Minimal Kinematics

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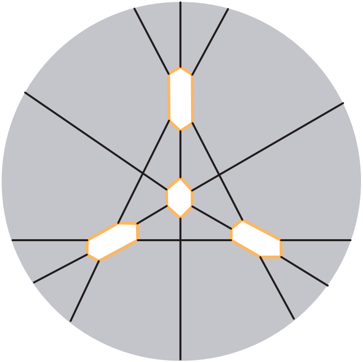


Amplitudes 2024

IAS Princeton, June 12, 2024

Points on a Line

Moduli space of n distinct points on the Riemann sphere $\mathbb{CP}^1$:

$$\mathcal{M}_{0,n} \;=\; \mathrm{Gr}(2,n)^o/(\mathbb{C}^*)^n$$

Point configurations are represented by $2 \times n$ matrices:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & x_1 & x_2 & \cdots & x_{n-3} & 1 \end{bmatrix}$$

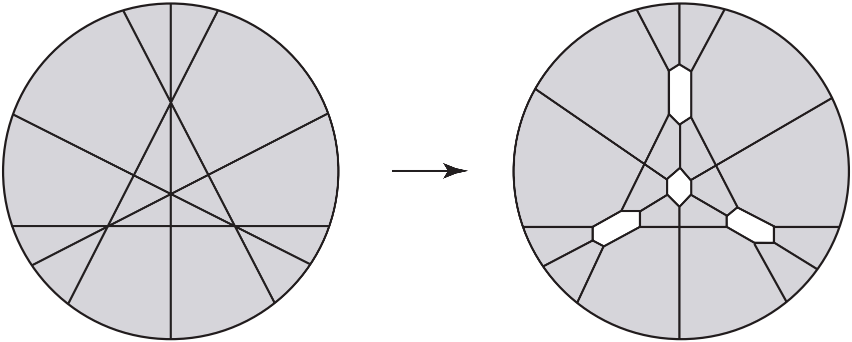
"Gauge fixing"

$\mathcal{M}_{0,n}$ is a very affine variety of dimension $n-3$. On the open Grassmannian $\mathrm{Gr}(2,n)^o$ all Plücker coordinates p_{ij} are nonzero.

N. Early, A. Pfister, B.St: *Minimal kinematics on $\mathcal{M}_{0,n}$*, `arXiv:2402.03065`

Hyperplanes

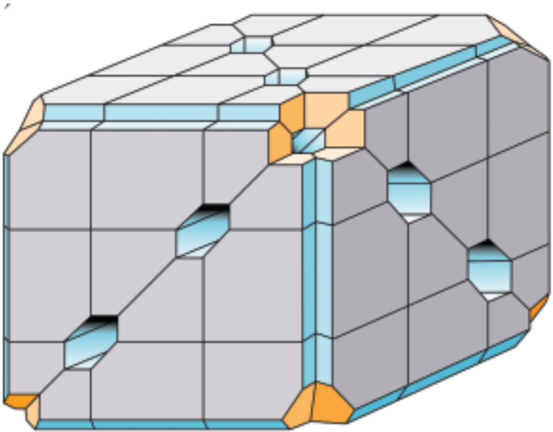
$\mathcal{M}_{0,n}$ is the complement of the hyperplane arrangement $\{x_i = x_j\}$.



The surface $\mathcal{M}_{0,5}$ is the complement of six lines in $\mathbb{CP}^2$.

The real surface $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ consists of 12 pentagons.

