Bootstrapping amplitudes using a chiral algebra

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Based on work by Roland Bittleston, K.C., Lance Dixon, Victor Fernandez, Anthony Morales, Natalie Paquette, Atul Sharma, David Skinner, Keyou Zeng

Overview

QCD amplitudes (without SUSY) are normally very very hard to compute.

Exact formulae for amplitudes are available for k loops with n legs at (k, n) in the range roughly

$$(0, n), \quad (1, n), \quad (2, n \leq 7), \quad (3, n = 4) \ldots$$

Computed by Parke-Taylor, Bern-Dixon-Kosower, Mahlon, Badger, Dunbar .... (See also talks by Tancredi, Abreu, Devoto, Xu).

I will explain a new method to compute very special amplitudes/form factors in massless QCD with special matter using chiral algebras.

In particular, this gives new formulae for n-point 2-loop all $+$ amplitudes at all n (consistent with the result from standard techniques for $n = 4$).

Take away from this talk

1. A (small) subsector of QCD with special matter is exactly solvable.
2. It has hidden algebraic structure which allows exact loop level computations.
3. This comes from relation to twistor space.

Self-dual QCD

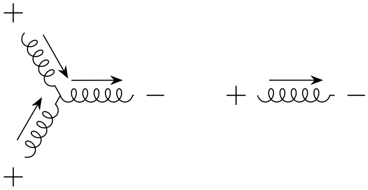
QCD can be written as a deformation of self-dual QCD:

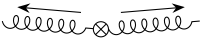
$$\int \mathrm{tr}(B \wedge F(A)_-) + \int \overline{\psi} \not{\partial}_A \psi \quad B \in \Omega^2_- \otimes \mathfrak{g}$$

We can deform SDQCD by adding the term

$$g^2_{YM}\,\mathrm{tr}(B^2)$$

Then it is equivalent to ordinary QCD in perturbation theory.
SDQCD has $++-$ vertex and $-+$ propagator



Adding $\mathrm{tr}(B^2)$ introduces a vertex 

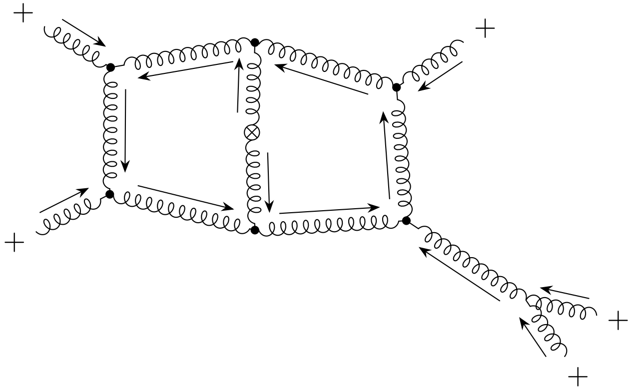
One can study QCD in perturbation around the self-dual sector. In $N = 4$ this is standard:

1. Form factors of $\mathrm{tr}(B^2)$ are the MHV vertex.
2. Form factors of $\mathrm{tr}(B^2)(x_1) \ldots \mathrm{tr}(B^2)(x_n)$ closely related to the "integrand".

The goal is to apply a similar program to non-supersymmetric gauge theory.

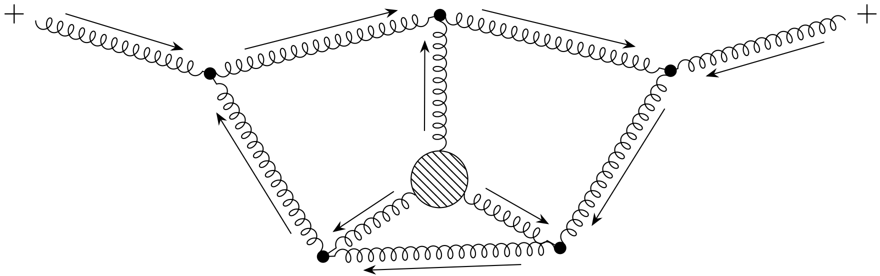
Without SUSY, form factors for $\mathrm{tr}(B^2)$ give massless QCD amplitudes up to $l \leq 2$ loops, with $2 - l$ states of negative helicity.

