Amplitudes Conference

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$w_{1+\infty}$ Symmetry in 4D Gravitational Scattering

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2312.08957 with Monica Pate

Review: Soft Graviton Theorems as Symmetries

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Soft graviton theorems are equivalently Ward identities for symmetries of the S-matrix

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$$\lim_{\omega \to 0} \omega \langle \text{out} | a_+(\omega \hat{q}) \mathcal{S} | \text{in} \rangle = S^{(-1)} \langle \text{out} | \mathcal{S} | \text{in} \rangle$$

$$S^{(-1)} = \sum_{k \in \text{in,out}} S_k(\hat{q}) = \sum_k \frac{\kappa}{2} \frac{\varepsilon^+_{\mu\nu} p^\mu_k p^\nu_k}{\hat{q} \cdot p_k}$$

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1401.76026 He, Lysov, Mitra, Strominger

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$$\boxed{S_k(z,\bar{z})|p_k\rangle \sim \delta(z,\bar{z})|p_k\rangle}$$

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These generalized symmetries also involve the insertion of soft gravitons

Symmetries from Subleading Soft Theorems

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1509.01406 Campiglia, Laddha

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Subleading soft theorem ⟷ superrotation (Virasoro) symmetry

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Extended BMS symmetry with Poincare global subgroup

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Algebra of further subleading soft graviton modes extends (chiral) Poincare to $w_{1+\infty}$

Tree-level, minimal coupling — 2103.03961 Guevara, EH, Pate, Strominger; 2105.14346 Strominger

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The graviton $\mathrm{w}_{1+\infty}$ algebra and symmetry action on hard **massless** particles were derived in celestial holography

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- Used basis of conformal primary operators that transform in highest-weight representations of $\mathrm{SL}(2,\mathbb{C})$ global conformal (Lorentz) transformations

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This talk: derivation of $\mathrm{w}_{1+\infty}$ symmetry action from momentum-space soft theorems for **massless and massive** hard particles, guided by conformal covariance

Soft Graviton Expansion

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Consider the low-energy expansion of an amplitude with an outgoing graviton:

$$\mathcal{A}(\omega\hat{q}; p_1, \cdots, p_n) = \frac{\kappa}{2} \sum_{\ell=-1}^{\infty} \omega^{\ell} \mathcal{A}^{(\ell)}(\hat{q}; p_1, \cdots, p_n)$$

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To isolate the ℓ term in the soft expansion, take the limit

$$\mathcal{A}^{(\ell)}(\hat{q};\{p_i\})\propto\lim_{\omega\to 0}\partial_\omega^{\ell+1}\left(\omega\mathcal{A}(\omega\hat{q};\{p_i\})\right)=(\ell+1)!\lim_{\epsilon\to 0}\epsilon\int_0^{\infty}\frac{d\omega}{\omega}\,\omega^{-\ell+\epsilon}\mathcal{A}(\omega\hat{q};\{p_i\})$$

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1705.01027 Pasterski, Shao

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Soft factors that generate the symmetry action on hard particles
should also transform covariantly with this weight

Review of Lorentz $\mathrm{SL}(2, \mathbb{C})$ Transformations

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Parametrize scattering data to facilitate analysis of global conformal transformations

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Massless Particles

$$p_k^\mu = \epsilon_k \omega_k \hat{q}^\mu(z_k, \bar{z}_k) \qquad \hat{q}^\mu(z,\bar{z}) = (1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z})$$

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Lorentz transformations act as Mobius transformations on the celestial sphere

$$z \to z' = \frac{az+b}{cz+d}, \qquad \bar{z} \to \bar{z}' = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}, \qquad ad-bc = \bar{a}\bar{d}-\bar{b}\bar{c} = 1$$

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Polarization vectors require a reference point $(z_0, \bar{z}_0)$ (equivalent to auxiliary spinor) to transform covariantly

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$$w_k \to w_k' = \frac{(aw_k+b)(\bar{c}\bar{w}_k+\bar{d}) + a\bar{c}y_k^2}{(cw_k+d)(\bar{c}\bar{w}_k+\bar{d}) + c\bar{c}y_k^2}$$

$$\bar{w}_k \to \bar{w}_k' = \frac{(\bar{a}\bar{w}_k+\bar{b})(cw_k+d) + \bar{a}cy_k^2}{(cw_k+d)(\bar{c}\bar{w}_k+\bar{d}) + c\bar{c}y_k^2}$$

$$y_k \to y_k' = \frac{y_k}{(cw_k+d)(\bar{c}\bar{w}_k+\bar{d}) + c\bar{c}y_k^2}$$

