Rationalizing Choice Functions by Multiple Rationales[#]

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Abstract

The paper presents a notion of rationalizing choice functions that violate the "Independence of Irrelevant Alternatives" axiom. A collection of linear orderings is said to provide a *rationalization by multiple rationales* for a choice function if the choice from any choice set can be rationalized by one of the orderings. We characterize a tight upper bound on the minimal number of orderings that is required to rationalize arbitrary choice functions, and calculate the minimal number for several specific choice procedures.

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1. Introduction

Imagine that you receive information on the choices made by a decision maker (DM) from all subsets of some set X. You know nothing about the context of these choices. You look for an explanation for the DM's behavior. You would probably look first for a single rationale explaining the behavior. Specifically, you would seek for a rationalizing ordering — that is, a linear ordering on X, such that for every choice set $A \subseteq X$, the DM's choice from A is the best element in A according to the ordering. You recall that the "Independence of Irrelevant Alternatives" axiom (IIA) — which requires that the chosen element from a set also be chosen from every subset that contains it — is a necessary and sufficient condition for the existence of such an explanation.

If you had more information about the context of the DM's choices — that is, the content of the alternatives and his possible considerations — you might assess this explanation further. For example, you would probably be somewhat skeptical towards this explanation if you discovered that the rationalizing ordering minimizes the DM's well being. However, in the absence of information about the context of the DM's behavior, you are likely to find an explanation by a rationalizing ordering persuasive.

Real-life choice procedures often violate IIA. When confronted with such a procedure, we tend not to give up on explanation by a rationalizing ordering so quickly. We search for ways to argue that the procedure does not "really" violate rationality. We have different ways of doing this. One way is to argue that the DM's choices originated not from a single rationale, but from *several* ones, each being appropriate for a subset of choice problems.

Formally, let X be a (finite) set of alternatives. Denote its cardinality by N. Let $P(X)$ be the set of all non-empty subsets of X. A *choice function* on X assigns to every $A \in P(X)$ a unique element $c(A)$ in A (we confine ourselves to choice functions and not correspondences). Our central new concept is the following: a K-tuple of strict preference relations $\{ \succ_k \}_{k=1,..,K}$ on X is a *rationalization by multiple rationales* (RMR) of the choice function c if for every A, the element $c(A)$ is $\succ_k$ -maximal in A for some k .

One possible interpretation of this explanation method is that the choice set conveys information about its constituent elements and given this information, the DM chooses what he thinks is the best alternative. In other words, the DM has in mind a partition of $P(X)$ and he applies one ordering to each cell in the partition. A cell is like a state of the world. The DM's behavior is rationalized after the state of the world is added to the description of the alternatives.

Clearly, every choice function has an RMR, since every choice set can be said to have its own relevant rationale. Among the many RMR's that can be given to a choice function, we propose to focus on those, which employ the *minimal* number of orderings. The meaningfulness of this explanation method for any particular choice function c can thus be assessed by $r(c)$ — the minimal number of orderings in any RMR of c. When $r(c) = 1$, the choice function is rational in the usual economic sense. The larger $r(c)$, the less meaningful is the rationalization by multiple rationales that can be given to c.

Let us now consider three specific choice functions, all of which violate IIA. Given the context in which they are presented, the RMR method will appear particularly appealing.

Luce and Raiffa (1957)'s Dinner

A customer chooses a main course from a restaurant's menu. He chooses chicken when the menu consists of steak and chicken only, yet goes for the steak when the menu consists of steak, chicken and frog's legs. More generally, a DM is characterized by an alternative x* and two orderings, $\succ_+$ and $\succ_-$; he applies $\succ_+$ whenever $x^* \in A$ and $\succ_-$ whenever $x^* \notin A$.

IIA is violated in this example. This violation can be attributed to the "epistemic value of the menu", to use Sen (1993)'s terminology: the presence of frog's legs in the menu conveys information about the quality of the other items. Rationality is violated if an alternative is described just by the name of the dish, and restored if quality is added to the description. RMR requires two orderings: one is applied when frog's legs are on the menu and another is applied when they are not. The low value of $r(c)$ reflects the intuition that RMR is a sensible way of rationalizing such a choice function.