E.g. two loop form factors of $\text{tr}(B^2)$ in SDQCD match two-loop all $+$ QCD amplitudes:



Form factors of an operator in SDQCD match those in massless QCD in a certain range of loop number and helicities of external states.

E.g.: form factors of $\mathrm{tr}(B^n)$ in SDQCD match form factors of $\mathrm{tr}(F_-^n)$ in massless QCD at l loops with $n - l$ external states of negative helicity.



Theorem (KC, Natalie Paquette)

When the matter representation R is such that

$$\mathrm{tr}_R(X^4) = \mathrm{tr}_{\mathfrak{g}}(X^4) \text{ for } X \in \mathfrak{g}$$

then form factors of SDQCD are the same as correlators of a certain 2d chiral algebra.

Chiral algebra means the structure of the holomorphic part of a 2d CFT, with holomorphic OPEs satisfying associativty/crossing symmetry.

Examples:

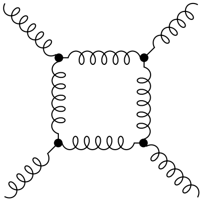
1. $SU(2)$, $N_f = 8$.
2. $SU(3)$, $N_f = 9$.
3. $SU(N)$, matter in $8F \oplus 8F^\vee \oplus \wedge^2 F \oplus \wedge^2 F^\vee$. Where $\wedge^2 F$ is the antisymmetric part of $F \otimes F$.
4. Any supersymmetric theory.

Anomaly cancellation

The trace identity

$$\mathrm{tr}_R(X^4) = \mathrm{tr}_{\mathfrak{g}}(X^4)$$

guarantees absence of an anomaly on twistor space coming from a box diagram



On space-time this is an anomaly to *integrability*.

Twistor space anomaly cancels $\iff$ one loop all $+$ amplitudes of SDQCD vanish.

Relationship to celestial holography

We call the chiral algebra the *celestial chiral algebra* as it can be viewed as living on the celestial sphere.

Chiral algebra for only $+$ helicity states was previously studied by Guevara, Himwich, Pate and Strominger.

Our approach differs from some approaches to celestial holography:

1. We only look at chiral algebras, not full CFTs.
2. We only study local CFTs.
3. Celestial chiral algebras exist in only *very special* situations related to integrability.

Chiral algebra operators

The chiral algebra is described in spinor helicity notation: null momenta $p_{\alpha\dot{\alpha}}$ are written in terms of a pair of spinors

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \widetilde{\lambda}_{\dot{\alpha}}$$

and we set

$$\lambda_\alpha = (1, z) \quad z \in \mathbb{CP}^1$$

The chiral algebra lives on the z-plane.

The chiral algebra has

Single particle states at $z \longleftrightarrow$ Single particle gauge theory states with momentum $\lambda = (1, z)$, $\widetilde{\lambda}$ arbitrary.

For each $\widetilde{\lambda}$, the chiral algebra has operators

$$\mathbb{J}_a^+[\widetilde{\lambda}](z), \quad \mathbb{J}_a^-[\widetilde{\lambda}](z)$$

corresponding to gluons of positive/negative helicity. These can be expanded in series

$$\mathbb{J}_a^+[\widetilde{\lambda}](z) = \sum \frac{1}{n!m!} \mathbb{J}_a^+[n,m](z) \widetilde{\lambda}_{\dot{1}}^n \widetilde{\lambda}_{\dot{2}}^m$$

Similarly for matter fields and $\mathbb{J}^-$, giving

$$\mathbb{J}_a^-[n,m](z) \quad \mathbb{M}_i^+[n,m](z) \quad \mathbb{M}_i^-[n,m](z)$$

A basis of operators in chiral algebra is given by normally ordered products of derivatives of these states, e.g.

$$: \mathbb{J}_{a_1}^-[n_1, m_1] \partial_z \mathbb{J}_{a_2}^+[n_2, m_2] \partial_z^3 \mathbb{M}_i^+[n_3, m_3] \cdots : (z)$$

