Two-loop integrals for the production of two heavy bosons and a jet at the LHC

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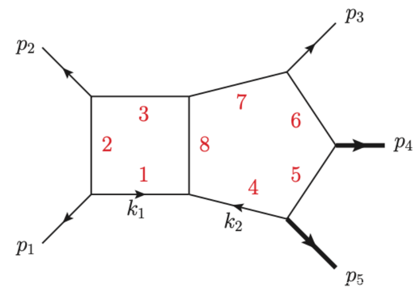
Content

Main goal: compute Feynman integrals to make their analytic structure transparent, and so that we can evaluate them in a stable and efficient way

Focus: Planar Feynman integrals for processes with five external particles, two of them massive, and with massless propagators



- This should be easy and boring!
    - Describing processes with 3 particles in the final state at 2nd order in perturbation theory
    - Functions that appear are the ones we've been saying we understand well for a long time!
    - Five-point 1-mass @ 2 loops was not that easy…
    - … but we have better tools and it actually was simple for five-point two-mass @ 2 loops!
    - First explorations in [2401.07632, Jiang, Liu, Xu, Yang, 24]
- **Bonus:** Double Lagrangian insertions in Wilson loop in $\mathcal{N} = 4$ sYM

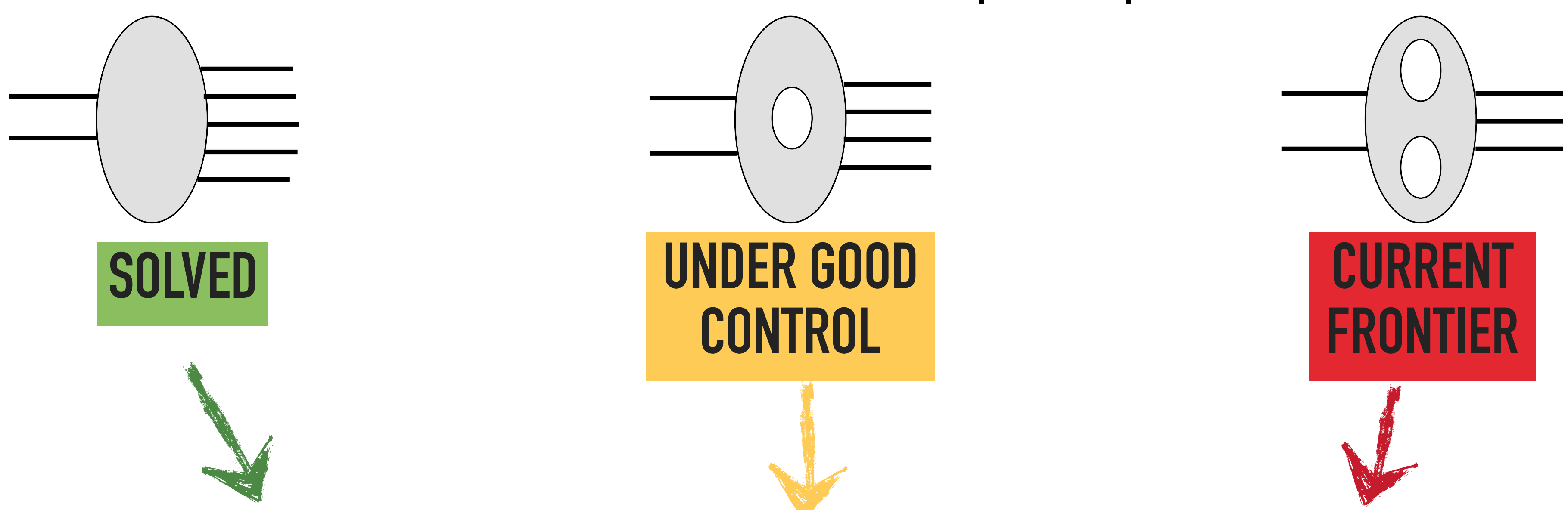
Motivation: Precision!

- Percent-level precision

$$\sigma = \sigma_{LO}\left(1 + \alpha_s\sigma_{NLO} + \alpha_s^2\sigma_{NNLO}\right) + \mathcal{O}(\alpha_s^3)$$

$\alpha_s\sigma_{NLO} \sim \mathcal{O}(10\%)$, $\alpha_s^2\sigma_{NNLO} \sim \mathcal{O}(1\%)$

- Amplitudes for NNLO corrections (five-point processes)



$$\sigma_{NNLO} = \sigma_{RR} + \sigma_{RV} + \sigma_{VV}$$

- Factorisation of work: amplitudes and phase-space integration

$$\sigma \sim \int \mathrm{d}\Phi \, |\mathscr{A}|^2$$

NB: Divergences appear, work in Dimensional Regularisation, $D = 4 \rightarrow D = 4 - 2\epsilon$

[See Chiara's talk for more details]

Amplitudes and Feynman Integrals

- Natural factorisation

$$\mathscr{A} = \sum c_i(\vec{p};\epsilon)\; m_i(\vec{p};\epsilon)$$

Master coefficients
- process/theory specific
- rational functions

Master integrals
- kinematic dependent
- `special' functions

1. Feynman integrals as vector spaces
   - ✓ Integration-by-parts (IBP) relations and master integrals

2. How to compute (multi-scale) Feynman integrals?
   - ✓ Differential equations and pure basis

   } Enough for formal studies, e.g., $\mathcal{N} = 4$ sYM

3. How to (efficiently) evaluate Feynman integrals?
   - ✓ Numerical methods and pentagon functions

   } Non-trivial, required for pheno studies

COMPUTING FEYNMAN INTEGRALS

Feynman Integrals as Vector Spaces: IBP relations

$$I(p_1,\dots,p_E;m_1^2,\dots,m_p^2;\nu;D)=\int\left(\prod_{j=1}^{L}e^{\gamma_E\epsilon}\frac{d^Dk_j}{i\pi^{D/2}}\right)\frac{\mathcal{N}(\{k_j\cdot k_l,k_j\cdot p_l\};D)}{\prod_{j=1}^{p}(m_j^2-q_j^2-i\varepsilon)^{\nu_j}}$$