Picture by Satyan Devadoss

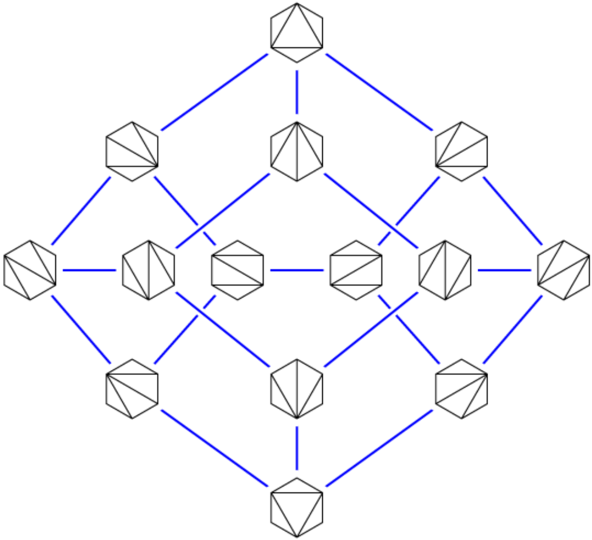


The threefold $\mathcal{M}_{0,6}$ is the complement of ten planes in $\mathbb{CP}^3$.

The real threefold $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ consists of 60 associahedra.

Picture by David Eppstein

$\mathcal{M}_{0,6}$



The 3-dim'l associahedron has 14 vertices, 21 edges and 9 facets.

Bernd studies Physics

Koba-Nielsen string integral

$$\phi_\epsilon(s) \;=\; \epsilon^{n-3} \int_{\mathcal{M}_{0,n}^{+}} \frac{1}{p_{12}p_{23}p_{34}\cdots p_{n1}} \prod_{1 \le i < j \le n} p_{ij}^{\epsilon \cdot s_{ij}}\, dp.$$

Parke-Taylor integrand $\mathcal{I}$

The *Mandelstam invariants* s_{ij} encode kinematic data.

They satisfy $s_{ii} = 0,\; s_{ij} = s_{ji}$ and *momentum conservation*

$$\sum_{j=1}^{n} s_{ij} = 0 \quad \text{for all } i.$$

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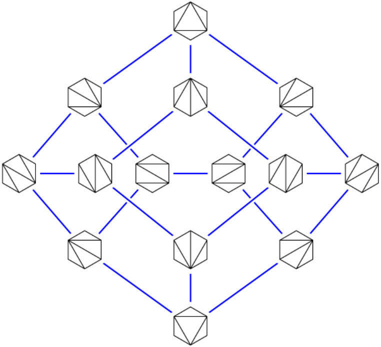
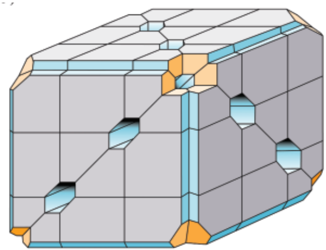
The leading singularity is a rational function of degree $3 - n$:

$$m_n \;=\; \lim_{\epsilon \to 0} \phi_\epsilon(s)$$

This expression is the biadjoint scalar amplitude in ϕ^3 theory:

$$m_6 = \frac{1}{s_{12}s_{34}s_{56}} + \frac{1}{s_{12}s_{56}s_{123}} + \frac{1}{s_{23}s_{56}s_{123}} + \frac{1}{s_{23}s_{56}s_{234}} + \frac{1}{s_{34}s_{56}s_{234}} + \frac{1}{s_{16}s_{23}s_{45}} + \frac{1}{s_{12}s_{34}s_{345}}$$
$$+ \frac{1}{s_{12}s_{45}s_{123}} + \frac{1}{s_{12}s_{45}s_{345}} + \frac{1}{s_{16}s_{23}s_{234}} + \frac{1}{s_{16}s_{34}s_{234}} + \frac{1}{s_{16}s_{34}s_{345}} + \frac{1}{s_{16}s_{45}s_{345}} + \frac{1}{s_{23}s_{45}s_{123}}$$





Physics meets Statistics

The *CHY scattering potential*

$$L(p) \;=\; \sum_{1\leq i<j\leq n} s_{ij}\log(p_{ij}).$$

is well defined on $\mathcal{M}_{0,n}$, by momentum conservation.