Chiral algebra dictionary

SDQCD states $\Longleftrightarrow$ chiral algebra operators

SDQCD form factors $\Longleftrightarrow$ chiral algebra correlators

SDQCD collinear singularities $\Longleftrightarrow$ chiral algebra OPEs

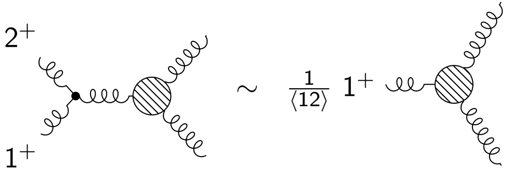
OPEs

OPEs are given by collinear singularities in SDQCD form factors.
At tree level:

$$\mathbb{J}^+[\widetilde{\lambda}_1](z_1)\mathbb{J}^+[\widetilde{\lambda}_2](z_2) \sim \frac{1}{\langle 12\rangle}\mathbb{J}^+[\widetilde{\lambda}_1+\widetilde{\lambda}_2]$$

$$\mathbb{J}^-[\widetilde{\lambda}_1](z_1)\mathbb{J}^+[\widetilde{\lambda}_2](z_2) \sim \frac{1}{\langle 12\rangle}\mathbb{J}^-[\widetilde{\lambda}_1+\widetilde{\lambda}_2]$$

$$\langle 12\rangle = z_1 - z_2$$

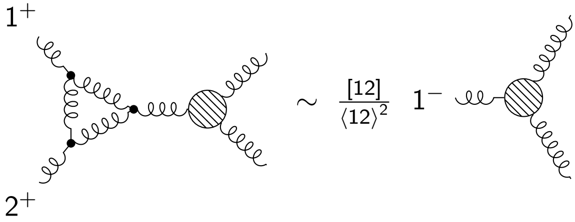


The $\mathbb{J}^- - \mathbb{J}^-$ OPEs are non-singular.

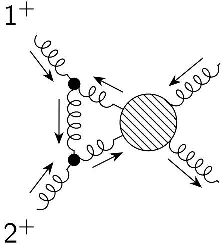
Loop level OPEs

At loop level there are corrections coming from the one loop splitting function:

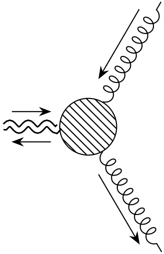
$$\mathbb{J}^+[\widetilde{\lambda}^{(1)}](z_1)\mathbb{J}^+[\widetilde{\lambda}^{(2)}](z_2) \sim \frac{[12]}{\langle 12\rangle^2}\mathbb{J}^-[\widetilde{\lambda}^{(1)}+\widetilde{\lambda}^{(2)}](z_1) + \text{lower order poles}$$



And also subleading corrections from non-splitting contributions:



$\sim \frac{[12]}{\langle 12 \rangle} \iint : 1^+ 1^- :$

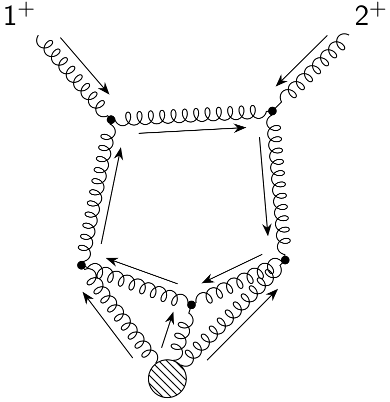


This gives

$$\mathbb{J}^+[\widetilde{\lambda}_1](z_1)\mathbb{J}^+[\widetilde{\lambda}_2](z_2) \sim \frac{[12]}{2\,\langle 12 \rangle} \int_{s,t=0}^{1} \mathrm{d}s\mathrm{d}t : \mathbb{J}^-[s\widetilde{\lambda}_1 + t\widetilde{\lambda}_2]\mathbb{J}^+[(1-s)\widetilde{\lambda}_1 + (1-t)\widetilde{\lambda}_2)] : (z_1) + \dots$$

There are also higher loop contributions to the OPE:

$$\mathbb{J}^+\mathbb{J}^+ \sim :\mathbb{J}^-\mathbb{J}^-\mathbb{J}^- :$$



Key question:

Do the OPEs dictated by collinear singularities of SDQCD form factors lead to an associative (= crossing symmetric) chiral algebra?