[Tkachov; Chetyrkin, Tkachov, 81]

$$\int d^Dk_i\frac{\partial}{\partial k_i^\mu}\left[v^\mu\frac{\mathcal{N}(\{k_j\cdot k_l,k_j\cdot p_l\};D)}{\prod_{j=1}^{p}(m_j^2-q_j^2-i\varepsilon)^{\nu_j}}\right]=0$$

- Linear relations of integrals with different ν_j

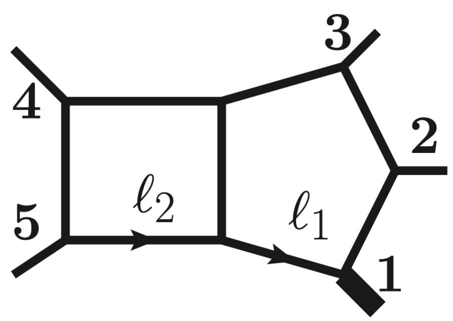
- Integrals in a family related by IBP relations, rational in scales and D
  - Reduce integrals to a set of master integrals
- The number of master integrals is always finite
  - Finite number of integrals needed to solve a family
- Each family defines a (finite dimensional) vector space
  - Like for any vector space, some bases are better than others
- Solved in several public codes
  - Kira, FIRE, NeatIBP, FiniteFlow, Reduze, LiteRed …
- Bottleneck in many applications
  - Only use (partial) analytics when it cannot be avoided
  - Bypass large analytic expressions with numerical evaluations (in finite fields)

Feynman Integrals as Vector Spaces

Example 1: five-point one-mass scattering at two loops ; Planar VS Non-Planar

- ✓ Depend on 6 variables
- ✓ Penta-boxes:

  [2005.04195]



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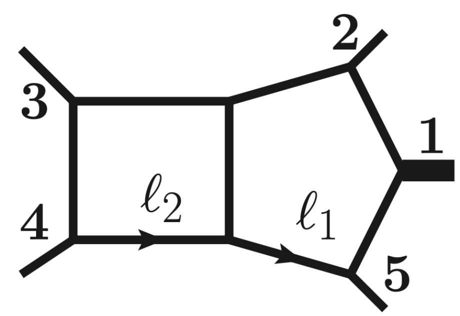

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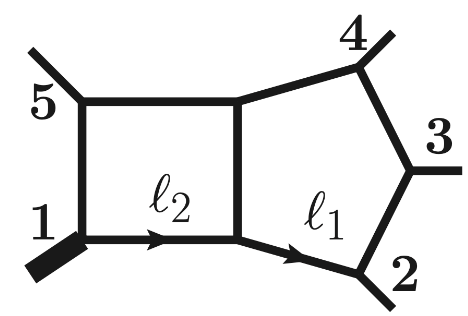

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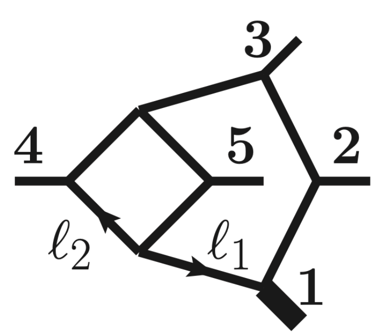
- ✓ Hexa-boxes:

  [2107.14180]



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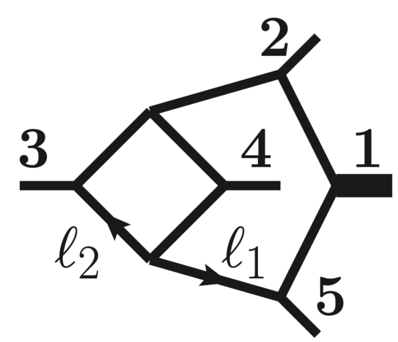

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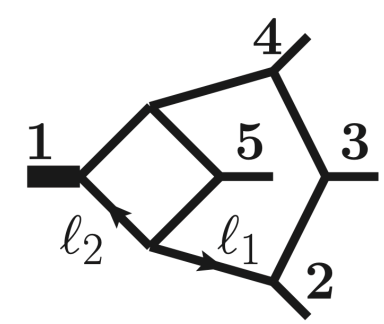

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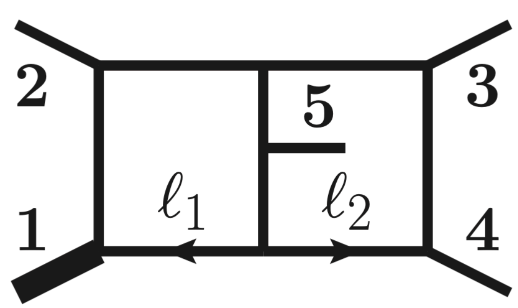
- ✓ Double pentagons:

  [2306.15431]



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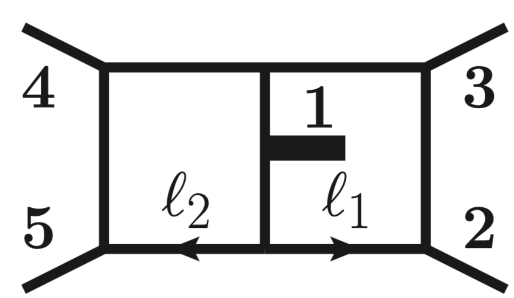

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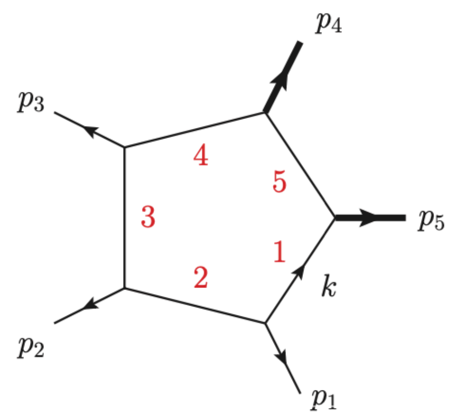
Feynman Integrals as Vector Spaces