The $(\mathbf{u}, \mathbf{v})$ procedure

An individual aims to be as moral as possible. However, if pursuing this goal leads to harmful consequences from an egotistic point of view, he abandons morality and adopts the best selfish alternative. More generally, this procedure is based on three primitives: two numerical functions, u and v, defined on the set X, and a number v^*. When choosing an element from the set A, the DM maximizes u as long as the optimization of u leads to a consequence, whose v-value is above v^*. Whenever the v-value of the u-maximal element in A falls below v^*, the DM switches to the v-maximal element.

Typically, such a choice violates IIA. Consider two sets, $A \subset B$. The u-maximal element in A may have a v-value above v^* and will therefore be chosen from A. At the same time, the u-maximal element in B may have a v-value below v^*; hence, the choice from B will be the v-maximal element. This element may belong to A and yet be different from the u-maximal element in A. RMR typically requires two orderings (induced by u and v).

The (u, v) procedure has a "twin" procedure, in which the DM maximizes u as long as this leads to a consequence, whose u-value is above some number u^*; otherwise, he maximizes v. Like the original (u, v) procedure, the verbal description of the "twin" procedure employs two rationales. However, any choice function induced by the "twin" procedure can be rationalized by a single ordering, which is lexicographic with respect to the pairs $(max\{u(x), u^*\}, v(x))$: x is preferred to y if $u(x) > u(y)$ and $u(x) > u^*$, or if $u(x), u(y) \leq u^*$ and $v(x) > v(y)$.

"The best from among the popular" procedure

A recruiting officer has an ordering over all possible candidates for an academic job. He classifies the candidates according to their field. Given a set of candidates, he chooses the best one from the most popular field in the set. More generally, a DM partitions the alternatives in X into J categories, $\{X_j\}_{j=1,\ldots,J}$ (where $|X_j| \geq 2$ for every $j = 1, \ldots, J$). He also has in mind a basic ordering $\succ$ on X. Given a set of alternatives A, the DM chooses an alternative, which is the best in one of the categories with the highest cardinality. It is easy to see that $r(c) = J$. An RMR will utilize J orderings, where each $\succ_j$ is a modification of $\succ$, in which the elements comprising X_j are moved to the top.

We now turn to our analytical task in this paper: obtaining general bounds on $r(c)$, and computing it for a variety of choice procedures.

2. A Bound on $\mathbf{r}(\mathbf{c})$

A trivial upper bound on the minimal number of rationales that are needed to provide an RMR for any choice function is N, the cardinality of X. Any collection $(\succ_x)_{x \in X}$, in which $\succ_x$ is an ordering that ranks x first, is an RMR of all choice functions. Whatever the choice a is, it is explained "retrospectively" by the rationale $\succ_a$. Our first proposition employs a variant on this construction to establish a tighter upper bound on $r(c)$:

Proposition 1: $r(c) \leq N - 1$ for every choice function c.

Proof: Let c be a choice function. Let $b \neq c(X)$. For every $a \neq b$, let $\succ_a$ be an ordering, in which a is the top element and b is the second-best element. Consider a set $A \in P(X)$. If $c(A) = a$, then this choice can be rationalized by $\succ_a$. If $c(A) = b$, then $A \neq X$ and hence, there exists $x \in X$, such that $x \notin A$. Since $x \notin A$ and b is the second-best element in $\succ_x$, this ordering rationalizes $c(A) = b$. ■

Our next result shows that "almost all" choice functions need $N - 1$ orderings for an RMR.

Proposition 2: The proportion of choice functions that can be rationalized by less than $N - 1$ orderings tends to 0, as N tends to infinity.

The proof will make use of the following simple lemma:

Lemma 0: If $r(c) = K$, then there exists an RMR of c with K orderings, all of whose top elements are distinct.