Theorem (KC, Natalie Paquette)

If the trace identity

$$\mathrm{tr}_R(X^4) = \mathrm{tr}_{\mathfrak{g}}(X^4)$$

holds then yes!
In particular all OPEs are rational functions with poles only in $\langle ij \rangle$.
Otherwise, associativity fails.

Proof uses twistor theory. Trace identity implies that there is no anomaly on twistor space, so the theory lifts to a local holomorphic QFT on twistor space. From this we can build a chiral algebra.

Associativity implies a lot of relations among collinear singularities!

Conformal blocks and computations

We would like to "bootstrap" form factors of SDQCD.

Operators in chiral algebra can be of negative $2d$ conformal weight. For example $\mathbb{J}^+[n,m]$ of conformal weight $1 - \frac{1}{2}(n+m)$.

In a chiral algebra with states of negative conformal weight, correlation functions are ambiguous: with operators $\mathcal{O}_i$ of conformal weight w_i,

$$\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\dots\mathcal{O}_n(z_n)\rangle$$

has pole of order $-2w_i$ at $z_i = \infty$, and poles at $z_i = z_j$ from OPE. The part regular at $z_i = z_j$ is ambiguous.

Definition

A conformal block is a way of defining correlation functions in a chiral algebra compatible with OPEs and conformal weights.

Theorem

1. *Conformal blocks of the chiral algebra naturally in bijection with local operators of SDQCD.*
2. *Correlation functions in the conformal block are the same as form factors for the local operator.*

Abstract proof goes by twistor theory.

This gives a bootstrap algorithm:

1. Choose a local operator in SDQCD, e.g. $\mathrm{tr}(B^2)$.
2. Identify the corresponding conformal block (typically determined by symmetries/dimensional analysis).
3. Compute correlation functions using the familiar 2d bootstrap.

Example: operator $\mathrm{tr}(B^2)$ has conformal block determined by

$$\left\langle \mathrm{tr}(B^2) \mid \mathbb{J}^-[\widetilde{\lambda}_1](z_1)\mathbb{J}^-[\widetilde{\lambda}_2](z_2) \right\rangle = \langle 12 \rangle^2 \, .$$

We can use this to bootstrap amplitudes.

E.g. at tree level, trivial manipulations give Parke-Taylor:

$$\left\langle \mathrm{tr}(B^2) \mid \mathbb{J}^+[\widetilde{\lambda}_1](z_1) \dots \mathbb{J}^-[\widetilde{\lambda}_i](z_i) \dots \mathbb{J}^-[\widetilde{\lambda}_j](z_j) \dots \mathbb{J}^+[\widetilde{\lambda}_n](z_n) \right\rangle$$
$$= \frac{\langle ij \rangle^4}{\langle 12 \rangle \dots \langle n1 \rangle}.$$

This is a bit boring! Much more interesting: one and two loop form amplitudes.

The two loop all $+$ form factor is the correlation function

$$\mathcal{A}^{(2)}(1^+, \dots, n^+) = \left\langle \mathrm{tr}(B^2) \mid \mathbb{J}^+[\widetilde{\lambda}_1](z_1) \dots \mathbb{J}^+[\widetilde{\lambda}_n](z_n) \right\rangle$$

We can compute this for gauge group SU_N with matter $\wedge^2 F \oplus \wedge^2 F^\vee \oplus 8(F \oplus F^\vee)$.

