\left[\frac{1}{\tilde{\epsilon}} - \log\left(\gamma^2-1\right) + i\pi\right] + \mathcal{O}\left(\epsilon^1\right),$$

- Can we write this Fourier transform as iterated integrals?
- Can we compute these integrals using differential equations? [Henn]

# Summary and future directions

- The scattering waveform at NLO in momentum space

- Comparison to MPM results and interesting connections to BMS

- The analytic waveform in impact parameter space

"The fact that the road leading to the present successful EFT/MPM comparison had some bumps, which taught us interesting lessons, is another example of the useful synergy between amplitude-based, and classical perturbation-theory-based, approaches to gravitational physics."

✤ NNLO waveform
- What is the interesting physics we are going to learn?
- Challenge for the QCD community

✤ Peeling? The $1/r$ expansion of the waveform

✤ Resummation in $G$ and comparisons with Numerical Relativity

✳Continuous-spin particles from the on-shell approach [Brando Bellazzini, SDA, Marcello Romano]