Example 2: five-point two-mass scattering ; one VS two loops

- ✓ Depend on 7 variables

[Abreu, Chicherin, Sotnikov, Zoia, to appear]



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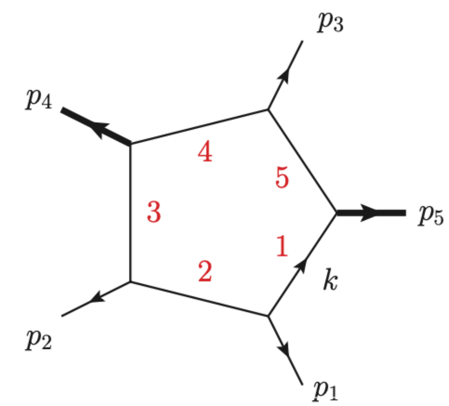

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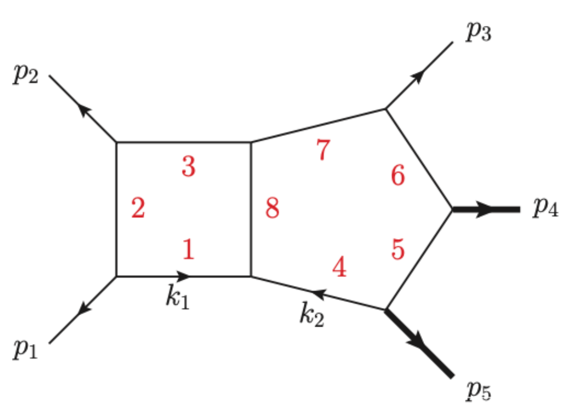

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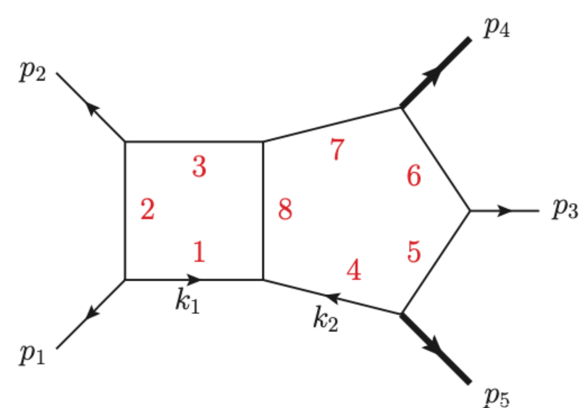

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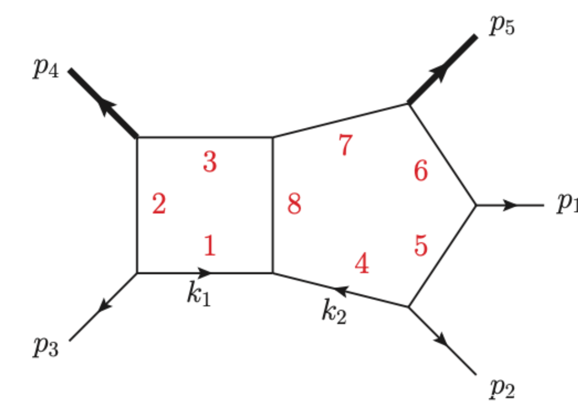

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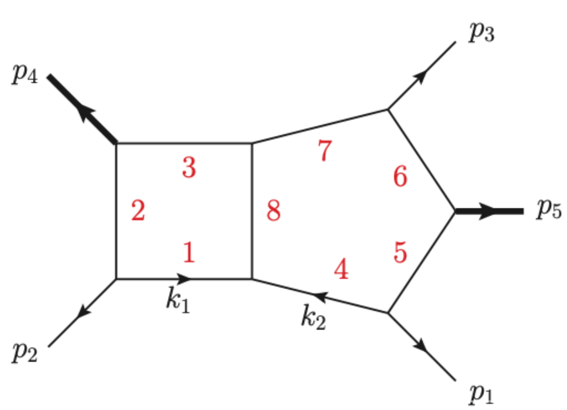

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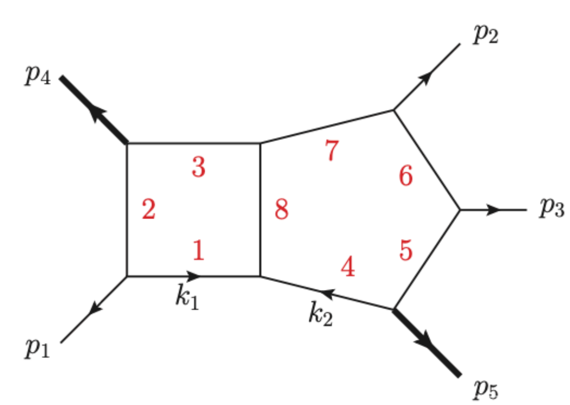

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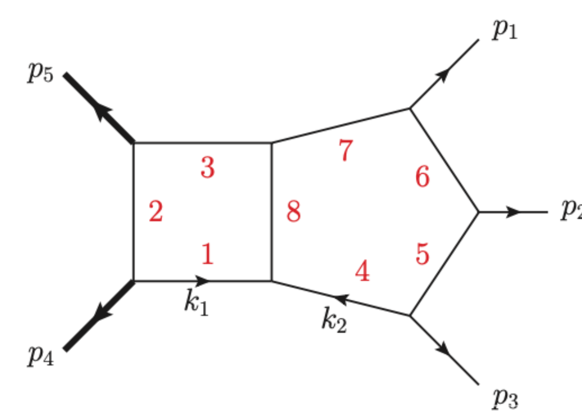

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[2401.07632, Jiang, Liu, Xu, Yang]

Computing Feynman Integrals: Differential Equations

- **Goal**: evaluate integrals around $D = 4$ dimensions (as expansion in ϵ)