Proof: Let $\{\succ_k\}_{k=1,..,K}$ be an RMR of c with $K \leq N$. Suppose that $\succ_i$ and $\succ_j$ have the same top element, a. Modify $\succ_i$ by moving a to the bottom. The new set of orderings is an RMR, since for every $A \in P(X)$: (1) If $c(A) = a$ was rationalized by $\succ_i$, it could also be rationalized by $\succ_j$; (2) If $c(A) \neq a$ was rationalized by $\succ_i$, then $a \notin A$ and therefore, $c(A)$ is still rationalized by the modified $\succ_i$. Continue modifying $\succ_i$ in this way until its top element appears on the top of no other ordering. This procedure is iterated for any other pair of orderings having the same top element, until the K orderings have distinct top elements. ■

Proof of Proposition 2: We will calculate an upper bound on the proportion of choice functions c, with $r(c) \leq N - 2$. We will then show that this upper bound tends to 0 as N tends to infinity.

Let Z be a set of triples (a, b, T), where a and b are two distinct elements in X and T is a subset of $X \backslash \{a, b\}$, which consists of no more than $(N - 2)/2$ elements. By Lemma 0, c has an RMR whose $r(c)$ orderings have distinct top elements. Thus, to every c with $r(c) \leq N - 2$, we can attach a triple $(a, b, T) \in Z$, such that there exists an RMR of c, for which (1) a and b are not top elements of any ordering and (2) T is the set of top elements in those RMR orderings, in which a is ranked above b. What is the proportion of choice functions c with $r(c) \leq N–2$, to which a *particular* triple $(a, b, T) \in Z$ can be attached in this way?

For $(a, b, T) \in Z$ to be attached to a choice function c with $r(c) \leq N–2$, it must be that for every $R \supseteq T \cup \{a, b\}$, $c(R) \neq a$. Otherwise, $c(R) = a$ could not be rationalized by any of the orderings: for any ordering, whose top element is in T, a is ranked below the ordering's top element which belongs to R; and for any ordering, whose top element is not in T, a is ranked below b which also belongs to R.

For a given R, the proportion of choice functions for which $c(R) \neq a$ is $(1 - 1/|R|) \leq (1 - 1/N)$. Therefore, the proportion of choice functions for which a is chosen from no superset of $T \cup \{a, b\}$ is at most $(1 - 1/N)^{2^u}$, where $u = (N - 2)/2$, since 2^u is a lower bound on the number of subsets of $X \backslash (T \cup \{a, b\})$. The cardinality of Z is smaller than $N^2 \cdot 2^N$. Hence, the proportion of choice functions having an *RMR* with $N - 2$ orderings is at most $(1 - 1/N)^{2^u} \cdot N^2 \cdot 2^N$, which tends to 0 as N tends to infinity. ■

Note that $(1 - 1/N)^{2^u} \cdot N^2 \cdot 2^N < 1$ for $N > 19$. Thus, the above proof implies that for $N > 19$, there is a choice function c, for which $r(c) = N - 1$. It is quite hard to construct such a choice function for an arbitrary N. In an earlier version of this paper, we used tools from projective geometry to construct such a choice function for $N = p^2 + p + 1$, where p is a prime number. (Details will be provided upon request).

3. The "second-best" and "median" procedures

In contrast to the examples presented in the introduction, we will now analyze two choice procedures, whose description does not explicitly involve multiple rationales. We will see that providing an RMR for these procedures requires a "large number" of orderings. In particular, $r(c)$ is not independent of N. The detailed calculations illustrate the type of analytical arguments involved in the application of the RMR method.

3.1. The "second-best" procedure

A DM has in mind an ordering $\succ$ on the set X. For every $A \subseteq X$, he chooses the second-best element in A, according to $\succ$. We will show that there is no way to rationalize the second-best procedure with a "small" (and independent of N) number of rationales.

Proposition 3: Let c be a choice function that follows the second best procedure. Then, $r(c) = log_2 N$, rounded off to the higher integer.

Proof: For the constructive part of the proposition, it is sufficient to deal with the case of N being a power of 2, since any RMR of a choice function over X will also rationalize the choice function restricted to a subset of X. The construction will be inductive. For $N = 2$, the single ordering that places $c(X)$ on top suffices.