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Parametrize scattering data to facilitate analysis of global conformal transformations

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$$p_k^\mu = \epsilon_k m_k \hat{p}_k^\mu, \qquad \hat{p}_k^\mu = \frac{1}{2y_k}\left(y_k^2 n^\mu + \hat{q}^\mu(w_k,\bar{w}_k)\right), \qquad \hat{p}_k^2 = -1, \qquad n^\mu \equiv \partial_z \partial_{\bar{z}} \hat{q}^\mu(z,\bar{z})$$

Lorentz transformations act as isometries of AdS_3

$$ds^2 = \frac{dy^2 + dw d\bar{w}}{y^2}$$

$$\begin{aligned} w_k \to w_k' &= \frac{(aw_k+b)(\bar{c}\bar{w}_k+\bar{d}) + a\bar{c}y_k^2}{(cw_k+d)(\bar{c}\bar{w}_k+\bar{d}) + c\bar{c}y_k^2} \\ \bar{w}_k \to \bar{w}_k' &= \frac{(\bar{a}\bar{w}_k+\bar{b})(cw_k+d) + \bar{a}cy_k^2}{(cw_k+d)(\bar{c}\bar{w}_k+\bar{d}) + c\bar{c}y_k^2} \\ y_k \to y_k' &= \frac{y_k}{(cw_k+d)(\bar{c}\bar{w}_k+\bar{d}) + c\bar{c}y_k^2} \end{aligned}$$

$$p_k^\mu(y_k,w_k,\bar{w}_k) \to p_k^\mu(y_k',w_k',\bar{w}_k') = \Lambda^\mu{}_\nu p_k^\nu(y_k,w_k,\bar{w}_k)$$

Subleading Soft Expansion

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Consider the low-energy expansion of an amplitude with an outgoing graviton:

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1404.4091 Cachazo, Strominger

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$$\mathcal{A}^{(\ell>0)}(\hat{q}; p_1, \cdots, p_n) = \sum_{k=1}^{n} S_k^{\prime\prime(\ell)} \mathcal{A}(p_1, \cdots, p_n) + \varepsilon_{\mu\nu} \mathcal{B}_{\ell}^{\mu\nu}(\hat{q}; p_1, \cdots, p_n), \qquad S_k^{\prime\prime(\ell)} = \frac{1}{(\ell+1)!} \frac{\varepsilon_{\mu\nu} (i\hat{q}_{\rho} \mathcal{L}_k^{\mu\rho})(i\hat{q}_{\sigma} \mathcal{L}_k^{\nu\sigma})}{\hat{q} \cdot p_k} \left(\hat{q} \cdot \frac{\partial}{\partial p_k} \right)^{\ell-1}$$

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"Universal" term $S_k^{\prime\prime(\ell)}$ from gauge invariance, "non-universal" $\mathcal{B}_{\ell}^{\mu\nu}$ subleading in $\hat{q}\cdot p_k$

Tree-level, minimal coupling

1801.05528 Hamada, Shiu; 1802.03148 Li, Lin, Zhang

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1812.06895 Guevara, Ochirov, Vines
1405.1410 He, Huang, Wen
1504.01364 Lipstein

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Note: division by $\varepsilon_+\cdot p_k$ implies this is not gauge invariant, and the dependence on the reference point means the "universal" partition is not strictly conformally invariant.

However, the gauge and conformal non-invariance will drop out at the level of symmetry action.

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Can prove that this action satisfies the $\mathrm{w}_{1+\infty}$ algebra

$$[\delta_m^p,\delta_n^q]=[m(q-1)-n(p-1)]\,\delta_{m+n}^{p+q-2}$$

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However, for $\ell > 1$, its primary descendant depends on z_0, so to find the correct symmetry action, we need to revisit the soft factor itself

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$$\partial_{\bar{z}}^{\ell+3} S_k^{\prime(\ell)}(z,\bar{z}) = \mathcal{N}_\ell \frac{p_k^4}{(\hat{q}\cdot p_k)^{\ell+4}} (F_- \cdot \mathcal{L}_k)^{\ell+1} \qquad \mathcal{N}_\ell = (-\sqrt{2})^{\ell-1}(\ell+3)(\ell+2)$$

However, for $\ell > 1$, its primary descendant depends on z_0, so to find the correct symmetry action, we need to revisit the soft factor itself

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Covariant Massive Soft Factors