The one-loop and tree level OPEs are sufficient to compute this

$$\mathbb{J}^+[\widetilde{\lambda}_1](z_1)\mathbb{J}^+[\widetilde{\lambda}_2](z_2) \sim \frac{1}{\langle 12 \rangle}\mathbb{J}^+[\widetilde{\lambda}_1 + \widetilde{\lambda}_2](z_1) + \frac{[12]}{\langle 12 \rangle^2}\mathbb{J}^-[\widetilde{\lambda}_1 + \widetilde{\lambda}_2](z_1)$$
$$+ \frac{[12]}{2\langle 12 \rangle}\int_{s,t=0}^{1} \mathrm{d}s\mathrm{d}t : \mathbb{J}^-[s\widetilde{\lambda}_1 + t\widetilde{\lambda}_2]\mathbb{J}^+[(1-s)\widetilde{\lambda}_1 + (1-t)\widetilde{\lambda}_2)] : (z_1) + \dots$$

OPEs reduce the computation inductively to the tree-level result.

The two loop four point trace ordered amplitude is:

$$\begin{aligned}\mathcal{A}^{(2)}(1^+,2^+,3^+,4^+) = &\left(6N-4-8N^{-1}\right)\left(\frac{1}{(4\pi)^4}\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}+\frac{1}{(4\pi)^4}\frac{[41][23]}{\langle 41\rangle\langle 23\rangle}\right)\\ &-\left(4+8N^{-1}\right)\frac{1}{(4\pi)^4}\frac{[13][24]}{\langle 13\rangle\langle 24\rangle}\\ &-\frac{2}{(4\pi)^4}\frac{[12][34](\langle 13\rangle\langle 24\rangle+\langle 14\rangle\langle 23\rangle)}{\langle 12\rangle^2\langle 34\rangle^2}\\ &+\frac{2}{(4\pi)^4}\frac{[14][23](\langle 13\rangle\langle 42\rangle+\langle 12\rangle\langle 43\rangle)}{\langle 14\rangle^2\langle 23\rangle^2}\end{aligned}$$

Then the n-point single trace amplitude is

$$\mathcal{A}^{(2)}(1^+,\ldots,n^+) = \sum_{1\le i<j<k<l\le n}\mathcal{A}^{(2)}(i^+,j^+,k^+,l^+)\frac{\langle ij\rangle\langle jk\rangle\langle kl\rangle\langle li\rangle}{\langle 12\rangle\ldots\langle n1\rangle}$$

The double-trace amplitude has been computed by Dixon-Morales, and the triple-trace amplitude vanishes – so the result is available in full colour.

Verifying two loop computations

Theorem (Dixon, Morales, 2406.xxxx)

The four-point two-loop amplitude computed using the chiral algebra matches the amplitude computed using Feynman diagrams.

Subtleties:

1. Dim. reg. is not so good for computations starting with SDQCD.
2. In dim. reg. there are IR divergences. The chiral algebra formula computes the finite part.
3. With a mass regulator, there are no IR divergences and the match is exact.

The computation is based on techniques from 0001001, 0201161, 0202271 by Bern, de Freitas, Dixon, Kosower, Wong.

Generalizations

In principle this method can be applied to compute certain form factors at even higher loops. Difficulties:

1. Knowledge of OPEs beyond one loop is required. All order formulae for OPEs were computed by
   1. K. Zeng, by direct computation in the supersymmetric case.
   2. N. Paquette and V. Fernandez, who show that all OPEs are determined by tree-level and one loop OPEs by associativity.
2. Even knowing the OPEs, the bootstrap method is quite complicated (but much simpler than Feynman diagrams).

We would also like to compute form factors of multi-point insertions $\mathrm{tr}(B^2)(x_1)\dots\mathrm{tr}(B^2)(x_n)$. This is currently out of reach except when all x_i are very close.

Including gravity

Bittleston-Sharma-Skinner and Bittleston perform a similar analysis including SDGR. There are pure gravity, pure gauge, and mixed anomalies. With matter in R_f scalars in R_s and gauge fields in $\mathfrak{g}$ anomalies cancel if

$$\begin{aligned} \dim R_s - 2\dim R_f + \dim \mathfrak{g} + 2 &= 0 \\ \mathrm{tr}_{R_s}(X^4) - 2\,\mathrm{tr}_{R_f}(X^4) + 2\,\mathrm{tr}_{\mathfrak{g}}(X^4) &= 0 \\ \mathrm{tr}_{R_s}(X^2) - 2\,\mathrm{tr}_{R_f}(X^2) + 2\,\mathrm{tr}_{\mathfrak{g}}(X^2) &= 0 \end{aligned}$$

Then there is a chiral algebra at loop level.