Let $N = 2^{n+1}$ and let the set Y consist of the 2^n $\succ$-top elements in X. Denote $Z = X \backslash Y$. By the inductive assumption there is a profile of n orderings on Y, $(\succ_k)_{k=1,\ldots,n}$, which is an RMR for c defined on Y. Likewise, there is a profile of n orderings on Z, $(\succ'_k)_{k=1,\ldots,n}$, which is an RMR for c defined on Z. We will now construct $n+1$ orderings $\{P^k\}_{k=1,\ldots,n+1}$, which will provide an RMR for c defined on X. In each of the orderings P^k, $k = 1,\ldots,n$, the elements of Y appear on top, ranked by $\succ_k$, followed by the elements of Z, ranked by $\succ'_k$. In the ordering P^{n+1}, the elements of Z appear on top, ranked by $\succ$, followed by the elements of Y, ranked by $\succ$ as well.

Let us check that this $(n+1)$-tuple of orderings indeed provides an RMR of c, defined on X. Consider a choice set A. If $|A \cap Y| \geq 2$, then $c(A) = c(A \cap Y)$, by the definition of c. Therefore, $c(A)$ can be rationalized by P^k, where $\succ_k$ is the ordering that rationalizes $c(A \cap Y)$ in the RMR for c over Y. If $A \cap Y = \phi$, then $c(A)$ can be rationalized by P^k, where $\succ'_k$ is the ordering that rationalizes $c(A \cap Y)$ in the RMR for c over Z. If $|A \cap Y| = 1$, then $c(A)$ is rationalized by P^{n+1}.

It remains to be shown that if $(\succ_j)_{j=1,\ldots,n}$ is an RMR of c, then $N \leq 2^n$. Let z be the $\succ$-minimal element in X. For each j, define $A(j) = \{x \in X; x \succ_j z\}$. We will show that without loss of generality, we can assume that if both x and y belong to $A(j)$ and $x \succ y$, then $x \succ_j y$. Assume the contrary, i.e., $y \succ_j x$. If $c(S) = x$ is rationalized by $\succ_j$, then $y \notin S$. By the definition of c, S includes exactly one element that is $\succ$-better than x and therefore, $c(S \cup \{y\}) = x$. Thus, there must exist some ordering $\succ_h$, $h \neq j$, which rationalizes $c(S \cup \{y\}) = x$ – that is, $x \succ_h w$ for every $w \in S \cup \{y\}$, $w \neq$ x. But this means that $c(S) = x$ can be rationalized by $\succ_h$. Therefore, we can transform $\succ_j$ by moving x down, such that $z \succ_j x$, and still have a profile of orderings which rationalizes c.

No element $x \neq z$ can be ranked above z in all n orderings; otherwise, $c\{x,z\} = z$ is not rationalized by any of the orderings. In addition, no two elements x,y satisfying $x \succ y \succ z$ are ranked above z in the same set of orderings; otherwise, $c\{x,y,N\} = y$ is not rationalized by any of the n orderings, since for every j either $x \succ_j y$ or $z \succ_j y$. Thus, for every element $x \neq z$, there is a unique proper subset of orderings, in which x is ranked above z. The number of proper subsets of the n orderings is $2^n - 1$. Therefore, $N - 1 \leq 2^n - 1$. ■

The literature contains a number of anecdotal references to the second-best procedure. In Sen (1993), a college tea party participant consistently chooses the second-largest slice of cake. McFadden (1999) mentions people's tendency to choose the second-cheapest wine on the wine list in restaurants. In these examples, choice behavior violates IIA when alternatives are described by their size or price alone. However, in both cases, the context suggests that the DM's have additional considerations. In Sen's example, the DM may be concerned with his perceived greediness, while in McFadden's example, he may believe that the cheapest wine on the wine list is not drinkable. Thus, in each case, rationality of choice behavior is restored if the description of alternatives is extended, so as to include perceived greediness or wine quality.