- Many ways to compute Feynman integrals
  - ✓ Analytic/numerical integration of parametric representation
  - ✓ Transform into differential equation problem [Kotikov, 91; Bern et al, 94; Remiddi, 97; Gehrmann, Remiddi 00]

- Let $\overrightarrow{\mathcal{I}}$ be a set of master integrals ; it is closed under differentiation

$$\partial_{x_i} \overrightarrow{\mathcal{I}}(x,\epsilon) = A_{x_i}(x,\epsilon)\, \overrightarrow{\mathcal{I}}(x,\epsilon)$$

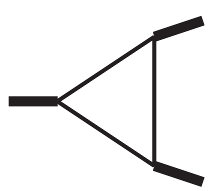
  - ✓ Derivatives change powers of propagators $\Rightarrow$ reduce to masters with IBPs
  - ✓ IBPs are rational in x and $D = 4 - 2\epsilon$ $\Rightarrow A_{x_i}(x,\epsilon)$ has rational entries
  - ✓ For generic $\overrightarrow{\mathcal{I}}$, not clear we gain a lot… but some bases are better than others!

Computing Feynman Integrals: Pure Bases

$$d\vec{\mathcal{J}}(x,\epsilon) = \epsilon\left(\sum_i A_i\, d\log W_i(x)\right)\vec{\mathcal{J}}(x,\epsilon)$$

[Henn, 13]

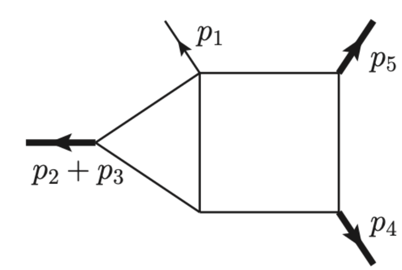
- A_i are matrices of rational numbers, all x dependence in W_i
- W_i give logarithmic singularities/branch cuts: symbol alphabet
- No general algorithm to find a pure basis (automated codes exist, with limitations)
- Leading singularities: this is where square roots appear!

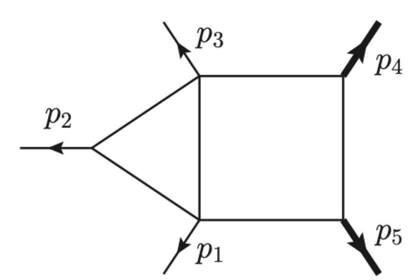


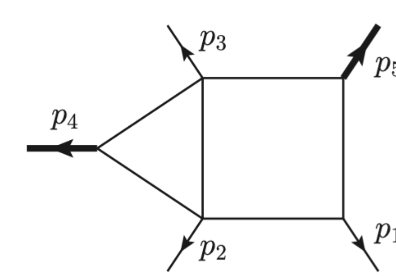
$$\sim \frac{1}{\sqrt{\Delta_3}}\mathcal{T}$$

- ✓ Determine Δ_3 without computing the integral
- ✓ Compute as residue of integrand

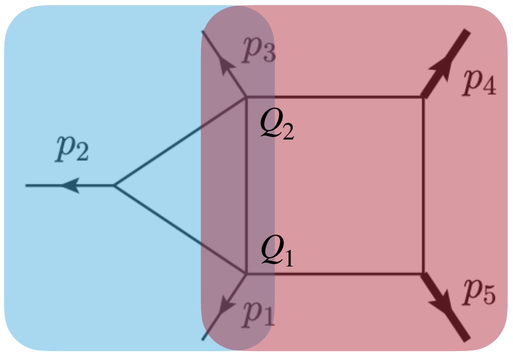
- 44 square roots for 2-loop 5-pt 2mass (10 for 2-loop 5-pt 1m)!
  - ✓ 3-point Gram Δ_3, degree 2: 7 permutations
  - ✓ 5-point Gram Δ_5, degree 4: 1 permutation
  - ✓ 4-point 3-mass root, degree 4: 18 permutations
  - ✓ New degree 4 root: 6 permutations
  - ✓ New degree 4 root: 12 permutations





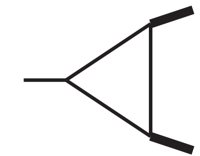
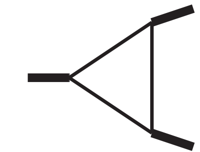
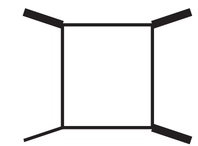
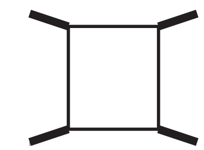


Computing Feynman Integrals: The New Roots



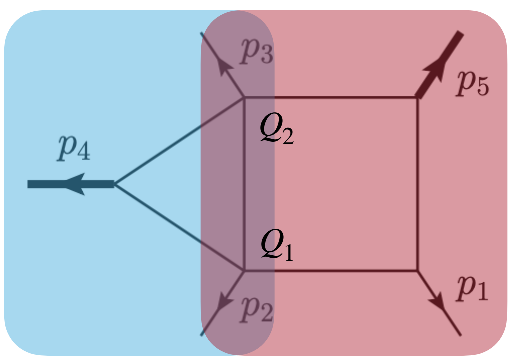
$$= \int d\ell_1 \frac{1}{D_1 D_2} \frac{1}{\sqrt{\Delta_{\square}(\ell_1)}}$$