Problem

SDGR has no local operators, so no form factors – there are conceptual difficulties with performing the analysis used for SDQCD.

Chiral algebras as large N limits

Sometimes SDQCD is controlled by a chiral algebra. Is this the large N limit of a 2d system?

Answer

Yes, on certain self-dual backgrounds.

Cleanest example: background is *Burns space*, conformally equivalent to $\mathbb{CP}^2 \setminus$ a point.

We can put a theory on Burns space consisting of:

1. A gauge-fixed SDYM for $\mathfrak{so}(8)$
2. "Scalar flat gravity" (a 4d cousin of Liouville)

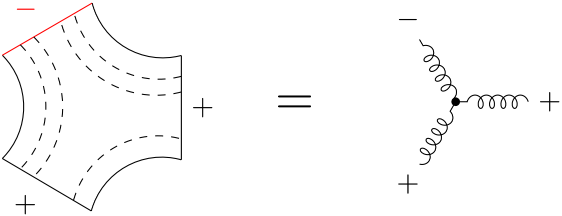
Conjecture (K.C., Paquette, Sharma)

Amplitudes for this theory on Burns space match large N correlators of a specific (very simple) chiral algebra.

The chiral algebra is the BRST reduction of some free $\beta - \gamma$ fields by Sp(N).

This is derived from a quite standard holographic analysis on twistor space and subject to numerous checks.

The duality implies that tree level amplitdues on $\mathbb{CP}^2 \setminus \{\text{a point}\}$ with n $+$ and one $-$ are given by planar Feynman diagrams in the dual CFT:



Can verify explicitly for 2-point amplitudes.

Flavour symmetry backgrounds

There are similar dualities for SDQCD chiral algebras in certain flavour symmetry background (R. Bittleston, K.C., K. Zeng).

Example

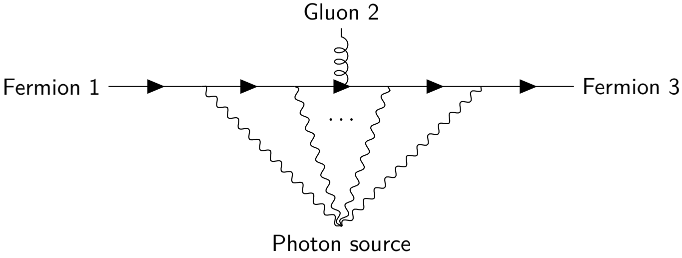
Sp(N) gauge theory, $N_f = 16$, background field for $SU(16)$ flavour symmetry given by

$$F_{11}(\mathcal{A}) = N\delta_{x=0} \quad N \in \mathfrak{u}(1) \subset \mathfrak{su}(16)$$

Amplitudes in this background are computed by correlators in an explicit $2d$ chiral algebra. This leads to nice formulae for these amplitudes.

With two fermions and one gluon the amplitude in presence of the strong background photon we find:

$$\frac{-\mathrm{i}}{\langle 12\rangle\langle 23\rangle}\sum_{a,b,c\geq 0}\frac{\langle 1\lambda\rangle^{a+b}\langle 2\lambda\rangle^{a+c}\langle 3\lambda\rangle^{b+c+2}[12]^a[13]^b[23]^c N^{2a+2b+2c+1}}{a!(a+b)!(b+c)!c!(a+b+c+1)!}$$



We also have recursive formulae for 2 fermions and n gluon amplitudes.

Summary

1. With special matter content, form factors of SDQCD are computed by a chiral algebra.
2. This leads to many new computations of form factors at loop level, and of amplitudes in self-dual backgrounds.
3. There are many more special amplitudes/form factors that this approach can in principle compute.