Given our *gauge fixing*, it suffices to sum over

$$S \;=\; \big\{(i,j) : 1\leq i<j\leq n-1\big\} \setminus \{(1,2)\}.$$

In **statistics**, the s_{ij} represent the data,
and $L(p)$ is the *log-likelihood function*.

Proposition (Varchenko 1995)

For general $s_{ij}\in\mathbb{C}$, the log-likelihood function $L(p)$ has $(n-3)!$ complex critical points $\hat{p}$. If the s_{ij} are real then all $\hat{p}$ are real.

Stringy Canonical Forms

Square the Parke-Taylor integrand, divide by the Hessian of the scattering potential, and sum over all $(n-3)!$ critical points ...

Theorem (CHY formula)

The biadjoint scalar amplitude equals

$$m_n \;=\; -\sum_{\hat{p}} \frac{\mathcal{I}^2}{\mathrm{Hess}(L)}(\hat{p}).$$

The number of summands is large: $(n-3)!$

Wouldn't be nicer to have only one critical point $\hat{p}$?

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Wouldn't be nicer to have only one critical point $\hat{p}$ *?*

This leads us to **Minimal Kinematics**

F. Cachazo and N. Early: *Minimal kinematics: an all k and n peek into* $\mathrm{Trop}_+ G(k,n)$, SIGMA Symmetry Integrability Geom. Methods Appl. (2021).

ML degree one

Theorem (Early-Pfister-St 2024)

Choices of minimal kinematics on the moduli space $\mathcal{M}_{0,n}$ are in bijection with 2-trees with vertex set $[n-1] = \{1, 2, \ldots, n-1\}$.

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Theorem (Early-Pfister-St 2024)

Choices of minimal kinematics on the moduli space $\mathcal{M}_{0,n}$ are in bijection with 2-trees with vertex set $[n-1] = \{1, 2, \ldots, n-1\}$.

Definition

For any subset T of $S = \{(i,j) : 1 \le i < j \le n-1\} \backslash \{(1,2)\}$, set

$$L_T \;=\; \sum_{(i,j) \in T} s_{ij} \cdot \log(p_{ij}).$$

T exhibits *minimal kinematics* if L_T has exactly critical point, which is a rational function in s, and T is inclusion-maximal.

Definition

We define a class of graphs inductively. The edge 12 is a *2-tree*.
Any *2-tree* on $[k]$ is obtained from a 2-tree on $[k-1]$
by selecting an edge ij and adding two new edges ik and jk.

2-trees

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & x_1 & x_2 & x_3 & 1 \end{bmatrix}$$

Every 2-tree T on $[n-1]$ has $2n-6$ edges in S.

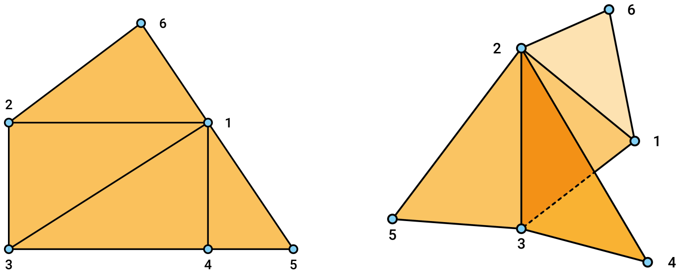
The number of 2-trees is $(2n-5)!!$. Up to symmetry, the number is

$$1, 1, \mathbf{2}, 5, 12, 39, 136, 529, 2171, 9368, 41534, \ldots \quad (A054581)$$
$$\text{for } n = 4, 5, \mathbf{6}, 7, 8, 9, 10, 11, 12, 13, \ldots.$$

For $n = 6$, there are two 2-trees:

$$T_1 = \{13, 23, 14, 34, 15, 45\} \quad \text{and} \quad T_2 = \{13, 23, 24, 34, 25, 35\}.$$

We visualize 2-trees as trees of $n-2$ triangles, with root 126:



Towards Horn

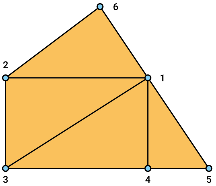
The six coordinates of the critical point $\hat{p}$ for the 2-tree T_1 are

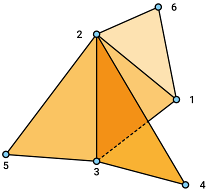
$$\hat{p}_{13} = \frac{s_{13}+s_{14}+s_{34}+s_{15}+s_{45}}{s_{13}+s_{23}+s_{14}+s_{34}+s_{15}+s_{45}} \qquad \hat{p}_{23} = -\frac{s_{23}}{s_{13}+s_{23}+s_{14}+s_{34}+s_{15}+s_{45}}$$

$$\hat{p}_{14} = \frac{(s_{13}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{15}+s_{45})}{(s_{13}+s_{23}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{15}+s_{34}+s_{45})} \qquad \hat{p}_{34} = \frac{(s_{13}+s_{14}+s_{34}+s_{15}+s_{45})s_{34}}{(s_{13}+s_{23}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{34}+s_{15}+s_{45})}$$

$$\hat{p}_{15} = \frac{(s_{13}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{15}+s_{45})s_{15}}{(s_{13}+s_{23}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{34}+s_{15}+s_{45})(s_{15}+s_{45})}$$

$$\hat{p}_{45} = -\frac{(s_{13}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{15}+s_{45})s_{45}}{(s_{13}+s_{23}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{34}+s_{15}+s_{45})(s_{15}+s_{45})}$$





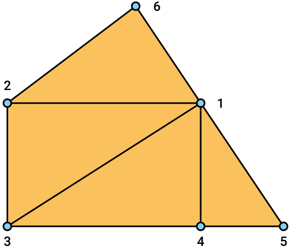
The six coordinates of the critical point $\hat{p}$ for the 2-tree T_2 are

$$\hat{p}_{13} = \frac{s_{13}}{s_{13}+s_{23}+s_{24}+s_{34}+s_{25}+s_{35}} \qquad \hat{p}_{23} = -\frac{s_{23}+s_{24}+s_{25}+s_{34}+s_{35}}{s_{13}+s_{23}+s_{24}+s_{34}+s_{25}+s_{35}}$$

$$\hat{p}_{24} = -\frac{(s_{23}+s_{24}+s_{25}+s_{34}+s_{35})s_{24}}{(s_{13}+s_{23}+s_{24}+s_{34}+s_{25}+s_{35})(s_{24}+s_{34})} \qquad \hat{p}_{34} = \frac{(s_{23}+s_{24}+s_{25}+s_{34}+s_{35})s_{34}}{(s_{13}+s_{23}+s_{24}+s_{34}+s_{25}+s_{35})(s_{24}+s_{34})}$$

$$\hat{p}_{25} = -\frac{(s_{23}+s_{24}+s_{25}+s_{34}+s_{35})s_{25}}{(s_{13}+s_{23}+s_{24}+s_{34}+s_{25}+s_{35})(s_{25}+s_{35})} \qquad \hat{p}_{35} = \frac{(s_{23}+s_{24}+s_{25}+s_{34}+s_{35})s_{35}}{(s_{13}+s_{23}+s_{24}+s_{34}+s_{25}+s_{35})(s_{25}+s_{35})}$$

Rational functions



For T_1, we set $s_{24} = s_{25} = s_{35} = 0$

in the biadjoint scalar amplitude

$$m_6 = \frac{1}{s_{12}s_{34}s_{56}} + \frac{1}{s_{12}s_{56}s_{123}} + \frac{1}{s_{23}s_{56}s_{123}} + \frac{1}{s_{23}s_{56}s_{234}} + \frac{1}{s_{34}s_{56}s_{234}} + \frac{1}{s_{16}s_{23}s_{45}} + \frac{1}{s_{12}s_{34}s_{345}} + \frac{1}{s_{12}s_{45}s_{123}} + \frac{1}{s_{12}s_{45}s_{345}} + \frac{1}{s_{16}s_{23}s_{234}} + \frac{1}{s_{16}s_{34}s_{234}} + \frac{1}{s_{16}s_{34}s_{345}} + \frac{1}{s_{16}s_{45}s_{345}} + \frac{1}{s_{23}s_{45}s_{123}}$$