$$= \int d\ell_2 \frac{1}{D_1 D_2 D_3} \frac{1}{Q_1^2(\ell_2) - Q_2^2(\ell_2)}$$

$\sim \frac{1}{Q_1^2 - Q_2^2}$ $\quad \sim \frac{1}{\sqrt{\Delta_3}}$

$\sim \frac{1}{st - Q_1^2 Q_3^2}$ $\quad \sim \frac{1}{\sqrt{\Delta_{\square}}}$

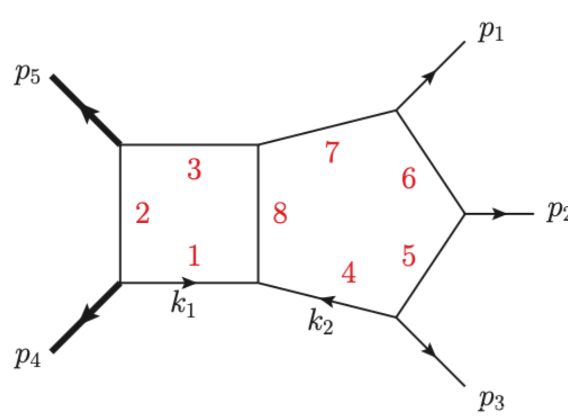


$$= \int d\ell_2 \frac{1}{D_1 D_2 D_3} \frac{1}{\sqrt{\Delta_3(\ell_2)}}$$

$$= \int d\ell_1 \frac{1}{D_1 D_2} \frac{1}{st(\ell_1) - Q_1^2(\ell_1) p_5^2}$$

✓ Need to work a bit harder to compute root…

✦ Side comment: one of the integrals comes with two roots!

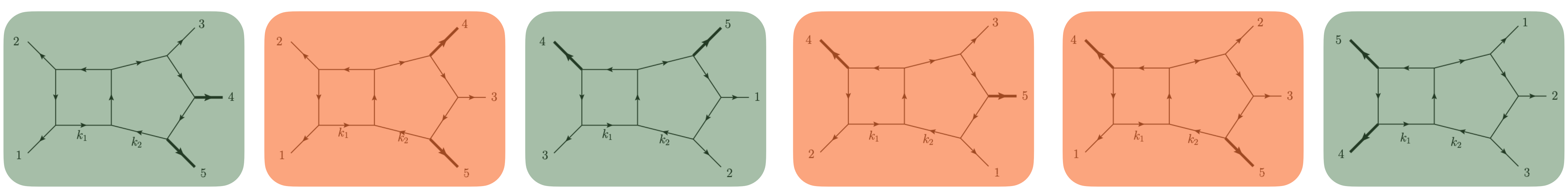


$$\sim \frac{1}{\sqrt{\Delta_3}\sqrt{\Delta_5}}$$

Computing Feynman Integrals: Alphabets and Letters

$$d\vec{\mathcal{J}}(x,\epsilon) = \epsilon\Big(\sum_i A_i \, d\log W_i(x)\Big)\vec{\mathcal{J}}(x,\epsilon)$$

- Getting diff. eq. relies on IBPs: difficult to do analytically…
- If the W_i are known, determine the A_i from numerical IBPs!
  - ✓ removes the IBP bottleneck, allows to attack multi-scale problems
- The W_i give singularities of Feynman integrals ⇒ Landau conditions
  - ✓ Factorisation of work: determine W_i without computing the differential equation!
  - ✓ Active area of research in Amplitudes area: coactions, solving Landau conditions, principal A-determinants, Gram determinants, Schubert problem, …
  - ✓ Two highlights: **[2311.14669, Fevola, Mizera, Telen]**, **[2401.07632, Jiang, Liu, Xu, Yang, 24]**
- Baikovletter [2401.07632] misses one of the new five-point roots
  - ✓ Not really an issue, we know it's there



Computing Feynman Integrals: Symbol Alphabet

$$d\vec{\mathcal{J}}(x,\epsilon) = \epsilon \left(\sum_i A_i \, d\log W_i(x) \right) \vec{\mathcal{J}}(x,\epsilon)$$

[Abreu, Chicherin, Sotnikov, Zoia, to appear]

family	dim(fam)	family	dim(fam)
Pa	16	PBmzz	105
Pb	15	PBzmz	104
PBmmz	94	PBzzm	104
PBmzm	87	PBzzz	127

Table 1: Number of master integrals in each family

family	dim($\mathcal{A}_{\text{fam}}$)	family	dim($\mathcal{A}_{\text{fam}}$)
Pa	43	PBmzz	80
Pb	39	PBzmz	96
PBmmz	85	PBzzm	82
PBmzm	52	PBzzz	104

Table 2: Dimension of the alphabet for each family

- ✓ Overall, 570 independent letters for planar two-loop five-point two-mass kinematics
- ✓ Even letters (215): polynomials/rational functions in the kinematic variables
- ✓ Odd letters in one square root (236): $W = \dfrac{P(\vec{s}) + Q(\vec{s})\sqrt{\Lambda(\vec{s})}}{P(\vec{s}) - Q(\vec{s})\sqrt{\Lambda(\vec{s})}}$
  - ‣ in this case, there are 44 different $\Lambda(\vec{s})$
- ✓ Odd letters in two square roots (119): $W = \dfrac{P(\vec{s}) + Q(\vec{s})\sqrt{\Lambda_1(s)}\sqrt{\Lambda_2(s)}}{P(\vec{s}) - Q(\vec{s})\sqrt{\Lambda_1(s)}\sqrt{\Lambda_2(s)}}$
- ✓ Most letters from `Baikovletter`, others (mostly odd) we determine ourselves

[Heller, von Manteuffel, Schabinger, 20]

[Abreu, Ita, Page, Tschernow, 20]

EVALUATING FEYNMAN INTEGRALS

Evaluating Feynman Integrals: Initial Condition

$$d\vec{\mathcal{J}}(x,\epsilon) = \epsilon\Big(\sum_i A_i\, d\log W_i(x)\Big)\vec{\mathcal{J}}(x,\epsilon)$$

- General solution singular at all $W_i = 0$ but Feynman integrals are not
  - Imposing this condition allows to determine the initial condition!

