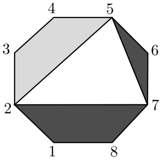
The magic number conjecture for the $m = 2$ amplituhedron and Parke-Taylor identities

Lauren K. Williams, Harvard



Based on: `arXiv:2404.03026`,

joint with Matteo Parisi, Melissa Sherman-Bennett, and Ran Tessler

Outline

- Tricolored subdivisions and partial cyclic orders
- Applications to Parke-Taylor identities and Parke-Taylor polytopes
- What is the amplituhedron?
- Magic number conjecture for the amplituhedron
- Proof of Magic number conjecture when $m = 2$

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Tricolored subdivisions and cyclic orders

- A *tricolored subdivision* τ of an n-gon is a subdivision of the polygon into smaller polygons (black, grey, or white) in which every edge connects two vertices of the n-gon.
- From each τ, can read off a *cyclic order* C_τ (is a cyclic analogue of partial order). To get C_τ from τ, read vertices of white (resp black) polygons clockwise (resp counterclockwise), and ignore the grey.
- The C_τ from our example requires that $(2,5,7)$, $(5,7,6)$, and $(1,8,7,2)$ are circularly ordered.
- A *circular extension* of C_τ is a total circular order compatible with C_τ. E.g. one circular extension of our example is: $(2,5,1,8,7,6,3,4)$.

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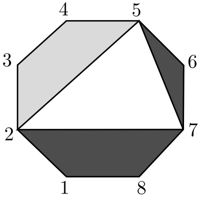

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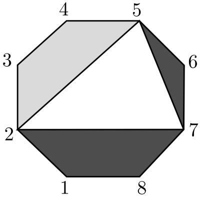

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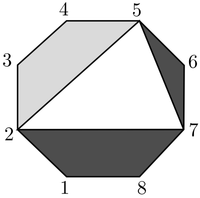

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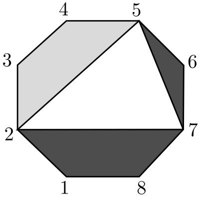

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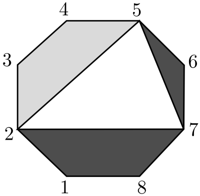

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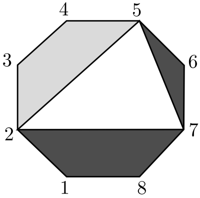

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The **Grassmannian** $Gr_{k,n}(\mathbb{C}) := \{V \mid V \subset \mathbb{C}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}$ by a full-rank $k \times n$ matrix C.

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(C)$ is the minor of the $k \times k$ submatrix of C in column set I.

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Grassmannian identities from tricolored subdivisions

- Given a permutation $w = w_1 \dots w_n$, define the *Parke-Taylor function*

$$\mathrm{PT}(w) := \frac{1}{P_{w_1 w_2} P_{w_2 w_3} \dots P_{w_n w_1}},$$

where the P_{ij} are Plücker coordinates on the Grassmannian $\mathrm{Gr}_{2,n}^{\circ}$.

We get the following identity.

Theorem (Parisi–ShermanBennett–Tessler–W)

Let τ be a tricolored subdivision with at least one grey polygon, and let C_τ be the cyclic partial order. Then

$$\sum_{w} \mathrm{PT}(w) = 0,$$

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The circular extensions of C_τ are $(1234), (1243), (1423)$,
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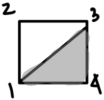

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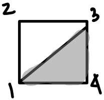
Example: 

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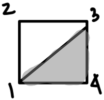

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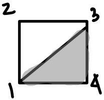

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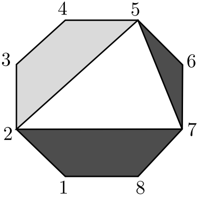
Tricolored subdivisions and Parke-Taylor polytopes



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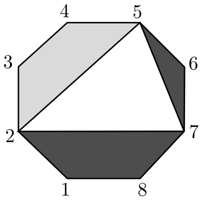

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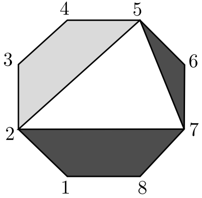

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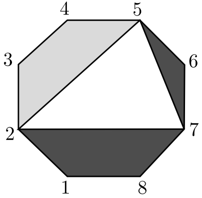