It should be stressed that in both anecdotes, the appeal of the rationalizations derives from the additional information concerning the DM's considerations, provided by the choice problem's description. In contrast, our rationalization method is "context-free" — it does not make use of any information other than choice behavior. Therefore, it is not surprising that the minimal RMR constructed in the proof of Proposition 3 bears no resemblance to the ad-hoc rationalizations we provided for the two anecdotes.

3.2 The "median" procedure

A voter orders all potential political candidates along a left-right axis. Given a set of candidates, he chooses the "median" candidate. Formally, let $X = \{1,\ldots,N\}$. For every set $A \subseteq X$ consisting of an odd number of elements, let $med(A)$ be the median element in A, with respect to the natural ordering. (We do not define the choice function for sets consisting of an even number of elements.) We will show that $r(med)$ is quite close to $N-1$.

Proposition 4: For all N, $N - 6\sqrt{N} < r(med) \leq N - \sqrt{N} + 1$

Proof: Let us show first that $r(med) \leq N - \sqrt{N} + 1$. Denote $t = \sqrt{N} - 1$. Let $(\succ_k)_{k=t+1,\ldots,N}$ be a profile of orderings satisfying: (1) k is the top element of $\succ_k$; (2) every $j \in \{1,\ldots,t\}$ is the second-best element in $(N-t)/t$ of the orderings. If $med(A) \in \{1,\ldots,t\}$, then the set A does not contain more than $t-1$ elements from among $\{t+1,\ldots,N\}$. Since $(N-t)/t > t-1$, the choice $med(A)$ is rationalized by one of the $(N-t)/t$ orderings, for which $med(A)$ is the second-best element.

For the proof that $N - 6\sqrt{N} < r(med)$, we need an auxiliary claim:

Claim: If $r(med) = K$, then there exists an RMR of med consisting of K orderings with distinct top elements, which are adjacent (with respect to the natural ordering).

Denote $r(med) = K$. By the claim, there is an element j and an RMR of med with K orderings, whose top elements are the members of $T = \{j, j+1,\ldots, j+K-1\}$. Let $j' = N - (j+K-1)$ and assume that $j' \leq j$, without loss of generality. Assume, contrary to the proposition, that $j > 3\sqrt{N}$. Let $Z = \{j - \sqrt{N},\ldots, j-1\}$. Since the number of orderings is less than N, there exists $z \in Z$ that is ranked above all other members of Z in no more than $\sqrt{N}$ orderings. Denote the set of top elements in those orderings by T'. Construct a choice set $Y = Z \cup T' \cup A$, where A is a subset of $\{1,\ldots, j - \sqrt{N} - 1\}$, such that $med(Y) = z$. For this to work, we need $|A|$ to satisfy $z - (j - \sqrt{N}) + |A| = |T'| + j - 1 - z$. Rearranging, we obtain $|A| = |T'| + 2j - 1 - 2z - \sqrt{N}$. There are enough elements to assign to A because $j - z \leq \sqrt{N}$ and $z - |T'| \geq 2\sqrt{N}$ and thus $|T'| + 2j - 1 - 2z - \sqrt{N} \leq j - \sqrt{N} - 1$.

Now, $med(Y) = z$ is not rationalized by any of the K orderings: it cannot be rationalized by any of the orderings, whose top elements belong to T' because these orderings always rank z below some $x \in T'$; and it cannot be rationalized by any of the other orderings because these rank z below some other element in Z. ■

Proof of claim: Let T be the set of top elements in the K orderings. By Lemma 0, we can assume that $|T| = K$. Let $\succ_a$ be the ordering, whose top element is a. The proof is by contradiction. Without loss of generality, there exists an element $x > N/2$, such that $x+1 \in T$ and $x \notin T$. Construct a profile of K orderings $\{\succ'_a\}$ as follows. For every $a \neq x+1$, obtain $\succ'_a$ from $\succ_a$ by exchanging the positions of x and $x+1$, in case $x \succ_a x+1$. Obtain the ordering $\succ'_{x+1}$ from $\succ_{x+1}$ by moving x to the top. We need to verify that we can rationalize the choice from every set A with the new profile of orderings.