Used for 5pt 1m @ 2loops, [Abreu, Ita, Moriello, Page, Tschernow, Zeng, 20, 21]

- AMFlow approach: [Liu, Ma, 22]
  - Go to (non-physical) limit where all integrals become tadpoles, known to 5 loops
  - Evolve back to physical points — Used for 5pt 1m @ 2loops, [Abreu et al, 23]
  - Obtain high-precision ($\mathcal{O}(100)$ digits) numerical evaluation at random point

- In our case: Euclidean/physical-region initial conditions $\{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, s_4, s_5\}$

$$X_{\text{eu}} = \left(-\frac{3}{2}, -3, -\frac{57}{8}, -\frac{23}{4}, -\frac{5}{8}, -11, -1\right) \qquad X_0 = (7, -1, 2, 5, -2, 1, 1)$$

  - 80 digits evaluations (took ~ 1 week). Sufficient for pentagon functions

Evaluating Feynman Integrals: Solving the DEs

- Trivial solution in terms of Chen iterated integrals, order by order in ϵ

$$[W_{i_1}, \ldots, W_{i_w}]_{\vec{s}_0}(\vec{s}) = \int_\gamma [W_{i_1}, \ldots, W_{i_{w-1}}]_{\vec{s}_0} \mathrm{dlog}\, W_{i_w}$$

  - γ connects $\vec{s}_0$ and $\vec{s}$; $[]_{\vec{s}_0} \equiv 1$
  - ✓ Formal solution, not trivial to evaluate…

- Numerical solution [Moriello, 19 ; Hidding, 20; Armadillo et al, 22; Liu, Ma, 22]
  - ✓ Start from known initial condition, and evolve along path
  - ✓ Generalised power-series solution with finite convergence radius

$$\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{N_{i,k}} \mathbf{c}_k^{(i,j_1,j_2)} (t - t_k)^{\frac{j_1}{2}} \log(t - t_k)^{j_2}$$

  - ✓ High-precision, but slow…

- Write solution in terms of special functions (multiple polylogarithms, …) …

For planar 5pt 1m @ 2loops, [Canko, Kardos, Papadopoulos, Smirnov, Syrrakos, Wever 20-22]

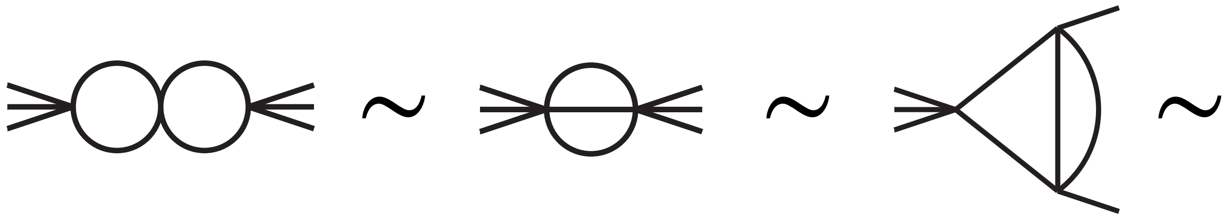
- Roots make it hard/impossible, and not the most convenient representation
  - ✓ Introduces spurious singularities
  - ✓ complicated branch cut structure means expression only valid in small region

Evaluating Feynman Integrals: Pentagon Functions

- Master integrals are linearly independent *before expansion in ϵ*

[Gehrmann, Henn, Lo Presti, 18]
[Chicherin, Sotnikov, 20]

- After expansion in ϵ, there are new relations:



$$\sim r_0 + r_1\,\epsilon\,\ln(s) + r_2\,\epsilon^2\,\ln^2(s) + \ldots$$

- Make relations explicit: build basis of special functions at each order in ϵ

- Improved algorithm for two-loop five-point one-mass processes

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 23]

1. Solve in terms of Chen iterated integrals, order by order in ϵ

$$[W_{i_1}, \ldots, W_{i_w}]_{\vec{s}_0}(\vec{s}) = \int_\gamma [W_{i_1}, \ldots, W_{i_{w-1}}]_{\vec{s}_0} \mathrm{dlog}\, W_{i_w}$$

   ✓ Simple algebra for Chen iterated integrals (with dlog kernels)!

2. Choose components of Feynman integrals as pentagon functions

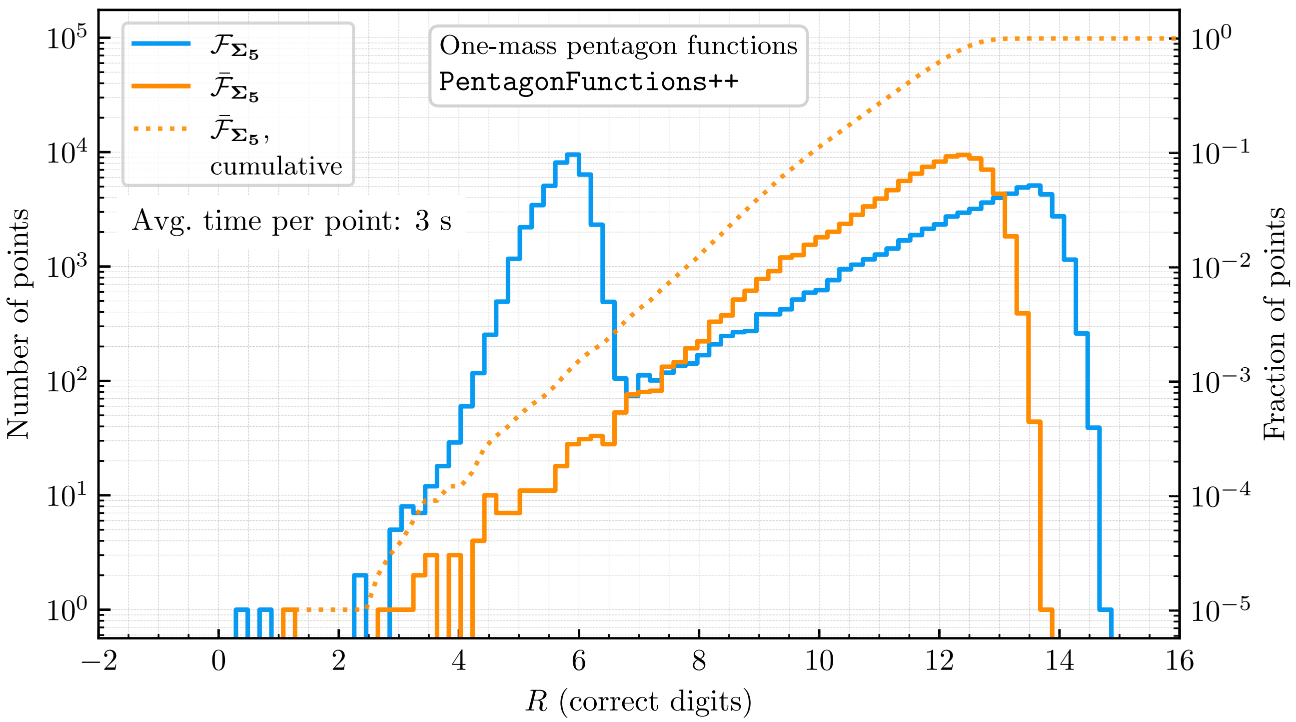
3. Use \`symbol technology' to write all integrals in terms of basis