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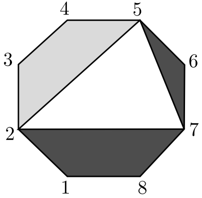

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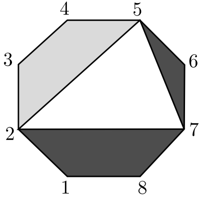

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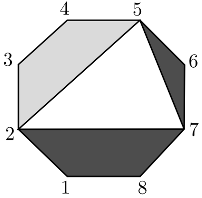

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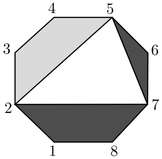

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Theorem (Parisi–Sherman-Bennett–Tessler–W.)

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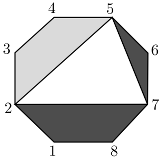

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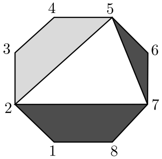

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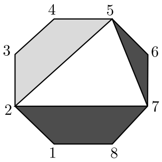

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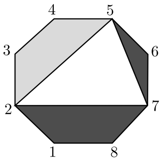

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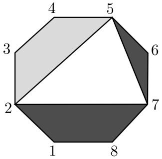

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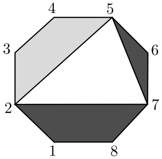

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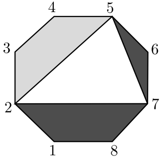

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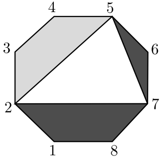

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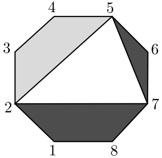

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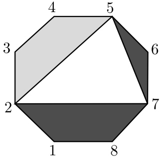

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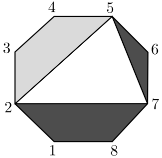

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Recall: the **Grassmannian** $Gr_{k,n}(\mathbb{C}) := \{V \mid V \subset \mathbb{C}^n, \dim V = k\}$
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$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(C)$ is the minor of the $k \times k$ submatrix of C in column set I.

The *matroid* associated to $C \in Gr_{k,n}$ is $\mathcal{M}(C) := \{I \in \binom{[n]}{k} \mid p_I(C) \neq 0.\}$

Gelfand-Goresky-MacPherson-Serganova '87 introduced the *matroid stratification* of $Gr_{k,n}$.

Given $\mathcal{M} \subset \binom{[n]}{k}$, let $S_{\mathcal{M}} = \{C \in Gr_{k,n} \mid p_I(C) \neq 0 \text{ iff } I \in \mathcal{M}\}$.

Matroid stratification: $Gr_{k,n} = \sqcup_{\mathcal{M}} S_{\mathcal{M}}$.

However, the topology of matroid strata is terrible –
Mnev's *universality theorem* (1987).

What is the positive Grassmannian?

Background: 1994 Lusztig total positivity for G/P, 1997 Rietsch, 2006 Postnikov preprint on *totally non-negative* (TNN) or "positive" Grassmannian.

Let $Gr_{k,n}^{\geq 0}$ be subset of $Gr_{k,n}(\mathbb{R})$ where Plucker coords $p_I \geq 0$ for all I.

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(Postnikov, see also Rietsch) If $S_{\mathcal{M}}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. So we have *positroid cell decomposition*

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- the recurrence of Britto–Cachazo–Feng–Witten (2005) expresses scattering amplitudes as sums of rat'l functions of momenta. Indiv terms have "spurious poles" – singularities not present in amplitude.
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- If $m = n - k$, $\mathcal{A}_{n,k,m}(Z) = Gr_{k,n}^{\geq 0}$.
- If $k = 1$ and $m = 2$, $\mathcal{A}_{n,k,m} \subset Gr_{1,3}$ is equivalent to an n-gon in $\mathbb{RP}^2$.
- For $k = 1$ and general m, get cyclic polytope in $\mathbb{RP}^m$.

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What is the amplituhedron?

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ (max minors > 0).

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Have $Gr_{k,n}^{\geq 0} = \sqcup_\pi S_\pi$ cell complex, and $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ a continuous surjective map onto km-dim'l amplituhedron $\mathcal{A}_{n,k,m}(Z)$.

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We will work with all-Z tilings, coming from collections of cells that give tilings for all Z.