The resulting specialized amplitude equals

$$\frac{(s_{13}+s_{14}+s_{15}+s_{34}+s_{45})(s_{14}+s_{15}+s_{45})s_{15}}{s_{23}(s_{13}+s_{14}+s_{15}+s_{23}+s_{34}+s_{45})s_{34}(s_{14}+s_{15}+s_{34}+s_{45})s_{45}(s_{15}+s_{45})}.$$

Everything is a product of **positive linear forms**! Why?

Positive linear geometry

Theorem (Kapranov-Huh)

For a very affine variety $X \subset (\mathbb{C}^*)^m$, the following are equivalent:

- *X has Euler characteristic ± 1.*
- *The log-likelihood function on X has a unique critical point $\hat{p}$.*
- *X admits a Horn uniformization*

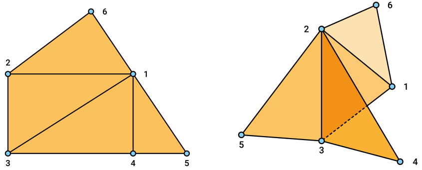
$$\hat{p} = \lambda \star (Hs)^H.$$

Here, H is an integer matrix with m columns and λ is a vector in $\mathbb{Z}^m$. The *Horn pair* (H, λ) is an invariant of the variety X.

E. Duarte, O. Marigliano, B. Sturmfels: *Discrete statistical models with rational maximum likelihood estimator*, Bernoulli **27** (2021) 135–154.

For us, $m = 2n - 6$ and s is the column vector of Mandelstam invariants.

Horn matrices



Example 3.2 $(n = 6)$**.** We consider the two 2-trees that are shown in (10). In each case, the Horn matrix has nine rows and six columns. We find that the two Horn matrices are

$$H_{T_1} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \quad \text{(columns } 13, 23, 14, 34, 15, 45\text{)} \quad \text{and} \quad H_{T_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \quad \text{(columns } 13, 23, 24, 34, 25, 35\text{)}.$$

For $H = H_{T_i}$, the column vector Hs has nine entries, each a linear form in six s-variables. Each column of H specifies an alternating product of these linear forms, and these are the entries of $(Hs)^H$. By adjusting signs when needed, we obtain the six coordinates of $\hat{p}$.

Horn uniformization for 2-trees

Given an edge ij in a 2-tree T, write $[s_{ij}]$ for the sum of all Mandelstam invariants s_{lm} where lm is a descendant of ij in T.

Corollary

The coordinates of the unique critical point $\hat{p}$ are

$$\hat{p}_{lm} \;=\; \pm \prod \frac{[s_{ik}]}{[s_{ik}] + [s_{jk}]},$$

where the product runs over ancestral triangles ijk of the edge lm.

Here *descendant* refers to the transitive closure of parent-child in constructing T: New edges ik and jk are children of old edge ij. Call ijk an *ancestral triangle* of lm if lm is a descandant of ik.

Back to Amplitudes

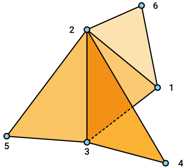
Theorem

The amplitude m_T associated with a 2-tree T equals

$$m_T \;=\; \prod_{ijk} \frac{[s_{ik}] + [s_{jk}]}{[s_{ik}] \cdot [s_{jk}]}.$$

Product over all triangles in T. Rational function of degree $3 - n$

We still need to define m_T.