Clearly, whenever $x, x+1 \notin A$, $med(A)$ is rationalized by $\succ'_a$ iff it is rationalized by $\succ_a$. If $x+1 \in A$ and $x \notin A$ then either $med(A) = x+1$, in which case $\succ'_x$ rationalizes $med(A)$, or $med(A) \neq x+1$, and then $med(A)$ is rationalized by some $\succ'_a$, where $\succ_a$ rationalizes $med(A - \{x+1\} \cup \{x\})$.

We are left with the case of $x \in A$. If $med(A) = x$, then $\succ'_{x+1}$ rationalizes the choice. If $med(A) \neq x$, then there are two possibilities:
(1) $x+1 \notin A$ Let $B = A - \{x\} \cup \{x+1\}$. Then, $med(B) = med(A) \neq x+1$. $med(B)$ is not rationalized by $\succ_{x+1}$. Therefore, there exists $a \neq x+1$, such that $\succ_a$ rationalizes $med(B)$. That is $med(B) \succ_a y$ for every $y \in A \backslash \{x, med(B)\}$ and $med(B) \succ_a x+1$. By construction, $med(A) = med(B)$ is the $\succ'_a$-optimal element in A.
(2) $x+1 \in A$ If $med(A)$ is rationalized by some $\succ_a \neq \succ_{x+1}$, then $med(A) \succ_a y$ for any other $y \in A$. By construction, the relative ranking between $med(A) \neq x$ and any other $y \neq x+1$ is not changed in $\succ'_a$. Since $med(A) \succ_a x$, we also have $med(A) \succ'_a x+1$. Now, assume that $med(A)$ is rationalized by $\succ_{x+1}$. Then, $med(A) = x+1$. Since $x > N/2$, at least two elements smaller than x are missing from A. Let B be the union of A and those two elements. Therefore, $med(B) = x$, which is rationalized by some $\succ_a$, $a \neq x+1$. By construction, $\succ'_a = \succ_a$. Therefore, $\succ'_a$ rationalizes $med(A)$. ■

The high value of $r(med)$ indicates that RMR is not a meaningful explanation of the median procedure, and even less so than in the case of the second-best procedure. Under RMR, the DM always applies a single rationale to any particular choice problem, although different rationales may fit different choice problems. The median procedure may be better explained by the existence of two fixed, diametrically opposed rationales, such that the DM tries to balance between them in the face of any choice problem.

4. Final Comments

The economist's standard way of explaining choice behavior is by seeking an ordering, whose maximization is consistent with the behavior. We proposed a variant of this approach: seeking a "small" number of orderings, such that the choice in every "case" (a choice set) is consistent with maximization with respect to one of those orderings. We extend the standard approach to choice functions that violate IIA by following the premise that the appeal of the explanation depends on the number of orderings involved.

We fully acknowledge the crudeness of this approach. The appeal of the RMR proposed for "Luce and Raiffa's dinner" does not emanate only from its small number of orderings (two), but also from the simplicity of describing in which cases each of them is applied. There is an element $x^* \in X$, such that all choices from sets that include x^* are governed by one rationale and all other choices are dictated by another rationale. More research is needed to define and investigate more "structured" forms of rationalization.

As emphasized in the introduction, our approach is “*context-free*”. We agree with Sen (1993) that if “motives, values or conventions” are missing from our description of the alternatives, then we’d better correct our model, whether or not IIA is violated. For example, if we knew that when the DM picks a slice of cake, he also cares about his perceived greediness, we would find it fitting to explain his choice behavior by adjusting the consequence space, so as to accommodate this additional motive and we would find an RMR-based explanation inappropriate, even if we discovered that $r(c)$ is low. Thus, although we accept that “there is no way of determining whether a choice function is consistent or not *without* referring to something external to choice behavior” (Sen (1993)), we believe that rationalization by a minimal number of rationales is an appropriate extension of the economist’s standard approach – especially in the absence of information about “something external”.

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