Motivation:
the "volume" of the amplituhedron computes scattering amplitudes;
AH-T conjectured that certain "BCFW cells" give a tiling of $\mathcal{A}_{n,k,4}(Z)$;
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Tilings of the amplituhedron

Tilings have been studied in special cases. Their cardinalities are interesting!

special case	cardinality of tiling of $\mathcal{A}_{n,k,m}$	explanation
$m = 0$ or $k = 0$	1	$\mathcal{A}$ is a point
$k + m = n$	1	$\mathcal{A} \cong \mathrm{Gr}_{k,n}^{\geq 0}$
$m = 1$	$\binom{n-1}{k}$	Karp-W.
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$m = 4$	$\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$	AH-T, EZ–L–T, EZ–L–P–SB–T–W
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Observation (Karp-Zhang-W)

$$\text{Let } M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}.$$

All known tilings of $\mathcal{A}_{n,k,m}$ for even m have cardinality $M(k, n-k-m, \frac{m}{2})$. Call this prediction the *Magic Number Conjecture*.

Remark: Consistent with results for $m=2, m=4, k=1$. Symmetries!

The number $M(a,b,c)$ counts: (In figure, $a,b,c = 2,4,3$.)

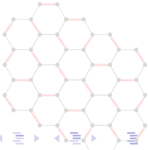
noncrossing lattice paths

plane partition

3	3	2	2
1	1	1	

rhombic tiling

perfect matching



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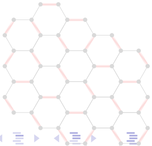

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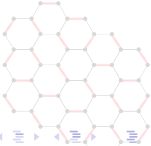

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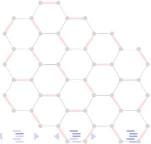

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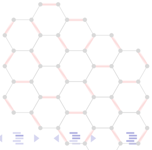

Tilings of the amplituhedron

Observation (Karp-Zhang-W)

Let
$$M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}.$$

All known tilings of $\mathcal{A}_{n,k,m}$ for even m have cardinality $M(k, n-k-m, \frac{m}{2})$. Call this prediction the *Magic Number Conjecture*.

Remark: Consistent with results for $m=2$, $m=4$, $k=1$. Symmetries!

The number $M(a,b,c)$ counts: (In figure, $a,b,c = 2,4,3$.)

noncrossing lattice paths

plane partition

3	3	2	2
1	1	1	

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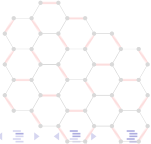

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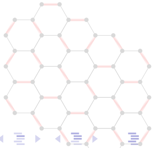

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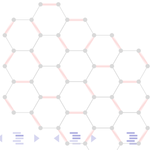

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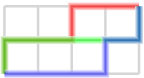

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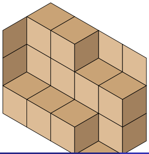

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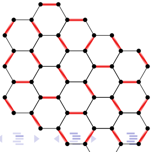

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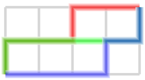

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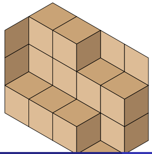

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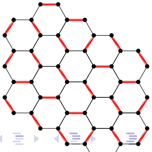

The magic number theorem for the $m = 2$ amplituhedron

Magic Number Theorem (P–SB–T–W)

All tilings of ampl. $\mathcal{A}_{n,k,2}(Z)$ have size $M(k, n-k-2, 1) = \binom{n-2}{k}$.

$k = 1$: Thm says that all triangulations of an n-gon have size $n-2$.

Ideas of the proof:

- There is a classification of tiles for the $m = 2$ amplituhedron using *bicolored subdivisions* (P–SB–W).
- Just as each Parke-Taylor polytope has a decomposition into w-simplices where w ranges over certain circular extensions, each tile has a decomposition into "w-chambers" where w ranges over certain circular extensions.
- Use above decompositions to define the P-T function of $\mathcal{A}_{n,k,2}(Z)$ and each tile, and show that this function is the same for ALL tiles.
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Tiles of the amplituhedron

Recall: $\overline{\tilde{Z}(S_\pi)}$ is a *tile* for $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ if $\tilde{Z}$ is injective on km-dim'l cell S_π. Lukowski–Parisi–Spradlin–Volovich conjectured:

Theorem (Parisi–Sherman-Bennett–W)

The tiles for $\mathcal{A}_{n,k,2}(Z) \leftrightarrow$ collections of **bicolored** subdivisions of an n-gon with total "area" k. To construct the cell S_π:

- Choose triangulation of black polygons into k black triangles.
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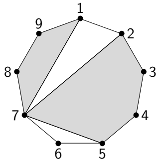

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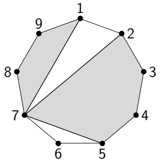

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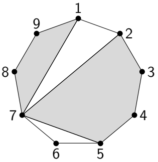

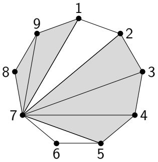

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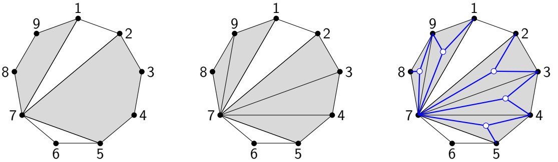


$$
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
* & 0 & 0 & 0 & 0 & 0 & * & 0 & * \\
0 & * & * & 0 & 0 & 0 & * & 0 & 0 \\
0 & 0 & * & * & 0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & * & * & 0 & * & 0 & 0
\end{bmatrix}
\end{array}
$$

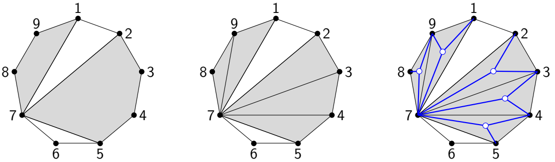
Tiles of the amplituhedron

Recall: $\overline{\tilde{Z}(S_\pi)}$ is a *tile* for $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ if $\tilde{Z}$ is injective on km-dim'l cell S_π. Lukowski–Parisi–Spradlin–Volovich conjectured:

Theorem (Parisi–Sherman-Bennett–W)

The tiles for $\mathcal{A}_{n,k,2}(Z) \leftrightarrow$ collections of **bicolored** subdivisions of an n-gon with total "area" k. To construct the cell S_π:

- Choose triangulation of black polygons into k black triangles.
- Put white vertex in every black triangle, connected to three vertices.
- Elements of S_π are the $k \times n$ *Kasteleyn matrices* with rows/columns indexed by the white and black vertices.



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Chambers of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$

Let $Z \in \text{Mat}_{n,k+2}^{>0}$. Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+2}$ sending $C \mapsto CZ$.
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- Inspired by matroid stratification, we define the *amplituhedron sign stratification* – decompose $\mathcal{A}_{n,k,2}(Z)$ into pieces based on the signs of twistor coordinates. (Parisi–Sherman-Bennett–W.; Karp-W.)
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The Magic Number Theorem for $\mathcal{A}_{n,k,2}(Z)$

- Given any region R of $\mathcal{A}_{n,k,2}(Z)$ that admits a tiling, we define its *weight function*

$$\Omega(R) := \sum \mathrm{PT}(\Delta^{Z}_{(w)}),$$

where the sum is over all w-chambers $\Delta^{Z}_{(w)} \subset R$.

- We prove that for any tile Z_τ of $\mathcal{A}_{n,k,2}(Z)$,

$$\Omega(Z_\tau) = (-1)^k \, \mathrm{PT}(\mathbf{I}_n),$$

where $\mathbf{I}_n$ is the identity permutation.

- It is known that there is a tiling of $\mathcal{A}_{n,k,2}(Z)$ consisting of $\binom{n-2}{k}$ tiles, so $\Omega(\mathcal{A}_{n,k,2}(Z)) = (-1)^k \binom{n-2}{k} \mathrm{PT}(\mathbf{I}_n)$.
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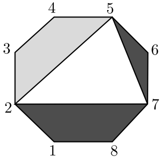

- The magic number conjecture for the $m = 2$ amplituhedron and Parke-Taylor identities `arXiv:2404.03026`, joint with Matteo Parisi, Melissa Sherman-Bennett, and Ran Tessler.
- "The $m = 2$ amplituhedron and the hypersimplex: signs, clusters, triangulations, Eulerian numbers, `arXiv:2104.08254`, joint with Matteo Parisi and Melissa Sherman-Bennett.