4. Implement in C++ code

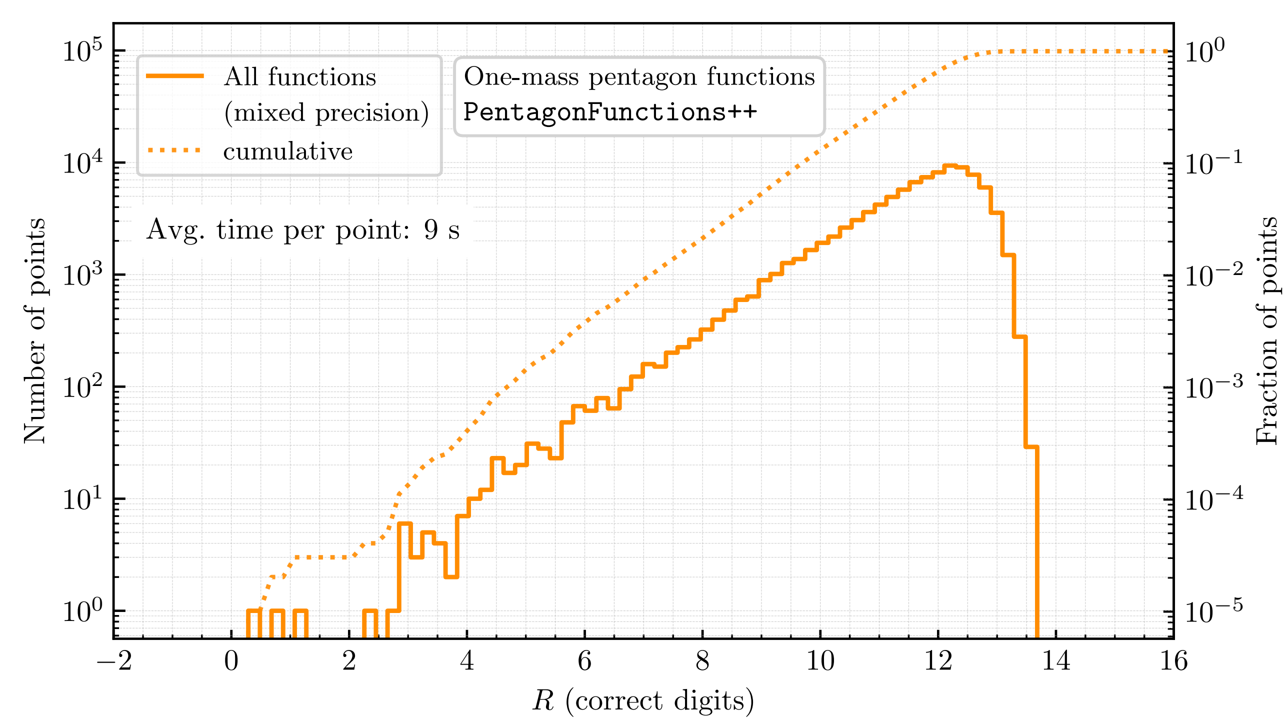
Evaluating Feynman Integrals: Pentagon Functions

Five-point one-mass scattering at two loops

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 23]



- Standard precision, no rescue system
- Precision loss because of square root



- With rescue system
- Easy to implement if good control of analytic structure

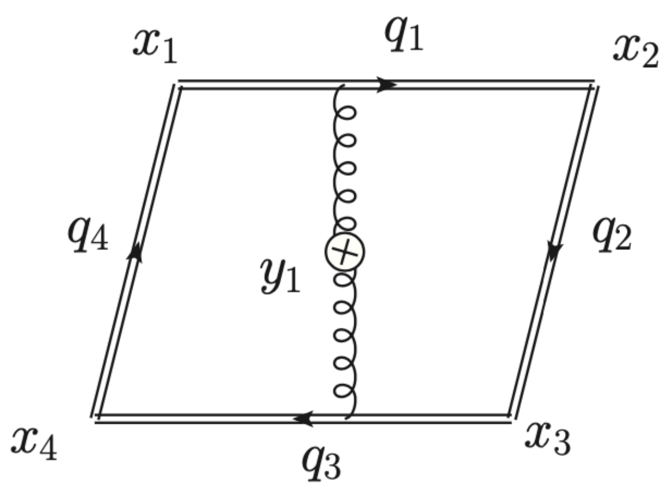
- Ready for phenomenological applications!