Example

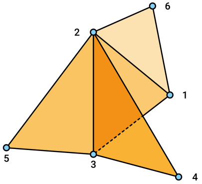
$$m_{T_2} \;=\; \frac{([s_{13}] + [s_{23}])}{[s_{13}]\,[s_{23}]} \cdot \frac{([s_{24}] + [s_{34}])}{[s_{24}]\,[s_{34}]} \cdot \frac{([s_{25}] + [s_{35}])}{[s_{25}]\,[s_{35}]}.$$

Matrix of Circuits

Let M_T be the $(n-2) \times n$ matrix whose rows are $p_{jk}e_i - p_{ik}e_j + p_{ij}e_k$ for $ijk \in T$.

Example

$$M_{T_2} = \begin{bmatrix} p_{23} & -p_{13} & p_{12} & 0 & 0 & 0 \\ 0 & p_{34} & -p_{24} & p_{23} & 0 & 0 \\ 0 & p_{35} & -p_{25} & 0 & p_{23} & 0 \\ p_{26} & -p_{16} & 0 & 0 & 0 & p_{12} \end{bmatrix}$$



The rows of M_T span the kernel of our $2 \times n$ matrix

The maximal minors of M_T are $\pm p_{ij} \cdot \Delta(M_T)$.

Lemma

For any 2-tree T, the gcd of the maximal minors of M_T equals

$$\Delta(M_T) = \prod_{ij} p_{ij}^{v_T(ij)-1},$$

where $v_T(ij)$ is the number of triangles containing ij. Degree $= n-3$.

Amplitude of a 2-tree

The *integrand* for a 2-tree T is

$$\mathcal{I}_T \quad = \quad \frac{\Delta(M_T)^2}{\prod_{ijk \in T} p_{ij} p_{ik} p_{jk}} = \prod_{ij} p_{ij}^{v_T(ij)-2}.$$

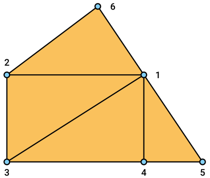
Rational function of degree $-n$ in Plücker coordinates.

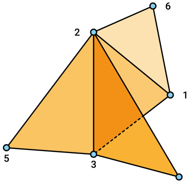
The *amplitude* for T is defined as

$$m_T \;=\; -\frac{(\mathcal{I}_T)^2}{\mathrm{Hess}(L_T)}(\hat{p}) \;=\; \prod_{ijk} \frac{[s_{ik}] + [s_{jk}]}{[s_{ik}] \cdot [s_{jk}]}$$

Corollary

If the 2-tree T is **planar** *then m_T is the restriction of m_n to T.*



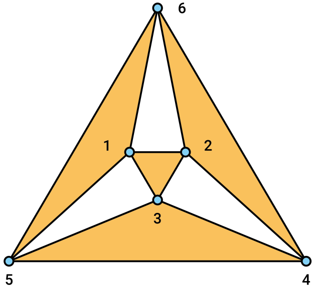


Hypertrees

... are collections triples $T = \{\Gamma_1, \ldots, \Gamma_{n-2}\}$ in $[n]$ such that

(a) each $i \in [n]$ appears in at least two triples, and

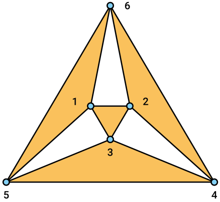
(b) $\left|\bigcup_{i \in S} \Gamma_i\right| \geq |S| + 2$ for all non-empty subsets $S \subseteq [n-2]$.



A-M. Castravet and J. Tevelev: *Hypertrees, projections, and moduli of stable rational curves*, Journal für die reine und angewandte Mathematik **675** (2013).

Hypertrees have the same number of triples as 2-trees, and M_T, $\mathcal{I}_T$ and m_T are defined as before. But 2-trees are not hypertrees.