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Soft factor $S_k^{\prime(\ell)}(z,\bar{z})$ can be split into two separately covariant pieces, one of which is independent of z_0

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$$f \equiv \hat{q}\cdot p_k, \quad f_0 \equiv \hat{q}_0 \cdot p_k, \quad \hat{q}_0 \equiv \hat{q}(z_0,\bar{z})$$

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Caution in massless limit

$$f \equiv \hat{q}\cdot p_k, \quad f_0 \equiv \hat{q}_0 \cdot p_k, \quad \hat{q}_0 \equiv \hat{q}(z_0,\bar{z})$$

$\mathrm{w}_{1+\infty}$ Action on Massive Particles

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Natural proposal for the "universal" soft factor for massive external particles at order $\ell > 1$

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$$\delta_m^p|p_k\rangle \equiv -\frac{1}{2}\int\frac{d^2z}{2\pi}\bar{z}^{p+m-1}\partial_{\bar{z}}^{2p-1}\begin{cases} S_k'^{(2p-4)}(z,\bar{z})|p_k\rangle, & p = \frac{3}{2}, 2, \frac{5}{2} \\ S_k^{(2p-4)}(z,\bar{z})|p_k\rangle, & p > \frac{5}{2}\end{cases}$$

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Can again prove that this action respects the $w_{1+\infty}$ algebra: $[\delta_m^p,\delta_n^q] = [m(q-1)-n(p-1)]\,\delta_{m+n}^{p+q-2}$

Summary

- Clarified origin of $\mathrm{w}_{1+\infty}$ symmetry action from tower of momentum-space soft factors
- Proposed new "universal" massive soft factors motivated by $\mathrm{SL}(2,\mathbb{C})$ covariance
- Discovered nontrivial symmetry action on massive particles
- Proved that symmetry action satisfies $\mathrm{w}_{1+\infty}$ acting on massive particles

Future Directions

- How is the symmetry algebra deformed by loops and higher-dimension operators?
- What theories solve $\mathrm{w}_{1+\infty}$ constraints?
- Organization of "non-universal" terms in soft expansion from symmetry principles?
- Simplification of symmetry action in *integer* massive conformal primary basis?
- Kinematic algebra description of symmetry action on massive particles?

More details

Massless $w_{1+\infty}$ Action

$$\delta^p_m|p_k\rangle \equiv -\frac{1}{2}\int \frac{d^2z}{2\pi}\ \bar{z}^{p+m-1}\partial_{\bar{z}}^{2p-1}S_k'^{(2p-4)}(z,\bar{z})|p_k\rangle$$

$$p = \bar{h} = \frac{\ell+4}{2}$$

$$1-p \leq m \leq p-1$$

Examples:

$p = \frac{3}{2} \to \ell = -1$ leading soft theorem, closed set of 2 modes: (chiral) translations

$$\delta^{\frac{3}{2}}_{-\frac{1}{2}}|p_k\rangle = \frac{\omega_k}{2}|p_k\rangle \qquad \delta^{\frac{3}{2}}_{\frac{1}{2}}|p_k\rangle = \frac{\omega_k\bar{z}_k}{2}|p_k\rangle$$

$p = 2 \to \ell = 0$ subleading soft theorem, closed set of 3 modes: $\mathrm{SL}(2,\mathbb{R})$ chiral half of Lorentz

Acting on scalars $\quad \delta^2_m|p_k\rangle = \bar{z}_k^{m-1}\left(\bar{z}_k^2\partial_{\bar{z}_k} - \frac{m+1}{2}\omega_k\partial_{\omega_k}\right)|p_k\rangle$

Action in the conformal primary basis follows directly: $\delta^q_n\phi_{\Delta_k}(z_k,\bar{z}_k) = \int_0^\infty \frac{d\omega_k}{\omega_k}\ \omega_k^{\Delta_k}\ \delta^q_n|p_k\rangle$ $\qquad -\omega_k\partial_{\omega_k} \mapsto \Delta_k$