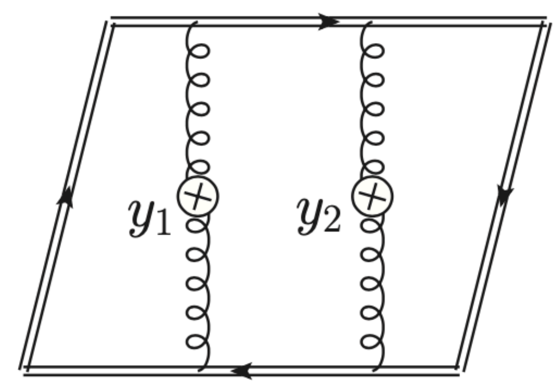
DOUBLE LAGRANGIAN INSERTIONS IN WILSON LOOP IN $\mathcal{N} = 4$ SYM

Lagrangian insertions in Wilson loop

$$F_l(x_1,\ldots,x_4;y_1,\ldots,y_l) = \frac{\pi^{2l}}{<W_\mathrm{F}>} < W_\mathrm{F}\,\mathscr{L}(y_1)\ldots\mathscr{L}(y_l) >$$

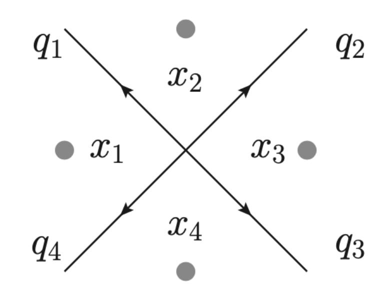
e.g., [Eden, Schubert, Sokatchev, 00]





- $l=1$: massless four-point kinematics

[Alday, Buchbinder, Tseytlin, 11]
[Alday, Heslop,Sikorowski, 12]
[Alday, Henn,Sikorowski, 13]



$$F_{l=1}(x_1,\ldots,x_4;x_0) = \frac{x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} \sum_{L\geq 0} (g^2)^{1+L} \tilde{F}^{(L)}_{l=1}\left(z = \frac{x_{24}^2 x_{10}^2 x_{30}^2}{x_{13}^2 x_{20}^2 x_{40}^2}\right)$$

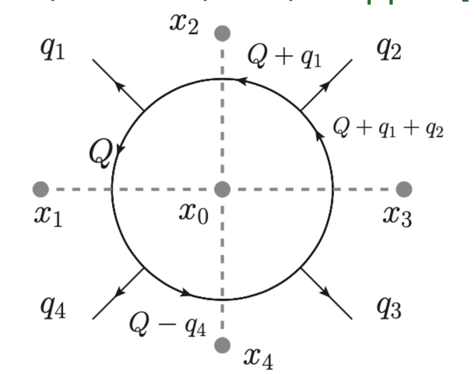
- ✓ Positivity conjecture: tested to $L=3$

[Arkani-Hamed, Henn, Trnka, 21]

$$(-1)^{L+1}\tilde{F}^{(L)}_{l=1}(z) > 0$$

Lagrangian insertions in Wilson loop

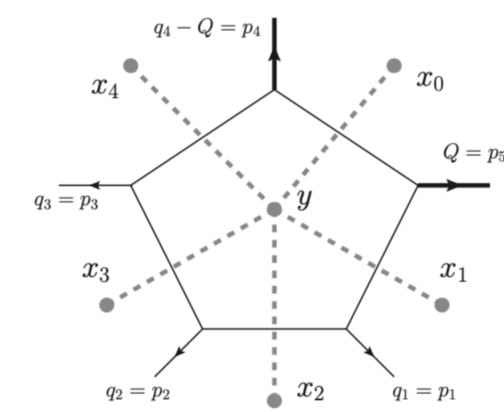
[Abreu, Chicherin, Sotnikov, Zoia, to appear]



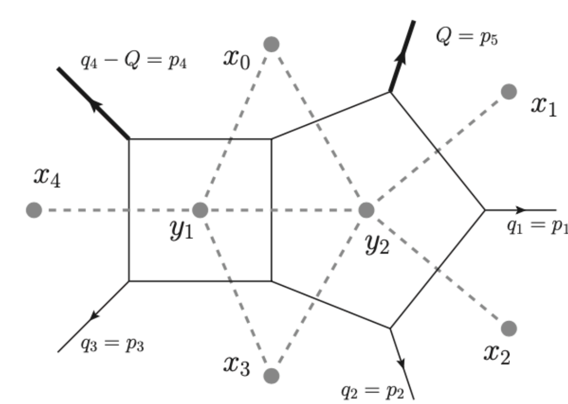
- $l = 2$: (degenerate) five-point two-mass kinematics

$$F_{l=2}(x_1,\ldots,x_4;x_0,x_{0'}) = \frac{x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}\frac{x_{13}^2 x_{24}^2}{x_{10'}^2 x_{20'}^2 x_{30'}^2 x_{40'}^2}\sum_{L\geq 0}(g^2)^{2+L}\tilde{F}^{(L)}_{l=2}(\mathbf{z})$$

  - ✓ $L = 0$: rational function $\quad \tilde{F}^{(0)}_{l=2} = \dfrac{1}{x_{13}^2 x_{24}^2}\left[x_{13}^2(x_{20}^2 + x_{40}^2) + x_{24}^2(x_{10}^2 + x_{30}^2)\right]$



  - ✓ $L = 1$: weight 2 function $\quad \tilde{F}^{(1)}_{l=2}(\mathbf{z}) = \sum_{i=1}^{7} r_i(\mathbf{z}) f_i^{(1)}(\mathbf{z})$
  - ✓ $L = 2$: weight 4 function $\quad \tilde{F}^{(2)}_{l=2}(\mathbf{z}) = \sum_{i=1}^{64} r_i(\mathbf{z}) f_i^{(2)}(\mathbf{z})$



- Finite and pure results: tests our integrals

- New conjecture (in kinematic region defined by amplituhedron): $\quad (-1)^L \tilde{F}^{(L)}_{l=2}(\mathbf{z}) > 0$
  - ✓ Holds very non-trivially for $L = 1$, working on $L = 2$ check

SUMMARY AND OUTLOOK

Summary and Outlook

- We have mature tools that allow us to push the state of the art
  - ✓ Pheno-ready integrals available for 5pt massless and 5pt one-mass processes
  - ✓ Progress in two-loop five-point two-mass processes was much faster
- New results obtained with pheno in mind leading to new formal studies
  - ✓ Lagrangian insertions in Wilson loop
- Are pentagon functions actually a good basis?
  - ✓ We know that they are not at one loop
  - ✓ Include rational factors to make them have better behaved limits
- New challenges ahead: what if singularities are not all dlogs?
  - ✓ Elliptic integrals and beyond!
  - ✓ A lot of developments, but still missing heavy machinery

THANK YOU!