On-shell diagrams



Example (Irreducible hypertree)

$T = \{123, 345, 156, 246\}$ has the matrix

$$M_T = \begin{bmatrix} p_{23} & -p_{13} & p_{12} & 0 & 0 & 0 \\ 0 & 0 & p_{45} & -p_{35} & p_{34} & 0 \\ p_{56} & 0 & 0 & 0 & -p_{16} & p_{15} \\ 0 & p_{46} & 0 & -p_{26} & 0 & p_{24} \end{bmatrix}$$

The *hypertree divisor* is an irreducible surface in $\mathcal{M}_{0,6}$, defined by

$$\Delta(M_T) = p_{12}p_{35}p_{46} - p_{13}p_{26}p_{45}.$$

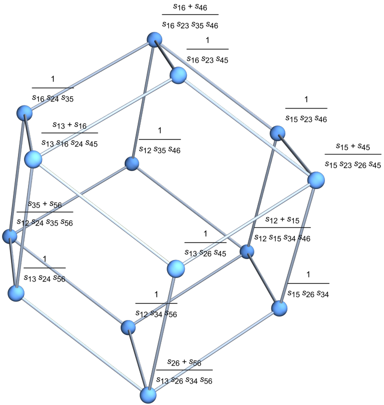
Hypertrees are **on-shell diagrams** in physics:

S. Franco, D. Galloni, B. Penante and C. Wen: *Non-planar on-shell diagrams*, Journal of High Energy Physics **6** (2015)

J.Tevelev: *Scattering amplitudes of stable curves*, Geometry & Topology (2024)

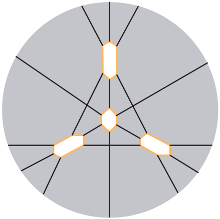
Hypertree amplitude

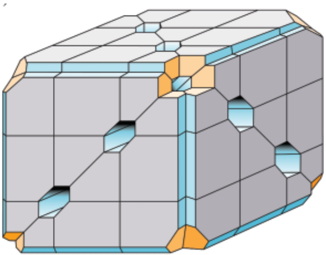
Summing over the **two critical points** of L_T yields



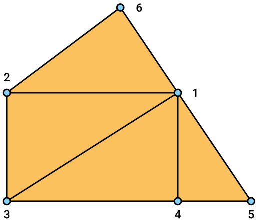
$$m_T = \frac{1}{s_{16}s_{24}s_{35}} + \frac{1}{s_{16}s_{23}s_{45}} + \frac{1}{s_{13}s_{26}s_{45}} + \frac{1}{s_{15}s_{23}s_{46}} + \frac{1}{s_{12}s_{35}s_{46}} + \frac{1}{s_{13}s_{24}s_{56}} + \frac{1}{s_{12}s_{34}s_{56}} + \frac{1}{s_{15}s_{26}s_{34}}$$
$$+ \frac{s_{15}+s_{45}}{s_{15}s_{23}s_{26}s_{45}} + \frac{s_{12}+s_{24}}{s_{12}s_{24}s_{35}s_{56}} + \frac{s_{12}+s_{15}}{s_{12}s_{15}s_{34}s_{46}} + \frac{s_{13}+s_{34}}{s_{13}s_{26}s_{34}s_{56}} + \frac{s_{13}+s_{16}}{s_{13}s_{16}s_{24}s_{45}} + \frac{s_{16}+s_{46}}{s_{16}s_{23}s_{35}s_{46}}.$$

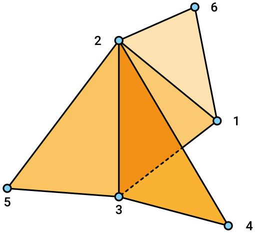
Conclusion

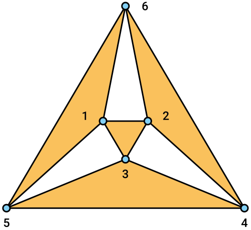




Physics of scattering amplitudes inspires combinatorics of $\mathcal{M}_{0,n}$.







Minimal kinematics builds on Horn uniformization