Can prove by induction that this action satisfies the $w_{1+\infty}$ algebra

$$[\delta^p_m, \delta^q_n] = [m(q-1) - n(p-1)]\ \delta^{p+q-2}_{m+n}$$

Base cases: $p = \frac{3}{2}, 2, \frac{5}{2}$

$$\delta^2_m : \delta^q_k \to \delta^q_{k+m}$$

$$\delta^{\frac{5}{2}}_m : \delta^q_k \to \delta^{q+\frac{1}{2}}_{k+m}$$

Massive $w_{1+\infty}$ Action

$$p = \bar{h} = \tfrac{\ell+4}{2}$$

$$1 - p \leq m \leq p - 1$$

$$\delta^p_m|p_k\rangle \equiv -\frac{1}{2}\int \frac{d^2z}{2\pi}\bar{z}^{p+m-1}\partial_{\bar{z}}^{2p-1}\begin{cases} S'^{(2p-4)}_k(z,\bar{z})|p_k\rangle, & p = \frac{3}{2}, 2, \frac{5}{2} \\ S^{(2p-4)}_k(z,\bar{z})|p_k\rangle, & p > \frac{5}{2} \end{cases}$$
$$= -\frac{\mathcal{N}_{2p-4}}{2}\int \frac{d^2z}{2\pi}\bar{z}^{p+m-1}\frac{p_k^4}{(\hat{q}\cdot p_k)^{2p}}(F_-\cdot\mathcal{L}_k)^{2p-3}|p_k\rangle.$$

Leading: (chiral) translations

$$\delta^{\frac{3}{2}}_{-\frac{1}{2}}|p_k\rangle = \frac{\epsilon_k m_k}{4y_k}|p_k\rangle \qquad \delta^{\frac{3}{2}}_{\frac{1}{2}}|p_k\rangle = \frac{\epsilon_k m_k \bar{z}_k}{4y_k}|p_k\rangle$$

Subleading: $\mathrm{SL}(2,\mathbb{R})$ Killing vectors on $(y_k, w_k, \bar{w}_k)$ hyperboloid

$$\delta^2_m|p_k\rangle = \frac{1}{2}\bar{w}_k^{m-1}\left(2\bar{w}_k^2\partial_{\bar{w}_k} + (m+1)\bar{w}_k y_k\partial_{y_k} - m(m+1)y_k^2\partial_{w_k}\right)|p_k\rangle$$

Splitting the Soft Factor

To separate the soft factor and isolate the covariant piece independent of z_0, use completeness:

$$1 = \frac{p_k^2}{p_k^2} = \frac{(\partial_z \hat{q} \cdot p_k)(\partial_{\bar{z}} \hat{q} \cdot p_k) - (\hat{q} \cdot p_k)(n \cdot p_k)}{p_k^2}$$

For example, when $\ell = 2$

$$S_k'^{(2)}(z,\bar{z}) = \boxed{\frac{\mathcal{N}_2}{6!}\left[\frac{1}{p_k^2}(\hat{q}\cdot p_k)^6 \partial_{\bar{z}}\left(\frac{(F_+ \cdot \mathcal{L}_k)^3}{(\hat{q}\cdot p_k)^6}\right)\right.} - \left.\frac{1}{p_k^2}(\hat{q}_0 \cdot p_k)^6 \partial_{\bar{z}}\left(\frac{(F_+ \cdot \mathcal{L}_k)^3}{(\hat{q}_0 \cdot p_k)^6}\right)\right]$$

$$\partial_{\bar{z}}^5 \left[\frac{\mathcal{N}_2}{6!}\frac{1}{p_k^2}(\hat{q}\cdot p_k)^6 \partial_{\bar{z}}\left(\frac{(F_+ \cdot \mathcal{L}_k)^3}{(\hat{q}\cdot p_k)^6}\right)\right] = \mathcal{N}_2 \frac{p_k^4}{(\hat{q}\cdot p_k)^6}(F_- \cdot \mathcal{L}_k)^3$$

Follows primary descendant pattern

There is an analogous separation of terms at every order:

$$S_k'^{(\ell)}(z,\bar{z}) = \frac{\mathcal{N}_\ell}{(2\ell+2)!} f^{2\ell+2} \frac{1}{p_k^{2\ell-2}} \partial_{\bar{z}}^{\ell-1}\left[\frac{(F_+ \cdot \mathcal{L}_k)^{\ell+1}}{f^{\ell+4}}\right]$$
$$- \frac{\mathcal{N}_\ell}{(\ell+3)!(\ell-2)!}\frac{1}{p_k^{2\ell-2}} \sum_{j=0}^{\ell-2} \binom{\ell-2}{j} \frac{(-1)^j}{j+\ell+4} f^{\ell-2-j} f_0^{\ell+4+j} \partial_{\bar{z}}^{\ell-1}\left[\frac{f^j (F_+ \cdot \mathcal{L}_k)^{\ell+1}}{f_0^{\ell+4+j}}\right]$$

$$f \equiv \hat{q} \cdot p_k,$$
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